Revised Edition

ELECTRICITY AND MAGNETISM

Dr. K.K. TEWARI

S. CHAND

ELECTRICITY AND MAGNETISM

For B. Sc. (Pass & Honours) and Engg. Students of Various Universities
&
As per UGC Syllabus

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PREFACE

This book entitled "Electricity & Magnetism" covers the syllabi of B.Sc. (Pass & Honours) and Engineering students of various Universities in India, and is written purely in S.I. Units (rationalised MKS system of units) with a complete vector treatment.

The S.I. system of units (rationalised MKS system of units) has rapidly come into almost universal use and has many practical advantages. On this emphasis the under-graduate courses have also been revised. In electricity and magnetism, particularly, the transformation of system of units simply by directly co-relating the final results from C.G.S. to RMKS does not lead to clear understanding, and is at times confusing and misleading. In fact, entire approach changes with change in system of units. Hence, a description of the subject purely in S.I. Units or RMKS system is needed.

The mathematical description of the book is based on the methods of vector analysis. Vector-analysis provides an efficient short-hand for writing Physics and at the same time makes it possible to visualise the physical meaning of concepts and laws distinctly and exactly. Hence, the vector treatment becomes necessary.

This book is based on lectures on the subject given by the author with about fifty per cent additional material. The emphasis is on the basic Physics with some instructive, stimulating and useful applications. Since the course of particular topic varies greatly between different Universities, hence, I have tried to organise the material, so that it is easily possible to select courses of different lengths and difficulty.

To make the students familiar, different systems of units are discussed in Ch. 1. Unless one is used, the vector treatment is somewhat difficult, hence, I have introduced vectors in Chs. 2 and 3, particularly, the physical meaning of div, grad and curl are explained. The Chs. 4–14 cover static electric and magnetic fields, steady currents, changing electric and magnetic fields, dynamics of charged particles, polarisation and magnetisation. In addition to the general aspects of electromagnetic fields, a discussion of the sources, development of forces and torques in such fields, the general features of the energy storage, energy flow and energy transfer and conversion are added.

Maxwell's equations are derived from the experimental laws of Gauss, Ampere and Faraday in Ch. 15. The A.C. theory and its applications are discussed in Chs. 16 and 17. The circuit analysis and its theorems are dealt in Ch. 18 separately.

Keeping in view the recent trend of various examinations, particularly competitive ones, illustrative objective questions are added at the end of each chapter in addition to problems and review questions.

Special features of the books are:—

(*i*) Use of S.I. Units throughout the book,

(*ii*) A full vector treatment of the subject,

(*iii*) A comparative study of electro-statics and magneto-statics as summarised in table 14.2,

(*iv*) Physical approach in treatment of polarisation and magnetisation,

(*v*) Proper explanation of the origin of fields $\vec{E}$ and $\vec{D}$ or $\vec{B}$ and $\vec{H}$ in terms of microscopic charge distribution or circulating currents to avoid any confusion,

(*vi*) An analogous study of polarisation and magnetisation, as shown in table 14.3,

(*vii*) Gradual and inter-related approach for development of the subject,

(*viii*) Large number of worked examples,

(*ix*) Objective questions for self-assessment,

(*x*) Simple and effective use of language.

In the end, I would request my esteemed readers to kindly send me their valuable suggestions for the improvement of the book and to notify me of any errors they might come across while going through it. Both will be gratefully acknowledged.

Dr. K.K. TEWARI

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And finally I am greatly thankful to my wife Uma, whose constant encouragement helped me to complete this book and to my children Master Saurabh and Gaurav for their patience and understanding during the preparation of the manuscript.

Dr. K.K. TEWARI

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CHAPTER 1

UNITS AND DIMENSIONS

1. PHYSICAL QUANTITIES AND PROPERTIES OF PHYSICAL EQUATIONS

All physical phenomena are represented by simple and compact expression of mathematics; physical concepts are designated by symbols, the relationships between concepts are expressed by formulae, and the correlations between phenomena are represented by equations. These equations are necessary for a reasonable discussion of the laws of physics, *so mathematics is the language of physics.* However, the physical formulae and equations are characterised by special properties and form a special class of mathematics. The knowledge of these properties is essential for an accurate formulation and intelligent application of the laws of physics. Particularly, in electricity and magnetism, this knowledge is essential for the understanding of the relations between different systems of electric and magnetic measurables used.

1.1 Quantitative Study of Physics

A quantitative study of physics starts with four preliminary procedures given below :

1. Selection of *basic,* or *primary, measurables* (basic objects of measurements) and specification of properties to be used for their identification.
2. Selection of instruments for the measurement of basic measurables.
3. Selection of standards and units for the calibration of these instruments and
4. Selection of *derived* or *secondary measurables* and specification of rules for their measurements.

The first of these procedures defines the concepts while the second and third procedures make it possible to associate a definite magnitude with basic quantities, thus completing their definition.

The fourth procedure involved a specified mathematical operation to be performed upon the results of primary measurements and defines the derived or secondary quantities.

All physical phenomena are described with the aid of these four procedures in terms of a few primary quantities and few secondary quantities; the correlations between these quantities are expressed in the form of equations.

These equations represent two different things; firstly they represent correlation in the physical phenomenon and secondly they represent our approach to the quantitative description of the physical phenomenon i.e. our selection of measurables, standards and units. However, this selection is arbitrary, as one can express the same set of correlation by using different measurables, standards and units. But this arbitrariness in the selection of measurables, standards and units is possible, if the equations satisfy the certain properties given below.

1.2 Ratio Requirement

Let us investigate how physical quantities, formulae, and equations are affected by the possibility of choosing different standards of units of measurements. Obviously, the correlation between physical phenomena are determined by the very nature of these phenomena, and do not depend on our choice

of standards or units. Since the correlations between physical phenomena are independent of the choice of standards or units, all equations which describe these correlations, must be invariant to a change of the size of the standards or the size of the corresponding units. Hence, the ratio of any two physical quantities, each of which represents the same measurable, must not depend on the units in terms of which these quantities are expressed; for example, the ratio of two distances must not depend upon whether these distances are measured in feet or in metres only. Those quantities that satisfy this *ratio requirement* are considered physically meaningful.

If both the primary and secondary quantities satisfy the ratio requirement, then the value of any secondary quantity represents a power product of the values of the primary quantities. Therefore physical quantities may be written as products of two factors. The first factor is a number and is called the *numerical value* of the quantity. The second factor is a power product of symbols, designating basic units or, in a general case, a power product of symbols designating basic measurables. This power product is called the *dimensions* of the quantity. Conceptually, different quantities usually have different dimensions and therefore dimensions are frequently used for identification of quantities.

1.3 Dimensional Homogeneity of Physical Equations

As we have discussed in the preceding section, all physical equations must be invariant to a change in the size of units. That is possible when the equation is *dimensionality homogeneous,* which means that only quantities possessing the same dimensions may be added, subtracted, or equated. If an equation is dimensionally homogeneous, any change of units cancels out and does not influence the equation. Thus the arbitrariness in the choice of standards and units imposes a condition on the equations that it must be dimensionally homogeneous.

1.4 Dimensional Constants

Another important property of physical equations is the presence of dimensional constant in them. This property arises from the fact that more than one basic measurable is used in the investiga-tion of physical phenomena.

Adding one more basic measurable to a given set of measureables results in the appearance of at least one new dimensional constant in the equations correlating this new measurable with those already present. Thus, in electricity and magnetism, the introduction of current and voltage, as new basic measurables in addition to length, mass, and time, results in the appearance of two new universal dimensional constants.

Dimensions

$$\text{Permittivity of free space } \varepsilon_0 = \left[\frac{\text{Current . Time}}{\text{Voltage . Length}}\right]$$

Units

$$\left[\frac{\dfrac{\text{Coulomb/Time}}{\text{Newton. metre}}}{\text{Coulomb}} \cdot \frac{\text{Time}}{\text{Metre}}\right] \quad \text{or} \quad \varepsilon_0 = \left[\frac{\text{Coulomb}^2}{\text{Newton. metre}^2}\right]$$

$$\text{Permeability of free space } \mu_0 = \left[\frac{\text{Voltage . Time}}{\text{Current . Length}}\right]$$

$$\text{Units} = \frac{\text{Newton . metre}}{\text{Coulomb}} \cdot \frac{\text{Seconds}}{\text{Ampere . metre}}$$

and Weber is special name given to the unit of $\vec{B}$ as follows:

$$\frac{1 \text{ Weber}}{\text{Metre}^2} = \frac{1 \text{ Newton}}{\text{Coulomb (metre/sec.)}}$$

$$\text{Weber} = \frac{\text{Newton . metre . second}}{\text{Coulomb}}$$

hence unit of permeability of free space μ_0 is $\left(\frac{\text{Weber}}{\text{Ampere . metre}}\right)$.

However, these constants are not present if electromagnetic phenomenon are formulated only in terms of length, mass and time as in the case of electrostatic, electromagnetic or gaussian systems.

The *significance of dimensional constants is generally under-estimated.* Sometimes they are regarded as a nuisance introduced in physics because of the necessity of "taking care of units". In fact dimensional constants originate from experimentally established correlation between physical quantities and may well be regarded as concise formulation of physical laws. Physical laws express certain permanent correlations between quantities. These permanent correlations are usually implicitly represented by the corresponding dimensional constants. When a dimensional constant enters an equation, it makes that equation subject to the corresponding law.

To illustrate this, let us consider, the well known ohm.'s law for electric conductors. According to this law, the ratio (V/I) of the voltage applied to a conductor and the current in this conductor has under certain conditions a constant value (R) independent of of either V or I. The ratio V/I is called the resistance R of the conductor. Thus ohm's law states that under certain conditions R is constant, or that there exists a dimensional constant R. Thus the statement of a physical law is equivalent to the statement of the existence of a certain dimensional constant.

1.5 Different Systems of Units in Electromagnetics

1. *CGS System*: There are two systems of units in C.G.S. system, namely,
2. Electrostatic unit (e.s.u.)
3. Electromagnetic unit (e.m.u.)

In these systems the difficulty was that same quantity has not only two different units one in e.s.u. and other in e.m.u, but also the dimensions of the physical quantity was different in two systems. Because μ_0 and and ε_0 are not dimensional constants they are constant in CGS system. Maxwell did not agree with this that the same quantity may have two dimensions. He imposed a condition on μ_0 and ε_0 that the ratio $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ must have the dimensions of velocity and

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \text{ (velocity of light)}$$

2. UNRATIONALISED MKS SYSTEM OR MKS SYSTEM

In C.G.S. System the three new electrical units for charge, current and voltage were introduced. However, to satisfy the Maxwell condition, in MKS System the unit of current, the ampere has been selected as a basis for other electrical quantities. Hence in addition to three fundamental units metre, kilogram and second as suggested by Giorgi in 1901 for mechanics, the ampere was chosen as fourth fundamental unit. Therefore unrationalised MKS System for electromagnetism is MKS A.

3. RATIONALISED MKS SYSTEM

With the introduction of ampere as a fourth unit all the other quantities were assigned units. It was

found that the factors 4π, 2π etc. occur in various electrical and magnetic quantities which do not have spherical, circular, radial or cylindrical symmetry whereas they (4π, 2π factors) do not occur in for the quantities which have either of the symmetry given above such as

Capacitance of parallel plate capacitor $C = \frac{A}{4\pi d}$ or

Field inside a straight solenoid $H = 4\pi ni$. In fact these two systems do not have any symmetry *(i.e.* spherical, cylindrical, circular or radial).

On the other hand the capacitance of spherical capacitor $\left(C = \frac{ab}{b-a} \right)$ which has spherical symmetry but do not have the factor π.

To avoid this situation Heaviside and Lorentz suggested that *electric and magnetic units in the MKS system should be redefined such that the factor π should occur in equations possessing spherical, circular, cylindrical or radial symmetry. The factor π should not occur in system which do not have symmetry such redefined MKS system is called rationalised MKS system,* hence the value and units to the constants are as follows :

1. $\varepsilon_0 = \frac{1}{4\pi \times 9 \times 10^9} = 8.85 \times 10^{-12}$ coul^2/Newt. m^2
2. $\mu_0 = 4\pi \times 10^{-7} = 1.25 \times 10^{-6}$ Weber/Amp. metre
3. $C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ metres/sec.
4. Length – Metre, Mass – Kilogram, Time – Second
5. Unit of force – Newton
6. Unit of energy = joule
7. Unit of power = watt.

4. SI UNITS

In 1965, the eleventh general conference on weights and measures adopted the rationalised MKS system as such and added the ampere as the unit of current for electrical units, the degree Kelvin as the unit of temperature and the unit of luminosity as candella for international use. These units are called, *SYSTEM INTERNATIONAL* units or SI Units. *Therefore, for electrical units, the SI units are the same as rationalised MKS system of units.*

1.6 Transformation of Units and Measurables

With the aid of dimensional analysis we can convert physical quantities and equations from one system to another. In electricity and magnetism there are two kinds of systems of basic measurables, namely the electro-mechanical systems and the mechanical systems. The electro-mechanical systems are followings:

1. The length, mass, time, voltage and current, *i.e.* LMTVI.
2. The length, mass, time and current, *i.e.* LMTI, *i.e.,* Rationalised MKS written as RMKS or SI System.
3. The length, mass, time and charge, *i.e.,* LMTQ.

The fundamental units in the first system are usually the metre, the kilogram, the second, the volt and the ampere *i.e.* MKSVA. Similarly in LMTI the units are MKSA and in LMTQ the units are MKSC.

The mechnical system is represented by length mass time *i.e.* LMT system. The units in this system are usually CGS — the centimeter, the gram and the second which are also called absolute units. The

LMT system is subdivided into following three systems.

1. Electrostatics
2. Electro-magnetic
3. Gaussian System.

To convert an equation to a new system of basic measurables, the symbols designating physical quantities are replaced by the corresponding symbols of the new system. The numerical and dimensional constants are replaced by the corresponding constants of the new system and the additional dimensional constants of the new system are introduced into the equation to make it dimensionally homogeneous in the new system as given in table 1.1.

1.7 Basic Measurables in Electricity and Magnetism

At the time when the first quantitative investigation of electric and magnetic phenomena were conducted, no electric or magnetic instruments were known. Almost all quantitative information had to be obtained through measurements of mechanical quantities with the aid of mechanical instruments. For a long time, the three mechanical measurables length, mass and time were used as the only basic measurables and were even believed to constitute the ultimate system of basic measurables.

Later, it was found that electric and magnetic phenomena could be investigated much more easily with the aid of special electric and magnetic instruments. It was also found that the description of these phenomena became more simpler and clearer if new electric or magnetic basic measurables were used together with the old mechanical ones. Finally, it was realised that there can be no ultimate system of measurables and that one should therefore use the system which serves its purpose best.*

Modern scientists obtain their knowledge about electric and magnetic phenomena mainly through the measurements of two electric quantities — current and voltage — and use current meters and voltmeter as basic tools for experimental investigations of these phenomena. In fact, measurements of current and voltage are the basic sources of quantitative information in almost any branch of modern experimental sciences. It is therefore natural to adopt formally current and voltage as the new basic measurables and to use them alongside with other basic measurables such as, for instance, length, mass, and time. This system of five basic measurables (the LMTVI System), is now generally used in experimental physics and is also most appropriate for the mathematical representation of physical phenomena.

1.8 Current as a Basic Measurable

In order to use current and voltage as basic measurables, we must first of all specify the properties that are attributed to them and that are used for their identification and qualitative definition.

The characteristic properties of that which we call current, or, more accurately, electric current, oan be demonstrated with the aid of the three following experiments.

Magnetic Property

If a wire is placed near a compass needle and is then connected to the terminals of a battery, the needle deflects from its initial position, just as it would if a magnet were placed near it. This magnetic action is attributed to the electric current produced in the wire by the battery and is regarded as the first characteristic property of electric current.

Thermal Property

If several turns of wire are wound around a thermometer, and the wire is connected to a battery, the thermometer shows that the wire heats up. Also, this thermal action is attributed to the current in the wire and is regarded as the thermal characteristic property of current.

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*The common systems of basic measurables are discussed in article (1.5).

Chemical Property

If two wires are inserted in a glass containing water, and if one end of each wire is then connected to a battery, small bubbles of gas begin to rise from the submerged ends of the wires. As the two previous effects, this chemical action too is attributed to electric current and is regarded as its third characteristic property.

On the basis of these three experiments we shall define electric current qualitatively as that which manifests itself by the magnetic, thermal, and chemical action in the manner described above.*

Current Meters

Like any basic measurable, electric current must be defined quantitatively by specifying the instruments with which it can be measured. One can construct current meters or galvanometers based on the magnetic, thermal or chemical action of the electric current. The most widely used current meters are, however, on magnetic action. To these three actions, or properties, of current one should add the property of causing a characteristic physiological sensation which makes it possible to feel the current. This property of current can also be used for electric measurements — for instance, for measurements in bridge circuits employing the null method. This incidently, shows clearly that a quantitative study of physical phenomena is possible without mechanical instruments or devices.

1.9 Voltage as a Basic Measurable

The characteristic properties of that which we call voltage can be demonstrated with the aid of the following two experiments.

Force-producing property

If two pieces of aluminium foil suspended by threads one near the other are connected by two wires to the terminals of a battery, these pieces attract each other (the battery should have several hundred individual cells in order to make the motion of the foil visible under class-room conditions). If the battery is changed, the force of attraction changes. The ability of a battery to produce force is attributed to the voltage generated by the battery, and the force producing property is regarded as the first characteristic property of voltage.

Current-producing property

If a wire is connected to the terminals of a battery, electric current is produced in the wire. The ability of a battery to produce current is attributed to the voltage generated by the battery. The current-producing property is therefore regarded as the second characteristic property of voltage.

On the basis of these two experiments we shall define voltage qualitatively in terms of the force and current action in the manner described above.

Voltage Meters

For the quantitative definition of voltage, we must specify the instruments with which it can be measured. Both characteristic properties of voltage can be used for its measurements. Voltage meters based on the force producing property of voltage are called electrostatic voltage meters, or electrometers.

1.10 Why Current is taken as Basic Measurable than Charge in SI System (i.e. RMKSA System)

In 1965, the eleventh general conference on weights and measures adopted the rationalised MKS System as such and added the ampere as the unit of current for electrical units. It means that the current is taken as basic measurable rather than charge. In other words, the current is considered as basic measurable and charge as secondary measurable.

It must be noted that the separation of measurables into basic and secondary is merely a matter of practical experience and has nothing to do with the establishment of ranks or priorities of some

* A definition in physics is never an explanation. A qualitative definition is merely the statement of a verbal convention; a quantitative definition is the statement of the rules of measurements or calculations.

physical quantities or concepts relative to others. Thus, the fact that we consider current, rather than charge, as a basic measurable does not mean that we regard current as a quantity more important or fundamental than charge; *it merely means that we regard a direct measurement of current as more expedient or convenient than that of charge.* Because of this reason current is taken as basic measurable in SI System.

Ex. 1.1. *The coulomb law in the LMTVI system is expressed as,*

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

Convert this formula to the LMT electrostatics system.

Sol. To convert this formula to the LMT, electrostatics system, we see from Table 1.1 given below that no symbols need to be replaced in the above formula. To make the equation dimensionally homogeneous ε_0 must be replaced by $1/4\pi$.

$$F = \frac{q_1 q_2}{r^2}$$

1.11 Symbols, Rationalised MKS Dimensions and derived practical Units for Electrical quantities are given in Appendix 2.

TABLE 1.1

Relationship between Dimensional Constants in Different Systems of Units

LMT VI	*LMTI*	*LMTQ*	*LMT (esu)*	*LMT (emu)*	*LMT (Gaussian)*
ε_0	ε_0	ε_0	$1/4\pi$	$1/4\pi c^2$	$1/4\pi$
μ_0	μ_0	μ_0	$4\pi/c^2$	4π	4π

REVIEW QUESTIONS

1.1 What are different systems of units in electricity and magnetism?

1.2 What is RMKS system of units? Outline its merits, compared to other systems of units in electricity and magnetism?

1.3 What are special features of SI Units?

1.4 Derive the dimensions of the following quantities in RMKS ; *(i)* electric field, *(ii)* magnetic field, *(iii)* magnetic dipole moment, *(iv)* potential difference, *(v)* capacity *(vi)* magnetisation.

SELF-TEST OBJECTIVE QUESTIONS

1.1 In addition to mass, length and time the other three physical quantities in the international system of units are:

(*a*) Current, voltage and temperature

(*b*) Voltage, temperature and luminous intensity

(*c*) Current, temperature and luminous intensity

(*d*) Current, voltage and luminous intensity

1-2 Beside metre, kilogram and second, the other basic units ol SI System are:

(a) Ampere, volt, kelvin and candela

(b) Ampere, volt and candela

(c) Ampere, volt, candela and Newton

(d) Ampere, kelvin and candela

1.3 In order to express physical quantities precisely, wo need a:

(*a*) unit

(*b*) derived unit
(*c*) unit and a number
(*d*) rationalised system

1.4 The fact that we consider current, rather than charge, as a basic measurable, it merely means:
(*a*) that we regard current as a quantity more important than charge
(*b*) that we regard current as a quantity less important than charge
(*c*) that we regard current as a quantity more (or less) fundamental than charge
(*d*) that we regard a direct measurement of current as more expedient or convenient than that of charge.

1.5 To avoid the use of factor 4π in most of the calculation, one would use:
(a) absolute system
(*b*) rationalised system
(*c*) CGS System
(*d*) SI System

Ans. 1. (*c*), **2.** (*d*), **3.** (*c*), **4.** (*d*) and **5.** (*c*)

CHAPTER 2

VECTOR ANALYSIS (ALGEBRA)

In the last chapter we said that mathematics is the language of physics. This language becomes specially simple and clear if it is based on the methods of *vector analysis.* In fact, *vector analysis provides an efficient short-hand for writing physics, and at the same time makes it possible to visualise the physical meaning of the equations distinctly and exactly.* Particularly in electromagnetic theory the vector analysis is used as mathematical tool as well as a mode of thought, the fundamentals of vector analysis are given below.

2.1 Symmetry in Physical Laws

The word symmetry commonly used in physics has a special meaning. A picture is symmetrical if one side is somehow the same as the other side. *Precisely, a thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation.* For example, if we look at a base that is left and right symmetrical, then turn it 180° around the vertical axis it looks the same. Similarly the Newton's laws of motion do not change when the coordinates are moved from one place to other. Therefore, we say that the *laws of physics are symmetrical for translational displacements.* The same is true for rotational displacement. Not only Newton's laws, but all the other laws of physics known so far have the two properties which we call invariance (or symmetry) under translation of axes and rotation of axes. *On the basis of these properties a new mathematical technique has been developed for writing and using physical laws. A very powerful mathematical machinery has been devised called vector analysis and in fact this is the symmetry of physical laws.* The word *vector* in English is *vecteur* in French, *vektor* in German and *BekTOP* (pronounced vector) in Russian have the same meaning *i.e., A vector is a quantity having direction as well as magnitude.*

2.2 Scalar and Vector Quantities

Physical quantities are divided into two main classes : (1) Scalar quantities or *scalars* and (2) vector quantities or *vectors.*

1. Scalar Quantities

Physical quantities which have magnitude only but not associated with a direction or orientation in space are called scalars. Such quantities are specified by their numerical value and dimensions.

Examples of scalars are mass, volume, density, temperature, energy, electric potential and charge etc.

Scalars are manipulated by six elementary operations, namely, *addition* and *subtraction, multiplication* and *division, involation* (*i.e.* raising a number to a power) and *evoluation.* In each pair the second operation is the *inverse* of the first. Multiplication is extended addition and involation is extended multiplication similarly subtraction, division and evoluation are successively extended inverse operations. All these mathematical operations with scalars obey the rules of ordinary algebra by means of certain laws given below:

(*i*) **Law of Association:** According to this law, the sum or the product of a finite number of scalars is independent of the manner in which they may be grouped or associated.

For example,

$A + B + C = A + (B + C) = (A + B) + C = (A + C) + B$

and

$A \times (B \times C) = (A \times B) \times C = (A \times C) \times B$

i.e., the brackets are unnecessary in continuous sums and products.

(*ii*) **Law of Commutation:** It states that the result of addition or multiplication of a number of scalars is quite independent of the order in which they may be taken. In other words additions and multiplications can be made in any order. Thus,

$$A + B = B + A$$

$$A \times B = B \times A$$

(*iii*) **Law of Distribution:** It states that in both addition and multiplication the result is the same as the sum of individual products taken term by term. For example,

$$A \times (B + C) = A \times B + A \times C$$

and $(A + B) \times C = A \times C + B \times C$

(*iv*) Finally, a fourth law, fundamental in the logical discussion of inverse operations is usually stated as follows:

If $A \times B = 0$

then either $A = 0$ or $B = 0$

2. Vector Quantities

However, many *physical quantities which have magnitude as well as some direction or orientation are called vector quantities.* To specify such quantities completely, we require the statement of the direction or orientation in addition to the numerical value and dimensions.

Examples of such quantities are moment of inertia, acceleration, force, velocity, displacement, momentum, magnetic induction, magnetic and electric intensities etc. Moreover a vector must obey the well known parallelogram Law of Addition (Art. 2.7).

When is a Physical Quantity Represented by a Vector

A quantity that have magnitude and direction is not necessarily a vector quantity. As an example take the case of a finite rotation of a rigid body about a given axis : it has a magnitude (*viz.* the angle of rotation) and also a direction (*viz.* the direction of the axis of rotation). But it is not a vector quantity because the two finite rotations of the body do not add up according to vector law of addition, *i.e.* the resultant rotation is not the vector sum of the two finite rotations.

However, if the rotation is small or infinitely small say (δs), it may be regarded as a vector quantity because here the arc described by the body in a small interval of time (δt) is nearly a straight line and can be represented as a vector. Consequently the transverse velocity

$\frac{ds}{dt} = \left(\lim_{\delta \to 0} \frac{\delta s}{\delta t}\right)$ at the point may also be represented by a vector. By a similar reasoning

angular velocity

$$\frac{d\theta}{dt} = \left(\lim_{\delta t \to 0} \frac{\delta \theta}{\delta t}\right)$$

is a vector quantity. Hence for small rotation, angular velocity and angular momentum are vector quantities whereas for large or finite rotations these are not vectors.

Thus, to qualify as a vector, a physical quantity must satisfy two conditions.

1. It must satisfy the parallelogram law of addition.
2. It must have a magnitude and a direction independent of the choice of coordinate system.

2.3 Vector Notation

In ordinary writing a scalar quantity is represented in light face italics or by ordinary letters such as A where as a vector quantity on a blackboard or paper is represented by putting a wavy line underneath the letter as $\underset{\sim}{A}$ or by putting an arrow over the letter, as $\vec{A}$. In print, however, it is represented by the bold-faced type-letter **A**.

The magnitude of a vector being a scalar quantity is represented in italics. For example, A is the magnitude of vector $\vec{A}$. The magnitude of a vector is also called as modulus or mod $\vec{A}$. denoted as $|A|$.

2.4 Graphical Representation of a Vector

Graphically a vector is represented by an arrow whose length is equal to the magnitude of the vector (Fig. 2.1). The arrow is drawn on a choosen scale parallel to the direction of the vector. Thus, the length and direction of arrow represent the magnitude and the direction of the vector.

Fig. 2.1

2.5 Some Important Definitions About Vectors

***(i)* Unit Vector:** A vector of unit magnitude is called a unit vector. A unit vector in the direction of A is written as A read as A, hat or A, caret. Sometimes it is also denoted by bold faced small letter a. So that

$$\vec{A} = \hat{A}\,A \text{ or } \vec{A} = Aa \qquad ...(2.1)$$

***(ii)* Zero or Null Vector:** It is a vector of zero magnitude denoted by 0. For null vector initial and terminal points are coincident.

***(iii)* Proper Vectors:** Vectors other than null vectors are called proper vectors.

***(iv)* Negative Vector:** A vector having same magnitude but direction opposite to that of given vectors is called negative vector relative to that vector as shown in Fig. 2.2.

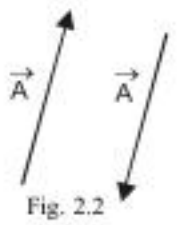

Fig. 2.2

***(v)* Co-Initial Vectors:** Vectors are called co-initial when they have common initial point.

***(vi)* Collinear or Parallel Vectors:** Those vectors, which have the same line of action or having lines of action parallel to the same line are called collinear vectors. Thus vectors having the same line of action (as $\vec{A}$ and $\vec{C}$) or parallel lines of action (as $\vec{A}$ and $\vec{B}$), in Fig. 2.3, are said to be collinear : they need not have the same direction.

However, if the directions of the two parallel vectors be the same, they are represented as *like vectors*. The like vectors have the following important property.

If $\vec{A}$ and $\vec{B}$ are two collinear or like vectors there exists a scalar n such that $\vec{B} = n\vec{A}$, the absolute value of n being the ratio of the two lengths of the two collinear or like vectors.

***(vii)* Equality of Vectors:** Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal, if they have same magnitude and direction. For example, if $\vec{A}$ and $\vec{B}$ are equal vectors, then

$$\vec{A} = \vec{B}$$

Although a vector may refer to a physical quantity defined at a particular point it does not necessarily have any particular location. Therefore, we can compare two vectors even though they may measure physical quantities defined at quite different points of space and time so that it is not necessary that the two vectors should have the same origin. Thus in Fig. (2.3) PQ and RS are two parallel lines in space and if $|\vec{A}| = |\vec{B}| = |\vec{C}|$

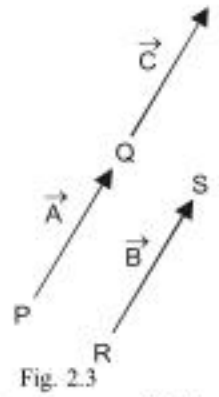

Fig. 2.3

Then $\vec{A} = \vec{B} = \vec{C}$

In fact such vectors are called *free vectors* to distinguish them from a *localised vsctor* which must pass through a specified point in space such as a position vector.

2.6 Multiplication and Division of a Vector by a Scalar

The product $n\vec{A}$ or $\vec{A}n$ of a vector $\vec{A}$ and a scalar n is defined as a vector whose magnitude is equal to $n|A|$, and whose direction is the same as that of $\vec{A}$, the relation $\vec{B} = n\vec{A}$ is represented by the arrow whose length is n times the length of the vector $\vec{A}$ and whose direction is parallel to $\vec{A}$. Fig. 2.4 shows veotor $\vec{A}, \vec{B}$ and $\vec{C}$ where,

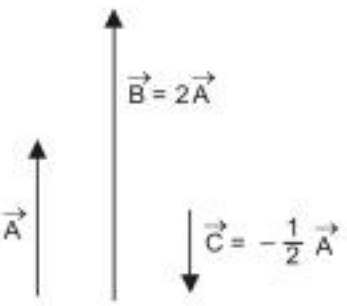

Fig. 2.4

$$\vec{B} = 2\vec{A}$$

$$\vec{C} = -\frac{1}{2}\vec{A}$$

If n is a scalar,

$$n(\vec{A} + \vec{B}) = n\vec{A} + n\vec{B} \qquad ...(2.2)$$

and $$(n + m)\vec{A} = n\vec{A} + m\vec{A}$$

so that multiplication of a vector by a scalar is distributive over addition of the vectors as well as over addition of the scalars. The division of vector $\vec{A}$ by a non-zero scalar m is simply the multiplication of vector $\vec{A}$ by $1/m$. Therefore the unit vector $\hat{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{A}$$

2.7 Addition and Subtraction of Two Vectors

1. Addition

1. The Parallelogram Law: Vectors are added in accordance with the parallelogram law of addition, according to which the sum or the resultant $\vec{R}$ of two vectors $\vec{A}$ and $\vec{B}$ is the diagonal of the parallelogram of which $\vec{A}$ and $\vec{B}$ are the two adjacent sides as shown in Fig. 2.5(*a*) *i.e.* $\vec{R} = \vec{A} + \vec{B}$.

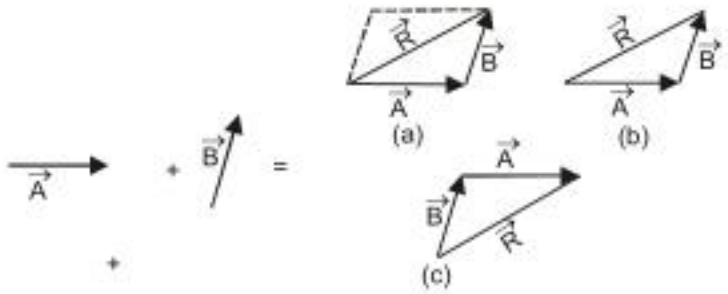

Fig. 2.5

2. The Triangle Law: Let $\vec{A}$ and $\vec{B}$ be the two vectors for obtaining their sum or resultant, $\vec{B}$ is carried parallel to itself until the tail of $\vec{B}$ coincides with the head of $\vec{A}$. Now the vector $\vec{R}$ drawn between the tail of $\vec{A}$ and the head of $\vec{B}$ is the sum $\vec{A} + \vec{B}$ as shown in Fig. 2.5(*c*) *i.e.*,

$$\vec{R} = \vec{A} + \vec{B}. \qquad ...(2.3)$$

The same result is obtained if the head of $\vec{B}$ is attached with tail of $\vec{A}$ as shown in Fig. 2.5(*c*) *i.e.*, the resultant $\vec{R}$ is the same irrespective of the order in which the vectors are taken.

Then,

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

So that, the vector addition is *commutative.*

2. SUBTRACTION

Vectors are subtracted with the aid of the negative vectors. The negative vector $-\vec{A}$ is defined as the vector whose magnitude is the same as that of $\vec{A}$ but whose direction is opposite to the direction of $\vec{A}$.

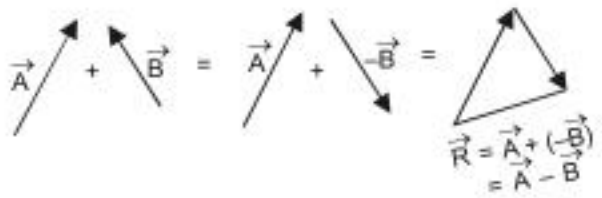

Fig. 2.6

The difference of two vectors $\vec{A}$ and $\vec{B}$ is defined as the sum of $\vec{A} + (-\vec{B})$. Thus the resultant $\vec{R} = \vec{A} + (-\vec{B})$ is shown in Fig. (2.6).

To subtract the vector $\vec{B}$ from $\vec{A}$, we reverse the vector $\vec{B}$ and the rule of addition applied.

2.8 Addition of More than Two Vectors

The Polygon Law of Vectors' Addition

The sum of more than two vectors is obtained in accordance with the polygon law of addition which states that if a vector polygon be drawn, placing the tail end of each succeeding vector at the head of the preceding one. Then their resultant $\vec{R}$ is drawn from the tail-end of the first to the head-end of the last as shown in Fig. 2.7 (*a*) and (*b*).

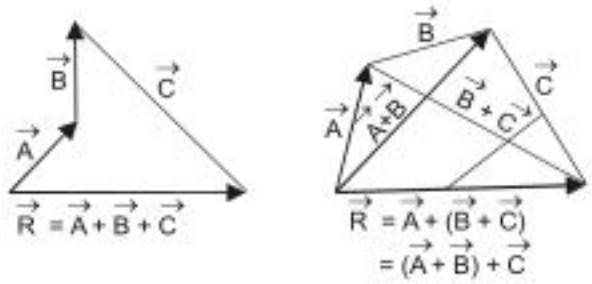

Fig. 2.7

The resultant from Fig. 2.7(*b*) is given by,

$$\vec{R} = \vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C} \qquad ...(2.4)$$

Showing that vector addition is also *associative* like scalar addition and is applicable to any number of vectors.

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2.9 Right Handed and Left Handed Coordinate System

In actual practice vectors are represented in a coordinate system and the one most generally used is the familiar cartesian coordinate system. They may be of two types— (*i*) right handed (Fig. 2.8*a*), (*ii*) left handed (Fig. 2.8*b*).

In the case of the right handed coordinate system, the thumb, the low finger and the middle finger of the right hand give the relative directions of three mutually perpendicular axes X, Y and Z respectively. Its mirror image (Fig. 28b) is the left handed system. In principle any one of the two coordinate systems may be employed

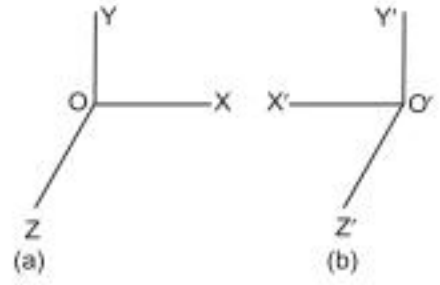

Fig. 2.8

but the right handed system is commonly used, because of convenience. The direction of axes in the right handed system are given by right hand screw rule according to which a rotation from X to Y axis through small angle moves the screw, in the positive direction of Z-axis, if seen from the origin.

2.10 Components of a Vector

Vector $\vec{A}$ can be decomposed into any number of vectors whose sum is $\vec{A}$. The vectors obtained after decomposition are called *components of vector* $\vec{A}$. The process of decomposition is the reverse of addition. The components most frequently used, are those *mutually perpendicular* or *rectangular ones* along the three coordinate axes X, Y and Z respectively, of the right handed cartesian coordinates system and they are called rectangular components of the vector.

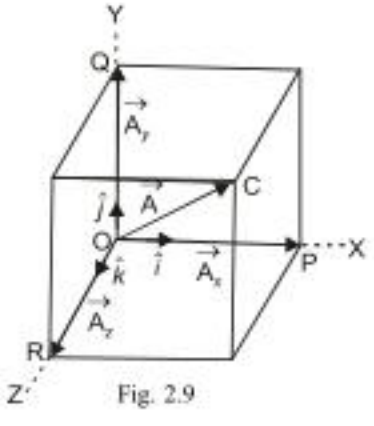

Fig. 2.9

Let us consider the origin of the coordinates system at the starting point of a vector $\vec{A}(=\vec{OC})$. Now construct a rectangular parallelopiped with the three edges which meet at O lying along the three axes such that $\vec{A}$ is the diagonal through O, as shown in Fig. 2.9.

Let A_x, A_y and A_z be the vector intercepts of A (or three components of A, or the resolved part of A) along X, Y and Z axes respectively. Then we have

$$\vec{A} = \vec{A_x} + \vec{A_y} + \vec{A_z} \qquad ...(2.5)$$

Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be unit vectors along X, Y, Z axes respectively. If $\mathbf{A}_x$, $\mathbf{A}_y$ and A_z are the magnitudes of the vectors $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ respectively, then, we have,

$$\vec{A_x} = \mathbf{A}_x, \ \vec{A_y} = A_y \hat{j} \text{ and } \vec{A_z} = \mathbf{A}_Z \hat{k}$$

therefore, the equation (2.5) can be written as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \qquad ...(2.6)$$

By the geometry (Theorem of Pythagorus in three dimension) of the Fig. 2.9, we have,

$$OC^2 = OP^2 + OQ^2 + OR^2$$

or $$A^2 = A_x^2 + A_y^2 + A_z^2 \qquad ....(2.7)$$

or $$A = [A_x^2 + A_y^2 + A_z^2]^{1/2} \qquad ...(2.8)$$

where A is the magnitude (or modulus) of vector $\vec{A}$. So that the magnitude of a vector is equal to the under root of the sum of the squares of its rectangular components.

The magnitude of the component vectors are given by

$$\vec{A_x} = \vec{A} \cos(\vec{A}, x), \ = \vec{A_y} = \vec{A} \cos(\vec{A}, y)$$

$$\vec{A_z} = \vec{A} \cos(\vec{A}, z) \qquad ...(2.9)$$

Thus, $$\cos(\vec{A}, x) = \vec{A}x / \vec{A} \qquad ...(2.10)$$

$$\cos(\vec{A}, y) = \vec{A}y / \vec{A} \text{ and } \cos(\vec{A}, z) = \vec{A}_z / \vec{A}$$

With the help of cosines given in equation (2.10) the direction of vector $\vec{A}$ can be determined, hence these are called *direction cosine of vector* $\vec{A}$. They are generally represented by the letters *l*, *m* and *n* respectively.

Putting the value of A_x, A_y and A_z from (2.9) in (2.8) we get,

$$\cos^2(\vec{A}, x) + \cos^2(A, y) + \cos^2(A, z) = 1$$

or $$l^2 + m^2 + n^2 = 1 \qquad ...(2.11)$$

Thus the *sum of the squares of the three direction cosines of a vector is equal to unity.*

The unit vector $\hat{A}$ in the direction of $\vec{A}$ is given by,

$$\hat{A} = \frac{\vec{A}}{A} = \frac{A_x}{A} + \hat{i} + \frac{A_y}{A} \hat{j} + \frac{A_z}{A} \hat{k}$$

But from equation (2.10) A_x/A, A_y/A and A_z/A are the direction cosines of $\vec{A}$.

Therefore, $\hat{A} = \cos(\vec{A}, x)\hat{i} + \cos(\vec{A}, y)\hat{j} + \cos(\vec{A}, z)\hat{k}$ ...(2.12)

The sum of two vectors can be expressed as

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k} \quad ...(2.13)$$

Its magnitude $|R| = [(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2]^{1/2}$ ...(2.14)

its direction cosines are $(A_x + B_x)/|R|$ etc.

In general, if we have a set of vectors $\vec{A}_1, \vec{A}_2, \vec{A}_3$ etc. their sum can be expressed in terms of rectangular components as follows :

$$\Sigma\vec{A} = (SA_x)\hat{i} + (SA_y)\hat{j} + (SA_z)\hat{k} \quad ...(2.15)$$

Thus the *addition of vectors reduces to the addition of the corresponding scalar components of these vectors.*

Similarly, the multiplication of a vector by a scalar reduces to the multiplication of the scalar components of this vector by the scalar. For example

$$n\vec{A} = (nA_x)\hat{i} + (nA_y)\hat{j} + (nA_z)\hat{k}$$

Thus, the *representation of vectors by means of components reduces the mathematical operations with vectors to purely algebraic operation with their components and eliminating the necessity of geometrical construction for calculations.*

2.11 Position Vector

If the coordinates of a point P be (x, y, z) relative to the origin of a cartesian coordinate system, then its distance from the origin can be specified by the vector $\overrightarrow{OP} = \vec{r}$ as shown in Fig. 2.10.

This vector is known as position vector of point P relative to O. We have,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad ...(2.16)$$

where, $r = [x^2 + y^2 + z^2]^{1/2}$ ...(2.17)

The direction cosines of r obviously are (x/r), (y/r) and (z/r). If there are two points P and Q, having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) and from the origin O the position vectors $\overrightarrow{OP} = \vec{r}_1$ and $\overrightarrow{OQ} = \vec{r}_2$ as shown in Fig. 2.11,

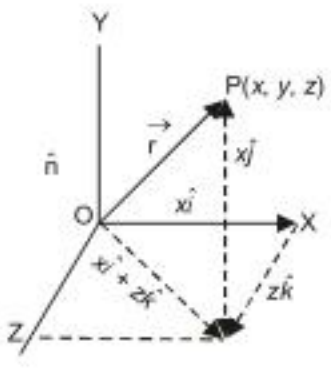

Fig. 2.10

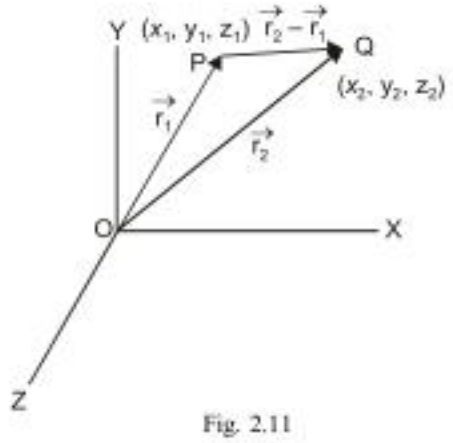

Fig. 2.11

we have $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

and therefore, $\vec{PQ} = \vec{r_2} - \vec{r_1}$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \qquad ...(2.18)$$

2.12 Coplanar Vectors

Vectors which are confined to the same plane, are called coplanar vectors. If $\vec{A}$ and $\vec{B}$ are two vectors in the same plane, then $\vec{R} = n_1\vec{A} + n_2\vec{B}$ will always be a vector in the plane containing $\vec{A}$ and $\vec{B}$. Thus the vector $\vec{R}$ is coplanar with $\vec{A}$ and $\vec{B}$, whatever be the values of the scalars n_1 and n_2.

However, in the case of three or more vectors, they are coplanar when they are parallel to the same plane, and any plane parallel to it is called as the plane of vectors.

If three vectors, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar vectors, then we have always three numbers n_1, n_2, n_3 such that

$$n_1 A + n_2 B + n_3 C = 0 \qquad ...(2.19)$$

Conversely, if it is given that $\vec{A}, \vec{B}, \vec{C}$ satisfy the condition (2.19) it can be proved that they are coplanar. The equation (2.19) can be rewritten as,

$$\vec{A} = \frac{n_2\vec{B}}{n_1} - \frac{n_3\vec{C}}{n_1}$$

Since the quantities $-n_2/n_1$ and n_3/n_1 are purely arbitrary scalars, the vector $\vec{A}$ must be coplanar with vectors $\vec{B}$ and $\vec{C}$.

If we express $\vec{A}, \vec{B}$ and $\vec{C}$ in components, the condition (2.19) for their coplanarity reduces to

$$\left.\begin{aligned} n_1A_x + n_2B_x + n_3C_x = 0 \\ n_1A_y + n_2B_y + n_3C_y = 0 \\ n_1A_z + n_2B_z + n_3C_z = 0 \end{aligned}\right\} \qquad ...(2.20)$$

On eliminating n_1, n_2, n_3 the Eq. (2.20) reduces to

$$A_x(B_y - C_yB_z) + B_x(C_yA_z - A_yC_z) + C_x(A_yB_z - B_yA_z) = 0 \qquad ...(2.21)$$

or in the determinant form, we have.

$$\begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ A_z & B_z & C_z \end{vmatrix} = 0 \qquad ...(2.22)$$

In fact, we shall see in Art. (2.22) that left hand side of Eq. (2.22) represents the volume of a parallelopiped whose sides are $\vec{A}, \vec{B}$ and $\vec{C}$. And if $\vec{A}, \vec{B}$ and $\vec{C}$ lie in the same plane, then the volume of parallelopiped is zero *i.e.,* Eq. (2.22) is the condition for coplanarity of the three vectors.

2.13 Polar and Axial Vectors

Vectors can be classified into two classes :

(*i*) Polar Vectors and

(*ii*) Axial Vectors.

A vector corresponding to a linear effect in its direction is called polar vector, for example linear velocity, linear momentum, force etc., on the other hand the vectors corresponding to the rotation about a certain axis are said to be axial vectors *e.g.* angular velocity, angular momentum, couple etc. An axial vector is represented by a straight line indicating the axis with a curved ring like arrow round it indicating the direction of rotation.

2.14 Invariance of the Magnitude of a Vector on the Rotation of the Reference Frame

Let us take a vector $\vec{A}$ as shown in Fig. 2.12, we have,

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Now, rotate the reference frame about the origin, say, in the anticlockwise direction without changing the direction of $\vec{A}$, *i.e.,* keeping it fixed in its position (w.r.t. any fixed objects) such that the coordinate axes take the position OX′, OY′ and OZ′, as shown in Fig. 2.12. Now, let new unit vectors be $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ and the new components

of A be A_x', A_y' and A_z'along x', y' and z' respectively. We have,

$$\vec{A} = A_{x'}\vec{i'}_1 + A_{y'}\hat{j}' + A_{z'}\hat{k}'$$

As the length of vector $\vec{A}$ is quite independent of the orientation of the coordinate system, or its frame of reference, hence,

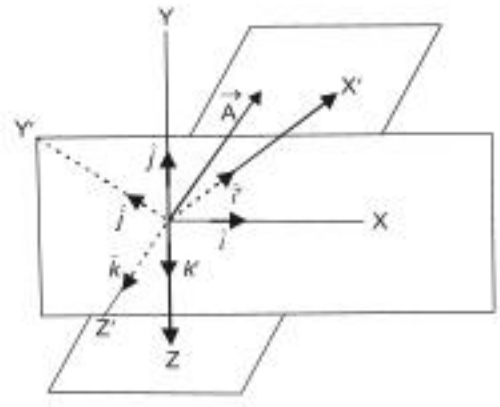

Fig. 2.12

$$A^2 = A_x^2 + A_y^2 + A_z^2 = A'_{x'} + A'_{y'}\, A'_{z'}$$

or, $$A = [A^2_x + A_y^2 + A_z^2]^{1/2} = [A'_{x'} + A'_{y'}\, A'_{z'}]^{1/2} \qquad \text{....(2.23)}$$

Thus the magnitude of a vector does not change by rotating the cartesian system (or the frame of reference) *i.e.* the invariance of the magnitude of a vector under the said condition.

2.15 Product of two Vectors

There are two particularly useful ways in which we define the product of two vectors. The first kind of product is called the *number* or the *scalar product* and the second kind is called the *vector product.* Both products are useful in physics. Both products satisfy the distributive law of multiplication.

2.16 Scalar or Dot Product of Two Vectors

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \cdot \vec{B}$ and is read as $\vec{A}$ dot $\vec{B}$. It is defined as a scalar equal to the product of the magnitudes of these vectors multiplied by the cosine of the angle between them as shown in Fig. 2.13a.

$$\vec{A} \cdot \vec{B} = AB \cos (\vec{A} \cdot \vec{B}) \qquad ...(2.24)$$

We note that $\cos (\vec{A} \cdot \vec{B}) = \cos (\vec{B} \cdot \vec{A})$ so that the scalar product is *commutative i.e.,*

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \qquad ...(2.25)$$

Further, the dot product AB cos θ of the vector $\vec{A}$ and $\vec{B}$ may be considered as either the multiplication of the magnitude of $\vec{A}$ and the projection (or the resolute) of $\vec{B}$ in the direction of $\vec{A}$ *i.e.,* B cos θ as shown in Fig. 2.13*b*.

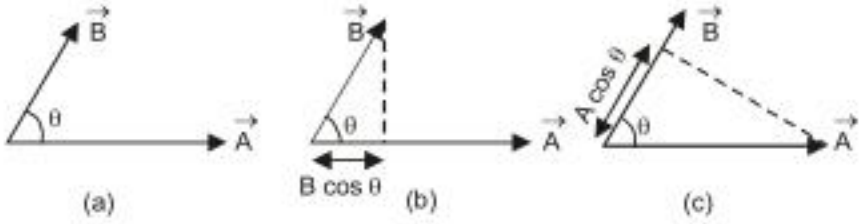

Fig. 2.13

$$AB \cos \theta = A (B \cos \theta)$$

The dot product also possesses the *distributive property i.e.,*

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \qquad ...(2.26)$$

In Fig. 2.14 vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and $\vec{C}$ are represented by $O\vec{a}$, $O\vec{b}$ and $b\vec{c}$ respectively, so that

$$O\vec{c} = \vec{B} + \vec{C}$$

O*d* and O*e* are the projection of vectors $\vec{B}$ and $\vec{B} + \vec{C}$ respectively along $\vec{A}$ and *de* is the projection of $\vec{C}$ over $\vec{A}$. Therefore,

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot (Oe) = \vec{A} \cdot (Od + de) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

If $\vec{A} \cdot \vec{B} = 0$ and $\vec{A} \neq 0$, $\vec{B} \neq 0$, we say $\vec{A}$ orthogonal to $\vec{B}$ or $\vec{A}$ and $\vec{B}$ are perpendicular

i.e., $\cos(\vec{A}, \vec{B}) = 0$, therefore,

If, $\vec{A} \cdot \vec{B} = 0, (\vec{A} \perp \vec{B})$ ...(2.27)

If $\vec{A}$ is parallel or antiparallel to $\vec{B}$

$$\cos(\vec{A} \cdot \vec{B}) = \pm 1$$

and therefore $\vec{A} \cdot \vec{B} = \pm AB, (\vec{A} \parallel \vec{B})$ ...(2.28)

where the minus sign holds if $\vec{A}$ and $\vec{B}$ are antiparallel. The dot product of a vector $\vec{A}$ with itself is

$$\vec{A} \cdot \vec{A} = A^2 \quad \text{...(2.29}a\text{)}$$

If follows that the dot product of the *cartesian unit vectors* are,

$$\left.\begin{aligned} &\hat{i} \cdot \hat{i} = , \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1, (\text{Angle is zero}) \\ \text{and} \quad &i \cdot j = j \cdot k = k \cdot i = 0, (\text{Angle is } \pi/2) \end{aligned}\right\} \quad \text{...(2.29}b\text{)}$$

Using the values of the dot products of the unit vectors given above we can write (by distributive property)

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad \text{...(2.30)}$$

that is *the dot product of the vectors is equal to the sum of the products of the corresponding cartesian components of these vectors.*

2.17 Vector or Cross Product of Two Vectors

The vector (or outer) product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$, read as $\vec{A}$ cross $\vec{B}$. It is also called the cross product of two vectors.

The vector, or cross, product of two vectors $\vec{A}$ and $\vec{B}$ is defined as a vector $\vec{R}$ whose magnitude is equal to the product of the magnitude of two vectors multiplied by the sine of the angle between them, and directed along a unit vector $\hat{n}$ which is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Thus we have,

$$\vec{R} = \vec{A} \times \vec{B} = AB \sin\theta \, \hat{n} \quad \text{...(2.31)}$$

$$0 \le \theta \le \pi$$

As shown in Fig. 2.15*a*

The direction $\vec{R}$ *i.e.,* that of the unit vector $\hat{n}$ is determined by the right hand screw rule. According to this rule, if a right hand screw is turned from the direction of $\vec{A}$ towards that of $\vec{B}$ through the smaller angle the direction of its advance is the direction of $\vec{R}$ *i.e.,* that of unit vector $\hat{n}$, as indicated in Fig. 2.15*b*.

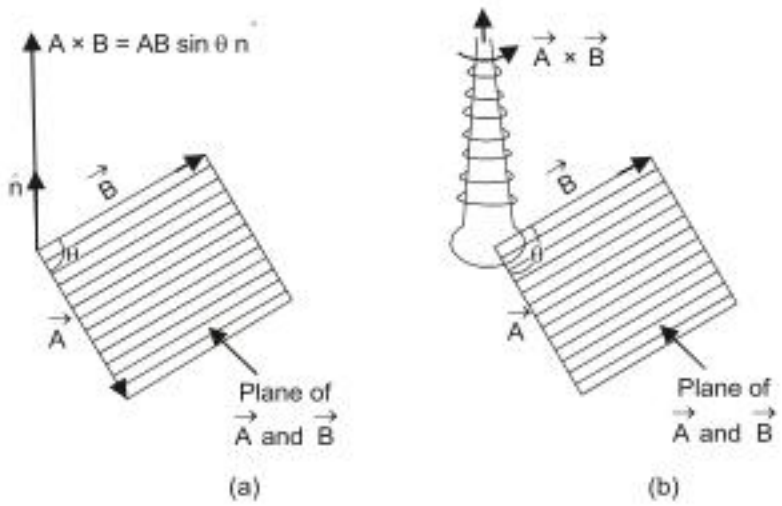

Fig. 2.15

2.18 Important Points About Vector-Product

(*i*) Since $0 \le \theta \le \pi$, sin θ cannot be negative, hence $|\vec{R}|$ or R cannot be negative.

(*ii*) The vector product is not commutative. From the definition of vector product it is evident that a change in the order of the factors in cross-product reverses the sign of product, because the sense of rotation of the screw is reversed *i.e.*,

$$\vec{B} \times \vec{A} = -AB \sin q = -\vec{A} \times \vec{B} \qquad ...(2.32)$$

So that the vector product is not *commutative.*

(*iii*) The vector product is *distributive,* thus

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(*iv*) The vector product is *associative,* if we multiply either $\vec{A}$ or $\vec{B}$ by a number or a scalar *m,* their vector product is also multiplied by the same number.

Therefore, $\vec{A} \times (m\vec{B}) = (m\vec{A}) \times \vec{B} = mAB \sin\theta\ \hat{n}$

Further, $(m\vec{A}) \times (n\vec{A}) = (mA\ nB) \sin\theta\ \hat{n}$

where *n* is another scalar.

Clearly we see that vector product is associative.

(*v*) If the two vectors $\vec{A}$ and $\vec{B}$ are parallel or collinear, θ = 0 or π. Thus, sin θ = 0. Then from definition,

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n} = 0.$$

Similar is the situation with two equal vectors. It follows that,

$$\vec{A} \times \vec{A} = 0 \qquad ...(2.33)$$

Therefore, the vector product of a vector with itself is zero.

If we apply this property to *unit vectors* $\hat{i}$, $\hat{j}$ and $\hat{k}$, then

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \qquad ...(2.34)$$

(*vi*) In case if $\vec{A}$ and $\vec{B}$ are perpendicular or orthogonal to each other, then,

$$\vec{A} \times \vec{B} = AB\hat{n}$$

because $\theta = \frac{\pi}{2}$ so $\sin\theta = 1$

Apply this property to *unit vector* $\hat{i}$, $\hat{j}$ and $\hat{k}$ *which are mutually orthogonal,* we get,

$$\left.\begin{aligned} \hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i} \\ \hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j} \\ \hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k} \end{aligned}\right\} \qquad ...(2.35)$$

(*vii*) Vector Product in terms of components. If we write vectors in terms of their rectangular components *i.e.*

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and $$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Now $$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

Since $$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

and $\hat{i} \times \hat{j} = -\hat{j} \times \hat{i}$ etc. as given above 2.35 we get,

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - B_zA_x)\hat{j} + (A_xB_y - A_yB_z)\hat{k}$$

This result can be written in the easily remembered determinant form *i.e.*,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad ...(2.36)$$

2.19 Applications of Cross Product

The Cross Product has many applications in physics as an illustration few are discussed below.

(*i*) **Force on a Moving Charge in a Magnetic Field:** If a charge q is moving with a velocity $\vec{v}$ at an angle θ with a magnetic field $\vec{B}$. Then the force acting on the charged particle is given by the experimentally verified following relation.

$$F = q\, v\, B \sin\theta$$

where F, v and B are the magnitudes of the force, velocity and magnetic field respectively. In vector form, the force

$$\vec{F} = q(\vec{v} \times \vec{B}) \text{ , (RMKS Units)} \qquad ...(2.37)$$

As shown in Fig. 2.16.

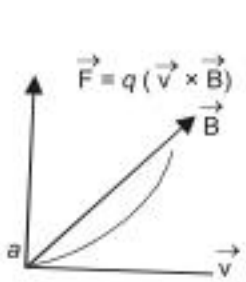

Fig. 2.16

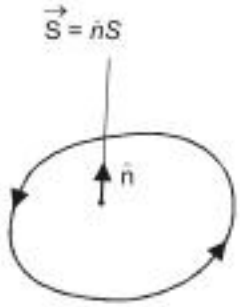

Fig. 2.17

Evidently $\vec{F}$ is perpendicular to both $\vec{V}$ and $\vec{B}$. This force is called Lorentz force and the relation 2.37 is called Lorentz force law. We shall discuss this in detail latter on.

(*ii*) **Vector Area:** The magnitude of a cross product $\vec{A} \times \vec{B}$ is AB sin θ, *i.e.,* the area of the parallelogram with sides $\vec{A}$ and $\vec{B}$, inclined at angle θ. Since in cross product $\vec{A} \times \vec{B}$, the shape of the figure is not specified, by the definition of the product, *any plane area of amount AB sin* θ *along a direction normal to the plane of* $\vec{A}$ *and* $\vec{B}$ *ie.,* a unit normal vector n can represent a vector product.

Thus the area is a vector quantity.

Consider a plane surface such as shown in Fig. 2.17. It can be regarded as having both magnitude and direction. If magnitude is represented by the amount of area and its direction is that of the normal to the plane. Thus,

$$S = \hat{n}S \qquad ...(2.38)$$

The sign to be attributed to a vector area is defined with reference to the order in which it is traced out as viewed from an external point such as O. For example if the periphery of the surface shown in Fig. 2.17, is traced in the counter-clockwise direction, then the positive direction .of the vector area is along the unit normal $\hat{n}$. The directions of tracing and of $\hat{n}$. are related by the right hand screw rule.

2.20 Triple Product of Vectors

With the help of the dot and cross products of two vectors we can determine the products involving several vectors. Among them two kinds of triple products which occur frequently in physical applications, namely,

(*i*) box, or triple scalar product $\vec{A} . (\vec{B} \times \vec{C})$, and

(*ii*) triple cross, or triple vector product $\vec{A} \times (\vec{B} \times \vec{C})$

2.21 Scalar Triple Product

Let the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ represent the edges of a parallelopiped as shown in Fig. 218. ($\vec{B} \times \vec{C}$) is a vector normal to the plane of $\vec{B}$ and $\vec{C}$. But the area of the base of the parallelopiped = BC sin ϕ which is the magnitude of the vector product $\vec{B} \times \vec{C}$ and its direction perpendicular to this area as shown in Fig. 2.18.

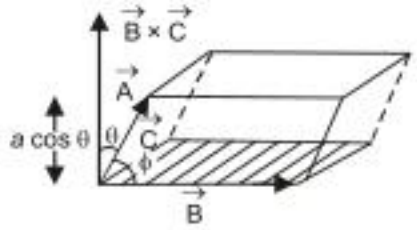

Fig. 2.18

The scalar product of $\vec{A}$ with $(\vec{B} \times \vec{C})$ is the product of this area and the projection of $\vec{A}$ along $(\vec{B} \times \vec{C})$. The projection of $\vec{A}$ along $(\vec{B} \times \vec{C})$ is A cos θ, *i.e.,* the vertical height of parallelopiped. Thus,

$$\vec{A} . (\vec{B} \times \vec{C}) = A \cos \theta \, (\vec{B} \times \vec{C})$$

= Vertical height of parallelopiped x area of its base

= Volume of parallelopiped ...(2.39)

Any face of this parallelopiped can be taken as the base and hence its volume can be represented by the following three expressions,

$$\vec{A} . (\vec{B} \times \vec{C}) = \vec{B} . (\vec{C} \times \vec{A}) = \vec{C} . (\vec{A} \times \vec{B}) \qquad ...(2.40)$$

To retain the volume with positive sign, the cyclic order of $\vec{A}$, $\vec{B}$ and $\vec{C}$ should be maintained because we know, $\vec{B} \times \vec{C} = \vec{C} \times \vec{B}$. As the order of terms in scalar product is meaningless, we have,

$$(\vec{A} \times \vec{B}) . \vec{C} = (\vec{B} \times \vec{C}) . \vec{A} = (\vec{C} \times \vec{A}) . \vec{B} \qquad ...(2.41)$$

It follows that the value of a scalar triple product depends on the cyclic order of the vectors and is independent of the position of the dots and the crosses. Hence, *the dot and the cross may be interchanged.*

So scalar triple product is often written as $[\vec{A}\,\vec{B}\,\vec{C}]$, *i.e.* in their cyclic order without showing the dots or crosses.

2.22 Some Important Conclusions from Scalar Triple Product

(*i*) Scalar triple product in terms of *Cartesian coordinates,* we have,

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A} \, (\vec{B} \times \vec{C})$$

or $$[ABC] = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) . \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$, (For $\vec{B} \times \vec{C}$ *sec* Eq. 2.36)

or $$[\vec{A}\,\vec{B}\,\vec{C}] = A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)$$

or $$[ABC]=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad (2.42)$$

(*ii*) The scalar triple product of the *unit vectors* $\hat{i}$, $\hat{j}$ *and* $\hat{k}$, we have

$$[\hat{i}\ \hat{j}\ \hat{k}] = (\hat{i} \times \hat{j}) . \hat{k}$$

or $$[\hat{i}\ \hat{j}\ \hat{k}] = \hat{k} . \hat{k}$$

or $$[\hat{i}\ \hat{j}\ \hat{k}] = 1$$

Similarly, $[\hat{j}\ \hat{k}\ \hat{i}] = 1$ and $[\hat{k}\ \hat{i}\ \hat{j}] = 1$

Hence, $$[\hat{i}\ \hat{j}\ \hat{k}] = [\hat{j}\ \hat{k}\ \hat{i}] = [\hat{k}\ \hat{i}\ \hat{j}] = 1 \quad ...(2.43)$$

But, $$[\hat{i}\ \hat{j}\ \hat{k}] = [\hat{k}\ \hat{j}\ \hat{i}] = [\hat{j}\ \hat{i}\ \hat{k}] = -1 \quad ...(2.44)$$

(*iii*) If two of the three vectors are equal, the scalar triple product is zero, say, we have three vectors, $\vec{A}$, $\vec{A}$ and $\vec{B}$, then,

$$[\vec{A}\ \vec{A}\ \vec{B}] = (\vec{A} \times \vec{A}) . \vec{B} = 0$$

Since $\vec{A} \times \vec{A} = 0$, see Eq. 2.33

(*iv*) If among the three, two vectors are parallel, then their scalar triple product is zero. If $\vec{A}$ and $\vec{B}$ be parallel we have the condition that

$\vec{B} = n\vec{A}$ where n is some scalar. Thus, we have,

$$[\vec{A}\ \vec{B}\ \vec{C}] = [\vec{A} \times \vec{B}] . \vec{C} = (\vec{A} \times n\vec{A}) . \vec{C}$$

$$= n(\vec{A} \times A) . \vec{C}$$

= 0, according to case (*iii*) given above

(*v*) If all the three vectors are coplanar, their scalar triple product is zero, since the volume of parallelopiped is zero. So that the condition for coplanarity of three vectors is that their scalar triple product should be zero. This is possible when two vectors are parallel.

2.23 The Vector Triple Product $\vec{A} \times (\vec{B} \times \vec{C})$

The cross product $(B \times C)$ is a vector normal to the plane containing $\vec{B}$ and $\vec{C}$. Therefore vector $\vec{A} \times (\vec{B} \times \vec{C})$ will be perpendicular to the plane containing $\vec{A}$ and $(\vec{B} \times \vec{C})$ *i.e.*, it will lie in the plane of $\vec{B}$ and $\vec{C}$ as shown in Fig. 2.19.

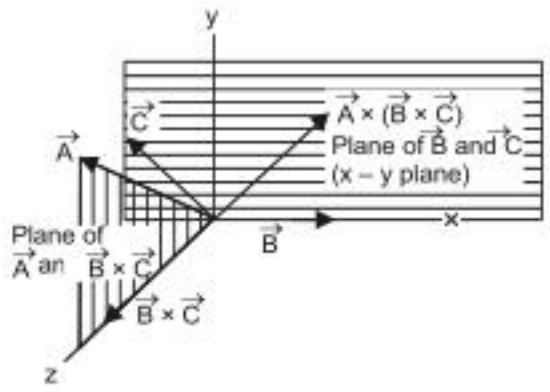

Fig. 2.19

Similarly, vector $\vec{B} \times (\vec{A} \times \vec{C})$ lies in the plane of $\vec{A}$ and $\vec{C}$ and is perpendicular to $\vec{B}$ and $\vec{C} \times (\vec{A} \times \vec{B})$ lies in the plane of $\vec{A}$ and $\vec{B}$ and is perpendicular to $\vec{C}$. Thus we see that

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq \vec{A} \times (\vec{B} \times \vec{C})$$

So vector multiplication is not *associative, i.e., the value of a vector triple product is entirely different if the order of the factors be interchanged.* In another words, the positions of the brackets are important. Let us evaluate the product $\vec{A} \times (\vec{B} \times \vec{C})$ of three vectors in terms of their components

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

then, $$\vec{B} \times \vec{C} = (B_yC_z - B_zC_y)\hat{i} + (B_zC_x - B_xC_z)\hat{j} + (B_xC_y - B_yC_x)\hat{k}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \text{ (say)}$$

Then, $$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A} \times \vec{v}$$

$$= (A_yv_z - v_yA_z)\hat{i} + (A_zv_x - \mathbf{A}_xv_z)\hat{j} + (A_xv_y - \mathbf{A}_yv_x)\hat{k} \qquad ...(2.45)$$

Its x-component $= [A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)]\hat{i}$

$$= [A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zC_zB_x]\hat{i}$$

On adding and subtracting $A_xC_xB_x\hat{i}$ on R.H.S.

Its x-component $= [A_yC_yB_x + A_zC_zB_x + A_xC_xB_x - A_yB_yC_x - A_zB_zC_x - A_xB_xC_x]\hat{i}$

$$= (\vec{A} \cdot \vec{C})B_x\hat{i} - (\vec{A} \cdot \vec{B})C_x\hat{i}$$

Similarly y-component $= (\vec{A} \cdot \vec{C})B_y\hat{j} - (\vec{A} \cdot \vec{B})C_y\hat{j}$

and z-component $= (\vec{A} \cdot \vec{C})B_z\hat{k} - (\vec{A} \cdot \vec{B})C_y\hat{k}$.

Hence, $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})(B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) - (\vec{A} \cdot \vec{B})(C_x \hat{i} + C_y \hat{j} + C_z \hat{k})$

$$= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \qquad ...(2.46)$$

Changing the order of multiplication in a vector triple product changes its value as shown here,

$$(\vec{A} \quad \vec{B}) \quad \vec{C} = -\vec{C} \quad (\vec{A} \quad \vec{B}) = (\vec{A} \cdot \vec{C})\vec{B} \quad (\vec{B} \cdot \vec{C})\vec{A} \qquad ...(2.47)$$

REVIEW QUESTIONS

2.1 State and explain whether the following are vectors or scalars.
(*a*) Temperature, (*b*) Strain, (*c*) Stress, (*d*) Torque, (*e*) Work, (*f*) Linear, Momentum, (*g*) Angular Momentum, (*h*) Electromotive force, (*i*) Potential, (*j*) Temperature gradient.

2.2 What is (*i*) a polar vector, (*ii*) an axial vector, (*iii*) Unit vector, (*iv*) a zero vector? What do you understand by the term orthogonal Unit vector triad?

2.3 What do you mean by position vector of a point? Express the position vector in terms of the orthogonal unit vector triad. What is the modulus of vector $\hat{r}$?

2.4 Define dot and cross-product of two vectors. If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along *x, y* and *z*-axis respectively, show that

$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$,

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

and $\hat{i} \times \hat{j} = \hat{k}$

2.5 Show that for any vector $\vec{A}$

$\vec{A} \cdot \vec{A} = A^2$ and $\vec{A} \times \vec{A} = 0$

2.6 If $\vec{A}$ and $\vec{B}$ are two vectors, then show

$\vec{A} \cdot \vec{B} = A_x A_x + A_y B_y + A_z B_z$

and $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

2.7 Deduce the condition for the vectors $2\hat{i} + 3\hat{j} - 4\hat{k}$ and $3\hat{i} + a\hat{j} - b\hat{k}$ to be parallel (or collinear).

2.8 Deduce the condition for the coplanarity of three vectors $\vec{A}, \vec{B}$ and $\vec{C}$.

2.9 Prove that if the triple scalar product of three vectors vanishes, the vectors are coplanar.

2.10 Show that if $\vec{a}$ and $\vec{b}$ are equal in magnitude and non-parallel $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

OBJECTIVE QUESTIONS

2.1 Two vectors are said to be equal if :
(*a*) their magnitudes are same
(*b*) direction is same
(*c*) originate from the same point
(*d*) they meet at the same point
(*e*) they have same magnitude and same sense of direction.

2.2 Two vectors $\vec{a}$ and $\vec{b}$ are parallel and have equal magnitudes then :

(*a*) they are equal
(*b*) they are not equal
(*c*) they may or may not be equal
(*d*) they have the same sense of direction
(*e*) they do not have the same direction.

2.3 A force is a

(*a*) unit vector (*b*) zero vector
(*c*) free vector (*d*) localised vector

2.4 The magnitude of a vector is never

(*a*) zero (b) unity
(c) negative (*d*) positive

2.5 If $\vec{a}$ is a non-zero vector and m is a non-zero scalar, then $m\vec{a}$ is a unit vector if :

(*a*) $m = = 1$, (*b*) $a = |m|$,
(*c*) $a = 1/|m|$

2.6 $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if :

(*a*) $\theta = \frac{\pi}{3}$, (*b*) $\theta = \frac{\pi}{4}$,

(*c*) $\theta = \frac{\pi}{2}$, (*d*) $\theta = \frac{2\pi}{3}$.

2.7 If, $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} = \vec{c}$. then $\vec{b}$ is called

(*a*) a complement of $\vec{c}$ (*b*) a component $\vec{c}$

(*c*) a projection of $\vec{c}$ (*d*) an addent of $\vec{c}$

2.8 Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular, if and only if $\vec{a} . \vec{b}$ equals

(*a*) unit vector (*b*) zero vector
(*c*) free vector (*d*) zero

2.9 If $[\hat{i}\ \hat{j}\ \hat{k}]$ be a set of orthonormal unit vector then fill-up the blanks

(*a*) $\hat{i} . \hat{i} + \hat{j} . \hat{j} + \hat{k} . \hat{k} =$ (*b*) $\hat{i} . \hat{j} + \hat{j} . \hat{k} + \hat{k} . \hat{i} =$

(*c*) $\hat{i} . \hat{i} = \hat{j} . \hat{j} = \hat{k} . \hat{k} =$ (*d*) $\hat{i} . \hat{j} = \hat{j} . \hat{k} = \hat{k} . \hat{i} =$

210 Let $\hat{i}, \hat{j}, \hat{k}$ he the unit vector along *x*-axis and z-axis respectively of the following six statements

$\hat{i}\,\hat{i} = 1; \hat{i}\,\hat{j} = -1; \hat{j}\,\hat{k} = 0$

$\hat{k}\,\hat{i} = -1; \hat{j}\,\hat{i} = 1; \hat{k}\,\hat{k} = 0$

(*a*) only two of them are correct (*b*) only three of them are correct
(*c*) only four of them are correct (*d*) only five of them are correct.

ANSWERS: 1. (*e*), **2.** (*c*), **3.** (*d*), **4.** (*c*), **5.** (*c*), **6.** (*d*), **7.** (*b*), **8.** (*d*), **9.** (*a*) 3 (*b*) 0, (*c*) 1, (*d*) 0, **10.** (*a*).

CHAPTER 3

VECTOR DIFFERENTIATION AND INTEGRATION

In the last chapter we said that vector analysis is a very powerful tool for visualising the physical meaning of the equations distinctly and exactly. Particularly, in electromagnetic theory change in field and potential can be understood clearly in terms of vector differentiation and integration which is discussed below.

3.1 Differentiation of Vector with Respect of Time

If a vector changes in magnitude as well as direction continuously with respect to some scalar variable, then such a vector is a function of this scalar variable. So that it can be differentiated with respect to that scalar variable and a new vector is obtained. And if the scalar variable is time, the result of differentiation of the vector with time is called time derivative of the vector.

Velocity and Acceleration: Let the position of a particle at any time t be specified by the vector $\vec{r}$ *i.e.* $\overrightarrow{OP}$ which is position vector and vector $\vec{r}$ is a function of variable t. As a time increase, the particle moves and the position vector changes in direction and magnitude, when time changes from t to $t+\delta t$, $\vec{r}$ becomes $\vec{r}+\delta\vec{r}$ at Q as shown in Fig. 3.1.

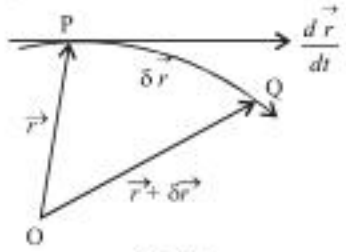

Fig. 3.1

The average rate of change of the position vector with time is given by,

$$\frac{\vec{r}+\delta\vec{r}-\vec{r}}{t+\delta t-t}=\frac{\delta\vec{r}}{\delta t}$$

This rate of change of $\vec{r}$ with time in the limit when $\delta t \to 0$ *is called as first time derivative, or the differential coefficient* and is represented by dr/dt. Therefore,

$$\frac{\delta\vec{r}}{\delta t}\lim_{dt\to 0}\frac{\delta\vec{r}}{\delta t}$$

The velocity of a particle is the rate of change of position w.r.t. time. Thus,

$$\vec{v}=\frac{d\vec{r}}{dt} \qquad ...(3.1)$$

However, if r is represented is cartesian components, as,

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Then, $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

or $$\vec{v} = v_x\hat{i} + v_y\hat{i} + v_z\hat{k} \qquad ...(3.2)$$

Similarly the second time derivative of the position vector gives the acceleration $\vec{a}$ of the particle. Thus, we get

$$\text{Acceleration} = \vec{a} = \frac{d^2r}{dt^2} = \frac{dv}{dt} \qquad ...(3.3)$$

or $$\vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

or $$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

or $$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \qquad ...(3.4)$$

Force $\vec{F}$ acting on the particle of mass m is given by Newton's second law of motion,

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} \qquad ...(3.5)$$

3.2 Differentiation of Sums and Products

(*i*) **Differentiation of Sum of the vectors :** The differential coefficient of sum $\vec{A} + \vec{B}$ both which are the function of t is equal to the sum of their individual coefficients.

Let $$\vec{R} = \vec{A} + \vec{B}$$

then for a change of time from t to $t + \delta t$, we have,

$$\vec{R} + \delta\vec{R} = (\vec{A} + \delta\vec{A}) + (\vec{B} + \delta\vec{B})$$

Hence, $\delta\vec{R} = \delta\vec{A} + \delta\vec{B}$ (Since $\vec{R} = \vec{A} + \vec{B}$)

dividing this by δt and putting the limit, we get,

$$\lim_{\delta t \to 0}\frac{\delta\vec{R}}{\delta t} = \lim_{\delta t \to 0}\frac{\delta\vec{A}}{\delta t} + \lim_{\delta t \to 0}\frac{\delta\vec{B}}{\delta t}$$

or $$\frac{d\hat{R}}{dt} = \frac{d\hat{A}}{dt} + \frac{d\hat{B}}{dt} \qquad ...(3.6)$$

so that such *differentiation is distributive* and holds good for any given number *i.e.*,

$$\frac{d}{dt}(\vec{A} + \vec{B} + \vec{C} + ...) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt} + \frac{d\vec{C}}{dt} + ... \qquad ...(3.7)$$

Similarly we can shown that for *difference of vectors*,

$$\frac{d}{dt}(\vec{A} - \vec{B}) = \frac{d\vec{A}}{dt} - \frac{d\vec{B}}{dt}$$

(*ii*) **Differentiation of Scalar Products of two Vectors, $\vec{A}.\vec{B}$**

Let $\vec{R} = \vec{A}.\vec{B}$, the increase δt in t gives,

$$\vec{R} + \delta\vec{R} = (\vec{A} + \delta\vec{A}).(\vec{B} + \delta B)$$

or $$\vec{R} + \delta\vec{R} = \vec{A}.\vec{B} + \vec{A}.\delta\vec{B} + \vec{B}.\delta A + \delta\vec{A}.\delta\vec{B}$$

Now, on neglecting the $\delta\vec{A}.\delta\vec{B}$ which is small, we get

$$\delta\vec{R} = \vec{A}.\delta\vec{B} + B.\delta\vec{A}, \qquad (\text{Since } \vec{R} = \vec{A}.\vec{B})$$

Dividing by δt and processing the limit, we get,

$$\frac{d\vec{R}}{dt} = \frac{d}{dt}(\vec{A}.\vec{B}) = \vec{A}.\frac{d\vec{B}}{dt} \vec{B}.\frac{d\vec{A}}{dt} \qquad ...(3.8)$$

So that, the operation is *commutative*.

(*iii*) **Differentiation of Vector Products $\vec{A} \times \vec{B}$**

Let $\vec{R} = \vec{A} \times \vec{B}$

As above, the change in time from t to $t + \delta t$ gives,

$$\vec{R} + \delta\vec{R} = (\vec{A} + \delta\vec{A}) \times (\vec{B} + \delta\vec{B})$$

or $$\vec{R} + \delta\vec{R} = \vec{A} \times \vec{B} + \vec{A} \times \delta\vec{B} + \delta\vec{A} \times \vec{B} + \delta\vec{A} \times \delta\vec{B}$$

Neglecting $\delta\vec{A} \times \delta\vec{B}$ and dividing by δt whence as before,

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{R}}{dt} = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B} \qquad ...(3.9)$$

in which the *order of vectors must be strictly maintained.*

(*iv*) **Differentiation of Triple Products**

In the case of triple products, we proceed as before to obtain the differential coefficients, thus,

(*a*) in scaler triple products, we have

$$\frac{d}{dt}[\vec{A}.(\vec{B} \times \vec{C})] = \frac{d\vec{A}}{dt}.(\vec{B} \times \vec{C}) + \vec{A}.\left(\frac{d\vec{B}}{dt} \times \vec{C}\right) + \vec{A}.\left(\vec{B} \times \frac{d\vec{C}}{dt}\right) \qquad ...(3.10)$$

(*b*) in vector product we have,

$$\frac{d}{dt}[\vec{A} \times (\vec{B} \times \vec{C})] = \frac{d\vec{A}}{dt} \times (\vec{B} \times \vec{C}) + \vec{A}. \times \left(\frac{d\vec{B}}{dt} \times \vec{C}\right) + \vec{A}. \times \left(\vec{B} \times \frac{d\vec{C}}{dt}\right) \qquad ...(3.11)$$

3.3 Scaler and Vector Fields

(*a*) **The Field.** If a physical quantity varies from point to point in space it can be expressed as a continuous function of the position of a point in a region of space, then such a function is *called the function of position or point function and the region in which it specifies the physical quantity is called the field.* There are two main kinds of the fields, corresponding to nature of physical quantities:

(*i*) Scaler fields corresponding to scaler quantities

(*ii*) Vector fields corresponding to vector quantities.

(*b*) **The Scaler Field** is represented by a continuous *scaler function* $\phi(x, y, z)$ giving the value of quantity at each point. In all practical cases, *the magnitude of such function does not change abruptly when it passes from any point to another close to it. For* example, distribution of temperature, magnetic and electrostatic potentials, density of any non-directed quantity in a given region of space. In any scaler field, we can draw surfaces corresponding to equal values of the scaler such as *isothermal, equidenisity* or *equipotential surfaces.* Such surface are called *level surface,* each level surface having its own constant value. Two level surface cannot cut each other because if they cut, scaler values corresponding to both must hold along their common line, which is contrary to our definition. Thus, scaler point function $\phi(x, y, z)$ *are single valued for uniform* at every point.

(*c*) **A Vector Field**. A vector field is represented at every point by a continuous vector function $\vec{F}(x, y, z)$. At any given point of field, the function $\vec{F}(x, y, z)$ is specified by a vector of definite magnitude and direction, both of which change continuously from point to point throughout the field region. *Examples* of vector fields: the distribution of velocity in a fluid (liquid or gas), the distribution of electric or magnetic field intensity etc.

Starting from any arbitrary point in the field, proceed an infinitesimal distance in the direction of the vector at that point so as to reach at a closely-neighbouring point. Proceeding in a similar way, we obtain in general, a curved line, called the *vector line*, *line of flow or flux line*. The tangent at any point of this curved line gives the direction of the vector at that point. For the representation of the magnitude of the vector at any point on a flux line, draw a very small surface perpendicular there to and choose a number of points per unit area upon this surface, which are numerically equal to the magnitude of the vector. Through each of these points flux lines can be drawn. Thus, the field is mapped out by flux lines. The direction of flux lines is that of vector function, their density *i.e.*, the number of lines crossing per unit area perpendicular to their direction, is the magnitude of the vector.

3.4 Partial Differentiation and Gradient

Partial Differentiation : If $\phi(x, y, z)$ is a scaler function of position in space, *i.e.* of coordinates x, y, z, then $\frac{\delta\phi}{\delta x}$ denotes the rate of change of ϕ with respect to x when y and z remain constant, is called partial differentiations. Similarly, $\frac{\delta\phi}{\delta y}$ and $\frac{\delta\phi}{\delta z}$ denote the rate of change of ϕ with respect to y and z respectively.

Gradient : The vector function $\hat{i}\frac{\delta\phi}{\delta x}+\hat{j}\frac{\delta\phi}{\delta y}+\hat{k}\frac{\delta\phi}{\delta z}$ is called the gradient of the scalar point function ϕ and is denoted by gradient ϕ so that,

$$\text{grad}\,\phi = \hat{i}\frac{\delta\phi}{\delta x}+\hat{j}\frac{\delta\phi}{\delta y}+\hat{k}\frac{\delta\phi}{\delta z}$$

And the differential,

$$d\phi = \frac{\delta\phi}{\delta x}dx+\frac{\delta\phi}{\delta y}dy+\frac{\delta\phi}{\delta z}dz$$

If $\vec{r}=\hat{i}x+\hat{j}y+\hat{k}z$ is the radius vector of the point in space from the origin, then,

$$d\vec{r} = \hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz$$

so. $$d\phi = (\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz).\left(\hat{i}\frac{\delta\phi}{\delta x}+\hat{j}+\frac{\delta\phi}{\delta y}+\hat{k}\frac{\delta\phi}{\delta z}\right)$$

$$d\phi = d\vec{r}.\text{grad}\,\phi$$

or $$d\phi = \text{grad}\phi.\,d\vec{r} \qquad ...(3.12)$$

The Operator $\vec{\nabla}$. In vector algebra, vector differential operator, denoted by ∇ (i.e., inverted Δ), read as del or nabla and is defined by,

$$\vec{\nabla} = \hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}$$

from definition given above,

grad $\phi = \left(\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}\right)\phi = \vec{\nabla}\phi$, a vector so the Eq. (3.12) becomes,

$$d\phi = \vec{\nabla}\phi\,.\,d\vec{r}$$

or $$d\phi = (\,\vec{\nabla}\,.\,d\vec{r}\,)\phi \qquad ...(3.13)$$

PHYSICAL MEANING OF THE GRADIENT OF THE SCALER FUNCTION

As we have discussed, that the scalar fields, such as temperature and electric potential, can be represented by level surfaces. Let us consider, two level surfaces s_1 and s_2 very close together and ϕ and $\phi + d\phi$ are the corresponding scalar function. Choose points A and B on the two surface as shown in Fig. 3.2. Let $\vec{r}$ be the position vector of A from the origin O, we have,

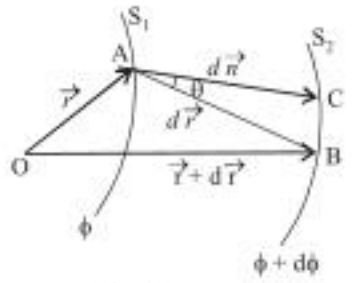

Fig. 3.2

$$\overline{OA} = \vec{r}$$
$$\overline{OB} = \vec{r} + d\vec{r}$$

hence, $$\overline{AB} = d\vec{r}$$

The least distance between the two surfaces s_1 and s_2 is $\overline{AC}$ in the direction of normal at $\overline{A}$. Let $\hat{n}$ be the unit vector along $\overline{AC}$ and $\overline{AC} = d\vec{n}$, we can write

$$d\vec{n} = dr\cos\theta = \hat{n}d\vec{r} \qquad ...(3.14)$$

The magnitude of the rate of increase at A of ϕ in the direction of AB will be $\frac{\delta\phi}{\delta r}$ and this rate of increase becomes greatest only when δr is minimum *i.e.*, along $\overline{AC}$ which is the least distance between surface. So that *greatest rate of increment is* $\frac{\delta\phi}{\delta n}$, in the direction of $\hat{n}$, therefore,

$$d\phi = \frac{\delta\phi}{\delta n}dn$$

Substituting from (3.14) we get,

$$d\phi = \frac{\delta\phi}{\delta n}\hat{n}.d\vec{r} \qquad ...(3.15)$$

But $d\phi$ is also given by, $d\phi = (\vec{\nabla}.d\vec{r})\phi$

$$d\phi = (\vec{\nabla}\phi.d\vec{r}), \text{ (see Eq. 3.13)} \qquad ...(3.16)$$

Equating these two values, given in Eqs.3.15 and 3.16, we get

$$\vec{\nabla}\phi.d\vec{r} = \frac{\delta\phi}{\delta n}\hat{n}.d\vec{r}$$

hence, $$\vec{\nabla}\phi = \frac{\delta\phi}{\delta n}\hat{n} = \text{grad}\phi, \text{ a vector}$$

Thus, *the gradient of scaler field grad ϕ or $\vec{\nabla}\phi$ is a vector whose magnitude at any point is equal to the maximum rate of increase of ϕ and directed along the direction in which this maximum change occurs i.e. perpendicular to level surface at that point.*

Ex: Suppose V is the potential due to static charges, then *electric field intensity* at any point is in the direction of the *greatest rate of decrease* of potential *i.e.*, normal to the equi-potential surface and its magnitude is equal to that of rate of decrease. Thus, we have,

$$\vec{E} = -\text{gradV} \qquad ...(3.17)$$

3.5 Operations with $\vec{\nabla}$

We have seen in preceding article (3.4) that $\vec{\nabla}\phi$ is a vector which is the result of operation of a scalar ϕ by $\vec{\nabla}$. In fact $\vec{\nabla}\phi$ is a *physical vector having a meaning i.e.*, it represents *the maximum rate of change of ϕ*.

Let us operate a vector with $\vec{\nabla}$, we can do this in two different ways :

1. $\vec{\nabla}$.(a vector) *i.e.*, a dot product of two vectors

2. $\vec{\nabla} \times$ (a vector) *i.e.*, a cross product of two vectors

Divergence:

The first products is some scalar because the dot product of two vectors is always a scalar. Thus if $\vec{F}$ represents a vector field, given by,

$$\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$$

and as we know $\vec{\nabla}$ is given by,

$$\vec{\nabla} = \hat{i}\nabla_x + \hat{j}\nabla_y + \hat{k}\nabla_z$$

or
$$\vec{\nabla} = \hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}$$

hence,
$$\vec{\nabla}.\vec{F} = \Delta_x F_x + \Delta_y F_y + \Delta_z F_z$$

or
$$\vec{\nabla}.\vec{F} = \frac{\delta F_x}{\delta_x} + \frac{\delta F_y}{\delta_y} + \hat{k}\frac{\delta F_z}{\delta_z} \qquad ...(3.18)$$

This sum is invariant under a coordinate transformation i.e., it is quite independent of the unit vectors that may be chosen. If we choose a different system (indicated by primes), we get*

$$\vec{\nabla}.\vec{F} = \frac{\delta Fx'}{\delta x} + \frac{\delta Fy'}{\delta y} + \hat{k}\frac{\delta Fz'}{\delta z} \qquad ...(3.19)$$

That is,
$$\vec{\nabla}.'\vec{F} = \vec{\nabla}.\vec{F}, \text{ for every point in space} \qquad ...(3.20)$$

So $\vec{\nabla}.\vec{F}$ is *scalar* field *which must represent some physical quantity. The scalar quantity* $\vec{\nabla}.$ (*a vector*) *is extremely useful in physics, is called the divergence so that,*

$$\vec{\nabla}.\vec{F} = \text{div } \vec{F} = \text{divergence of } \vec{F}, \text{ a scalar} \qquad ...(3.21)$$

Curl:

The cross product $\vec{\nabla} \times \vec{F}$ is called the curl of $\vec{F}$ and it is a vector so that

$$\vec{\nabla} \times \vec{F} = \text{curl } \vec{F} = \text{a vector}.$$

The curl $\vec{F}$ can be expressed in terms of the components by the useful rules of cross products as given below :

$$(\nabla \times F)_z = \nabla_x F_y - \nabla_y F_x = \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}$$

similarly,
$$(\nabla \times F)_x = \nabla_y F_z - \nabla_z F_z = \frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}$$

and
$$(\nabla \times F)_y = \nabla z F_x - \nabla x F_z = \frac{\delta Fx}{\delta z} - \frac{\delta F_z}{\delta x}$$

---

* We think of $\vec{F}$ as a physical quantity that depends on position in space and not strictly as a mathematical function of three variables, when $\vec{F}$ is differentiated w.r.t. x, y and z or w.r.t. x', y' and z', the mathematical expression for $\vec{F}$ must first be expressed as a function of the appropriate variables.

Combining these three we get,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ Fx & Fy & Fz \end{vmatrix} \qquad ...(3.22)$$

We shall discuss the physical meaning of divergence and curl in the next article. But, form the above discussion, we have three kinds of operation with $\vec{\nabla}$.

$$\begin{aligned} \vec{\nabla}\phi &= \text{grad } \phi = \text{a vector} \\ \vec{\nabla}.\vec{F} &= \text{div } \vec{F} = \text{a scalar} \\ \vec{\nabla}\times\vec{F} &= \text{curl } \vec{F} = \text{a vector} \end{aligned} \qquad ...(3.23)$$

3.6 Second Derivatives of Vector Functions of Fields

So far we have discussed the first derivative. Let us consider the second derivatives, as given below:

(*a*) $\vec{\nabla}.(\vec{\nabla}\phi)$ *i.e.* div. of gradient ϕ

(*b*) $\vec{\nabla}\times\vec{\nabla}\phi$ *i.e.* curl of gradient ϕ

(*c*) $\vec{\nabla}(\vec{\nabla}.\vec{F})$ *i.e.* gradient of a divergence $\vec{F}$

(*d*) $\vec{\nabla}.(\vec{\nabla}\times\vec{F})$ *i.e.* divergence of curl $\vec{F}$

(*e*) $\vec{\nabla}\times(\vec{\nabla}\times\vec{F})$ *i.e.* curl of a curl $\vec{F}$

Let us first discuss the second one, *i.e.*,(*b*)

$$\vec{\nabla}\times\vec{\nabla}\phi = (\vec{\nabla}\times\vec{\nabla})\phi$$

Since the cross products of two equal vectors is zero

i.e., $\vec{A}\times\vec{A} = 0$, (*see* Eq. 2.33)

so that we have ,

$$\text{curl (grad } \phi) = \vec{\nabla}\times\vec{\nabla}\phi = 0, \qquad ... (3.24)$$

for any scalar function ϕ.

Now, let us consider the fourth one *i.e.* (*d*)

$$\vec{\nabla}(\vec{\nabla}\times\vec{F}) = \text{div. (curl } \vec{F})$$

Since, we know, it is a combination similar to $\vec{A}.(\vec{A}\times\vec{B})$, where $\vec{A}\times\vec{B}$ perpendicular to $\vec{A}$ and has no component in the direction of A [*see* Art. 2.22 (*iii*)] *i.e.* if two of the three vectors are equal the product is zero. Therefore, we have,

$$\vec{\nabla}(\vec{\nabla}\times\vec{F}) = \text{div. (curl } \vec{F}) = 0 \qquad ...(3.25)$$

There are two important theorems based on the equations (3.24) and (3.25).

Theorem 1.

In physical problems we frequently find that the curl of some quantity, say of the vector field $\vec{A}$, is zero. Then certainly according to relation (3.24) $\vec{A}$ is the gradient of some quantity. The theorem is that if the curl $\vec{A}$ is zero, then $\vec{A}$ is always the gradient of the some scalar field ϕ such that $\vec{A}$ is equal to grad ϕ. In other words, we have,

$$\left.\begin{array}{ll} \text{If} & \vec{\nabla}\times\vec{A}=0 \\ \text{there is a} & \phi \\ \text{such that} & \vec{A}=\vec{\nabla}\phi \end{array}\right\} \qquad ...(3.26)$$

Theorem 2.

We have seen in Eq. (3.25) that the divergence of a curl of something is zero. If we come across a vector field $\vec{D}$ for which div. $\vec{D}$ is zero, we can conclude that $\vec{D}$ is the curl of some vector field $\vec{C}$,therefore, we can write,

$$\left.\begin{array}{ll} \text{If} & \vec{\nabla}\cdot\vec{D}=0 \\ \text{there is a} & \vec{C} \\ \text{such that} & \vec{D}=\nabla\times\vec{C} \end{array}\right\} \qquad ...(3.27)$$

3.7 The Laplacian Operator

The relation of a article (3.6) *i.e.*, div grad ϕ gives a new operator as given below :

$$\vec{\nabla}.(\vec{\nabla}\phi) = \vec{\nabla}.\vec{\nabla}\phi = (\vec{\nabla}.\vec{\nabla})\phi = \nabla^2\phi$$

It is read as del squared (or nabla squared) ϕ . It appears often in physics, it has been given a special name – the Laplacian operator

$$= \vec{\nabla}.\vec{\nabla}$$

$$\text{Laplacian} = \nabla^2 = \frac{\delta^2}{\delta x^2}+\frac{\delta^2}{\delta y^2}+\frac{\delta^2}{\delta z^2} \qquad ...(3.28)$$

As it is the dot or scalar product of two vectors $(\vec{\nabla}.\vec{\nabla})$, it is a scalar operator

Curl of the Curl:

Let us evaluate $\vec{\nabla}\times\vec{\nabla}\times\vec{F}$ *i.e.*, the curl of the curl, which can be written by the use of vector equality.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C})-(\vec{A}.\vec{B})\vec{C}$$

here $$\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F})-(\vec{\nabla}.\vec{\nabla})\vec{F}$$

or $$\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F})-\nabla^2\vec{F} \qquad ...(3.29)$$

Finally $\vec{\nabla}(\vec{\nabla}.\vec{F})$. It is a vector field

So we can write as our conclusion,

$$\left.\begin{array}{ll} (a) & \vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi = \text{a scalar field} \\ (b) & \vec{\nabla}\times(\vec{\nabla}\phi) = 0 \\ (c) & \vec{\nabla}(\vec{\nabla}.\vec{F}) = \text{a vector field} \\ (d) & \vec{\nabla}.(\vec{\nabla}\times\vec{F}) = 0 \\ (e) & \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F}) - \nabla^2\vec{F} \\ (f) & (\vec{\nabla}.\vec{\nabla})\vec{F} = \nabla^2\vec{F} = \text{a vector field} \end{array}\right\} \qquad ...(3.31)$$

3.8 The rate of flow of Flux of a vector field

If $\vec{v}(x, y, z)$ is a vector field, the scalar products of the vector operator and $\vec{v}$ is a scalar and is referred as the divergence of $\vec{v}$ *i.e.*, div $v = \vec{\nabla} \cdot \vec{v}$.

Let, $\vec{v}$ be the value of a vector field such as the velocity of a fluid at the mid point P (x, y, z) of a small rectangular parallelopiped with edges dx, dy and dz parallel to the three coordinate axes as shown in Fig. 3.3

On the face ABCD the value of the x-component of the vector

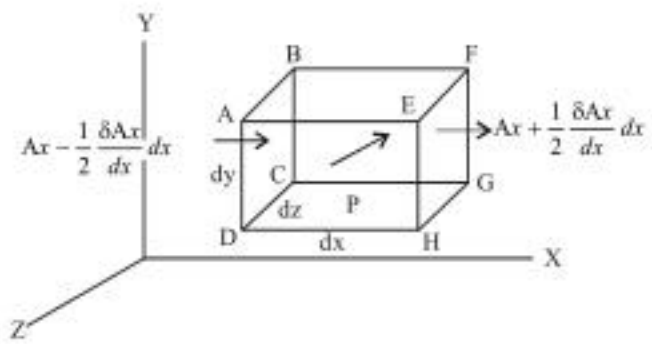

Fig. 3.3

$\vec{v}$ (here the velocity component along x-axis) at the middle of the face *i.e.* at $\left(x - \frac{dx}{2}\right)$ is

$$v_x - \frac{1}{2}\frac{\delta v_x}{\delta x}dx$$

As the face is infinitesimally small, this value can be taken as uniform over the face ABCD. Similarly for the face EFGH at $x + \frac{dx}{2}$, the x component is

$$v_x + \frac{1}{2}\frac{\delta v_x}{\delta x}dx$$

The flux through any face is defined as the scalar products of the vector area of the face and the vector $\vec{v}$ *i.e.*,

Flux = (component of velocity normal to the face) × (area of the face)

therefore, the flux (or fluid) entering per second through face ABCD of the parallelopiped

$$= \left(v_x - \frac{1}{2}\frac{\delta v_x}{\delta x}dx\right) dy\, dz, \text{ (ABCD} = dy\, dz)$$

Similarly, the flux (or fluid) leaving per second through the opposite face,

$$\text{EFGH} = \left(v_x + \frac{1}{2}\frac{\delta v_x}{\delta x}dx\right) dy\, dz$$

The flux is *positive* when the component of $\vec{v}$ and the outdrawn normal on the face of the element are in the *same direction*. Therefore, the excess of flux (or fluid) leaving the paralleloppied over that entering through faces ABCD and EFGH, *i.e.* along X-direction is

$$\left(v_x + \frac{1}{2}\frac{\delta v_x}{\delta x}dx\right) dy\, dz - \left(v_x - \frac{1}{2}\frac{\delta v_x}{\delta x}dx\right) dy\, dz$$

$$= \frac{\delta v_x}{\delta x}dx\, dy\, dz$$

By similar reasoning the contribution parallel to y-and z-direction are,

$$\frac{\delta v_y}{dy} dx\, dy\, dz \text{ and } \frac{\delta v_z}{dz} dx\, dy\, dz$$

The total net flux diverging from or leaving the element is

$$= \left(\frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} + \frac{\delta v_z}{\delta z} \right) dx\, dy\, dz$$

$$= (\vec{\nabla} \cdot \vec{v})\ dx\, dy\, dz$$

$$= (\text{div. } \vec{v})\ dx\, dy\, dz \qquad ...(3.31)$$

The volume of the paralleloppied is *dx dy dz. The amount of flux per unit volume is defined as the divergence of the vector* $\vec{v}$.

PHYSICAL MEANING OF DIVERGENCE

From the above discussion, we can conclude that if $\vec{v}$ be the velocity of the fluid at a point, then *Div* $\vec{v}$ *at a point inside the fluid is the rate at which the fluid is flowing away from the point per unit volume. In other words the div is the amount of flux. A positive value* of div $\vec{v}$ means that either the fluid is expanding (*i.e.* density decrease) or that the point is source of fluid. Similarly, the *negative value* of div $\vec{v}$ means the reverse *i.e.*, either the fluid is contracting or the point is the sink of fluid.

The divergence of current density (current per unit area) : $\vec{j}$ at a point gives the amount of charge flowing out per second per unit volume from a small closed surface surrounding the point.

Physical Meaning of div $\vec{v}$ = 0, everywhere or solenoidal Vector

When the divergence is everywhere zero, it means, the flux entering any element of space is exactly balanced by that leaving it and we may write

$$\text{div } \vec{v} = 0$$

which is true in many practical problems to understand clearly we consider Examples

Example 1. In a fluid div $\vec{v} = 0$, means that the fluid entering and leaving the element (or the paralleloppied) is the same, *i.e.*, there is no change in the density of field or in the other words the fluid is incompressible. If the fluxes entering and leaving an element are equal means that such fluxes have not been *generated* within this element. In the case of fluid this means that *there can be no sources or sinks in the field.*

Example 2. If $\vec{B}$ is magnetic field and its *div* $\vec{B}$ = *0, every where* means the lines of $\vec{B}$ could not have a source in the field of $\vec{B}$ from which they originate. That gives an *important conclusion that an isolated magnetic pole is not possible i.e. absence of mono-pole.*

Hence the lines of $\vec{B}$ should from a closed curve (like the magnetic filed of a current), or extend to infinity. A vector which satisfies this condition is called *Solenoidal vector*. In contrast for electric field $\vec{E}$

div $\vec{E} \neq 0$, *everywhere*, means the lines of $\vec{E}$ originate from a source (charge) in the field $\vec{E}$.

3.9 Vector Integration

The physical meaning of the divergence and curl operations is understand in terms of certain vector integrals and equations relating such integrals. Therefore, we shall deal herewith the three integrals, *viz. the line integral, the surface integral and the volume integral.*

LINE INTEGRAL

Let us consider a displacement from P to Q in a vector field $\vec{F}$. Further, let this displacement be divided into small elements dl_1, dl_2 etc. $\vec{F}$ is inclined at an angle θ to $\vec{dl}$ as shown in Fig. 3.4, such that it continuously varies in magnitude as well as direction as we proceed along the curve. The product of the length of the element and the component of $\vec{F}$ in the length direction is

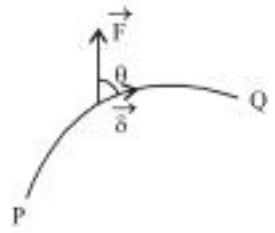

Fig. 3.4

$$F \cos\theta \, dl = \vec{F}.\vec{dl}$$

Hence, the integral of $\vec{F}$ from P to Q is

$$\int_P^Q F\cos\theta \, dl = \int_P^Q \vec{F} \, .\vec{dl} \qquad ...(3.32)$$

is called the integral of $\vec{F}$ from P to Q. Let us see, *what physically the line integral represents.* Since the vector field $\vec{F}$ can be regarded as a gradient of some scalar function ϕ, *i.e.* $\vec{F}$ = grad ϕ, and the gradient in fact represent the rate of change of a field quantity. Therefore the line integral is simply representing the sum of this *rate of change* and is equal to the total change between P and Q. Thus,

$$\int_P^Q \vec{F} \, .\vec{dl} = \int_P^Q \vec{\nabla}\phi.d\vec{l} = \int_P^Q \frac{\delta\phi}{\delta l}.d\vec{l} = \int_P^Q d\phi = \phi\ (Q) - \ \phi\ (P)$$

where ϕ (P) and ϕ (Q) are the scalar fields at P and Q respectively. Since the value of R.H.S. *i.e.* (ϕ (Q) − ϕ (P)) depends only on the values of ϕ at two points P and Q so that

$$\int_P^Q \vec{F}.d\vec{l} = \int_P^Q \vec{\nabla}\phi.d\vec{l} = \phi\ (Q) - \ \phi\ (P)$$

Any curve from P to Q — Any curve from P to Q

So that, the value of line integral depends only on the *position of the two points in the vector field* (which can be expressed as the gradient of a scaler field) and not upon the actual path taken between them, such vector field is called a *conservative field*. The *examples* of such fields are the familiar, electrostatics, magnetic and gravitational fields.

$$\int_P^Q \vec{F}.d\vec{l} = \int_P^Q (F_x dx + F_y dy + F_z dz) \qquad ...(3.33)$$

Example 1.

If vector $\vec{F}$ represents the force acting on a particle moving along curve from P to Q, the line integral $\int_P^Q F.d\vec{l}$ gives the total work done by the force in displacing the particle from P to Q.

Example 2.

If $\vec{F}$ represents the value of the electric (or magnetic field) intensity at P, then the integral $\int_P^Q \vec{F}.d\vec{l}$ gives the total work done on unit charge (or unit pole) during its motion from P to Q *i.e. the* potential difference between P and Q, independent of the path taken.

3.10 Line Integral of a Vector Field Around an Infinitesimal Rectangle

The curl of a vector filed $\vec{F}$ at a point is related to the line integral of $\vec{F}$ along the boundary C of an infinitesimal plane area around that point which encloses it. To establish the relation between these two quantities (*i.e.* curl and line integral), let us evaluate the line integral of the vector field $\vec{F}$ around an infinitesimal rectangle of sides PQ = δx and QR = δy in the X – Y plane (Fig. 3.5). Let the x and y-components of $\vec{F}$ at the point P be F_x and F_y respectively. Since, we are dealing

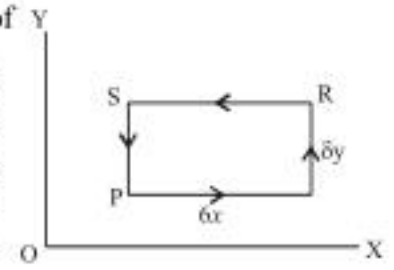

Fig. 3.5

with an infinite simal rectangle it means the sides are very small, so that the component of the vector at the middle point of any side may be taken as the average value for all points on it. Thus we have

The average value of x-component along the path PQ = $Fx + \frac{\delta Fx}{\delta x}\frac{\delta x}{2}$ and along the path RS = $F_x + \frac{\delta Fx}{\delta x}.\frac{\delta x}{2} + \frac{\delta Fx}{\delta y}\delta y$

Since the direction of the path RS is towards the negative direction of x, *therefore*, the integral of $\vec{F}$ along PQ plus line integral of $\vec{F}$ along RS is given by,

$$\int_P^Q \vec{F}.d\vec{l} + \int_R^S \vec{F}.d\vec{l} = \left(Fx + \frac{\delta Fx}{\delta x}\frac{\delta x}{2}\right)\delta x - \left(Fx + \frac{\delta Fx}{\delta x}.\frac{\delta x}{2} + \frac{\delta Fx}{\delta y}\delta y\right)\delta x$$

$$= \frac{\delta Fx}{\delta y}\delta x\delta y \qquad ...(3.34)$$

Again remembering that SP is towards the negative direction of y and proceeding in the same way, we get,

$$\int_Q^R \vec{F}.d\vec{l} + \int_S^P \vec{F}.d\vec{l} = \frac{\delta Fy}{\delta x}\delta x\delta y \qquad ...(3.35)$$

By adding (3.34) and (3.35) we get the line integral,

$$\oint_{PQRS} \vec{F}.d\vec{l} = \left(\frac{\delta Fy}{\delta x} - \frac{\delta Fx}{\delta y}\right)\delta x\delta y$$

As, we have shown in Eq. (3.22) that $\left(\frac{\delta Fy}{\delta x} - \frac{\delta Fx}{\delta y}\right)$ is the z-component of curl $\vec{F}$ or $\vec{\nabla} \times \vec{F}$. If the small area $\delta x\delta y$ may be regarded as the magnitude of the vector area $\delta S_z\hat{k}$, thus,

$$\oint_{PQRS} \vec{F}.d\vec{l} = (\text{curl } \vec{F})_z\, dS_z = (\text{Curl } \vec{F})_z.\, \delta S_z\hat{k}$$

This relation is valid for any infinitesimal area at P, if

$\delta S = \delta S_x\hat{i} + \delta S_y\hat{j} + \delta S_z\hat{k}$ with boundary C then we have

$$\oint_C \vec{F}.d\vec{l} = \text{curl } \vec{F}.\delta\vec{S} \qquad ...(3.36)$$

The maximum value of $\oint_C \vec{F}.d\vec{l} = |\text{curl } \vec{F}|\,\delta S$

or $$|\text{curl } \vec{F}| = \frac{\left[\oint_C \vec{F}.d\vec{l}\right]_{max}}{\delta S} \qquad ...(3.37)$$

Thus, *the magnitude of curl* $\vec{F}$ *at a point is the maximum value of the line integral of* $\vec{F}$ *per unit area along the boundary C of an infinitesimal plane area at that point*. And the direction of the curl is perpendicular to the plane to the infinitesimal area, corresponding to the maximum value of integral, there will be an orientation of the area for which the line integral is maximum *i.e.* the value of line integral depends upon the orientation of the given small vector area at the point.

3.11 Curl of a Conservative Field

Now take a closed path, consisting of path 1 from P to Q followed by path 2 from Q to P as shown in Fig. 3.6. As we know that the line integral in the case of conservative field is independent of the path between the points and depends only on the positions of these points. Thus, we get,

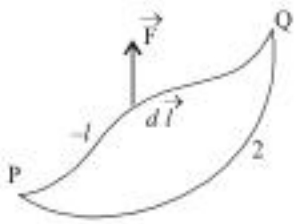

Fig. 3.6

$$\int_{1P}^{Q} \vec{F}.d\vec{l} = \int_{2P}^{Q} \vec{F}.d\vec{l}$$

the line integral around a close path from P to Q and Q to P

$$\oint_{\text{closed path}} \vec{F}.d\vec{l} = \int_{1P}^{Q} \vec{F}.d\vec{l} + \int_{2Q}^{P} \vec{F}.d\vec{l}$$

or $$\oint_{\text{closed path}} \vec{F}.d\vec{l} = 0, \left[\text{since } \int_{Q}^{P} = -\int_{P}^{Q}\right]$$

Therefore, for the conservative field the

$$\text{curl } \vec{F} = \frac{\left[\oint_C \vec{F}.d\vec{l}\right]_{max}}{S} \text{ is zero [}\textit{see}\text{ eq. (3.73)]}$$

or $$\text{curl } \vec{F} = 0 \qquad ...(3.38)$$

or $$\vec{\nabla} \times \vec{F} = 0$$

Such field is called a curl free or non-curl field.

Now according to relation (3.26), there is a scalar ϕ such that,

$\vec{F} = \nabla \phi$, that *is when the curl of a vector field is zero it can be expressed as the gradient of a scalar field* or the *lamellar vector*. The *fact* that for conservative field the curl $\vec{F}$ or the line integral around the closed path is zero or in other words in conservative field the value of line integral depends only on the location of these points allowed us to introduce the idea of potential. In electricity if the curl of electric field $\vec{E}$ is zero $(\vec{\nabla} \times \vec{E} = 0)$, then $\vec{E}$ is the gradient of scalar field *i.e.* potential.

3.12 The flux Through a Surface or Surface Integral

Let us imagine a smooth surface in the vector field then, consider a continuously varying $\vec{F}$ at a point P inside a small surface element of area *ds*. Draw the positive normal* of unit length *n* on the element and let θ be the angle between $\hat{n}$ and $\vec{F}$, as shown in Fig. 3.7.

The component of $\vec{F}$ perpendicular to *d*S is F cos $\theta = \vec{F} . \hat{n}$. Let us consider the flow of fluid, then the *net amount of flux going out through the surface is the "flux of velocity"* through surface. For any arbitrary closed surface, the net outward flow or flux is *the average outward normal component of the velocity times the area of the surface.* So that, we can write,

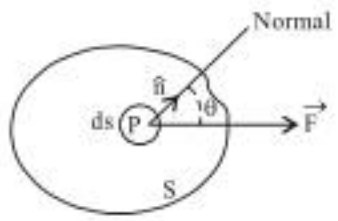

Fig. 3.7

Flux = (average normal component) (Surface area)

Here, the flux of $\vec{F}$ through the element $d\vec{S}$ is

$$\vec{F} . \hat{n} dS = F \cos \theta \, dS$$

The integral of this taken over the entire surface is called the *total flux or surface integral of* $\vec{F}$ through the whole surface S *i.e.*

$$\iint_S \vec{F} . \hat{n} dS = \iint_S \vec{F} \, Cos\theta dS = \text{Total fluid flowing outward through S} \quad ...(3.39)$$

In terms of cartesian components, we have

$$\iint_S \vec{F} \hat{n} \, dS = \iint_S (F_x dS_x + F_y dy + F_z dz)$$

$$(\text{as } \hat{n} dS = d\vec{S} = \vec{i} \, dS_x + \hat{j} \, dS_y + \hat{k} \, dS_z)$$

In the case *where the vector does not represent the flow of any-thing.* For example, *the electric field* $\vec{E}$, *the integral of the normal component of electric field over an area is still representing the flux*. In other words, it is *not necessary that the vector should represent the flow of secondary we write,*

$$\text{Flux of } \vec{E} \text{ through the surface} = \iint_S \vec{E} . \hat{n} dS$$

or $$\text{Flux of } \vec{E} \text{ through the surface} = \int_S \vec{E} . \hat{n} dS$$

(the sign $\iint_S$ is simply taken to distinguish from the line integral $\int_Q^P$, we shall represent flux by $\int_S$).

So that, we generalize the word "flux" to mean the "surface integral of t he normal component" of a vector.

---

* If the surface is closed, $\hat{n}$ is drawn outward from the enclosed volume. If the surface is unclosed, then $\hat{n}$ is drawn always on the same side of the surface, the normal on the opposite side being negative.

VOLUME INTEGRAL:

Let us consider a surface enclosing volume v in a vector field $\vec{F}$ and suppose that $\vec{F}$ is a vector point function at a point in a small volume element dv as shown in Fig. 3.8. Then the $\iiint_v \vec{F}.dv$ is known as the integral of the vector field $\vec{F}$ for entire volume v over the surface. In cartesian components

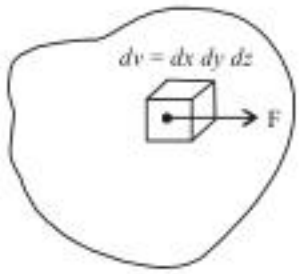

Fig. 3.8

$$\iiint_v \vec{F}.dv = \int_x \int_y \int_z (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\, dx\, dy\, dz \quad ...(3.40)$$

where $dv = dx\, dy\, dz$.

3.13 Gauss's Divergence Theorem

The theorem states that the normal surface integral of a vector function $\vec{F}$ over the boundary of a closed surface, which is the flux across S, is equal to the volume integral of the divergence of the function over the volume v enclosed by the surface, *i.e.*

$$\text{flux} = \iint_S \vec{F}.\hat{n}\, dS = \iint_s \vec{F}\,\hat{n}\, dS = \iiint_v \text{div}\ \vec{F}\ dv$$

$$= \iiint_v (\vec{\nabla}.\vec{F})\, dv$$

The *converse theorem is necessarily true. So that Gauss's divergence theorem is used to transform a volume integral into surface integral or the vice-versa.*

Proof:

Consider, that the volume v enclosed by the surface S is made up of a very large number of elementary volumes for convenience in the form of cubes or rectangular parallelopipdes adjoining each other. Let one such cube of volume dv, with its sides along the three axes, x, y, and z are dx, dy and dz respectively as shown in Fig. 3.9. If F_x, F_y and F_z are the component of F along x, y and z respectively, then the outward flux through the left is given by $-\int F_x dydz$, where the

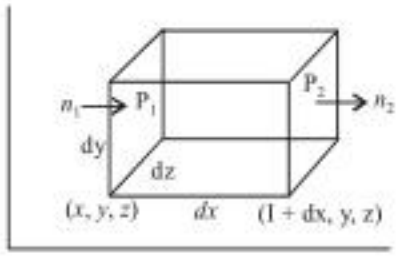

Fig. 3.9

integral extends over the area of face. As the cube considered is an infinitesimal one so that F_x at the middle point P, of the face may be taken for entire face. Let F_x at P_1 be $F_x(P_1)$. Then we have,

Flux through left face of the tube cube $= -F_x(P_1)\, dz\, dy$

where $|dz|\, dy$ is face area.

Similarly the flux through right face $= F_x(P_2)\, dy\, dz$, the flux is positive because $F_x(P_2)$ is positive, taking that F_x is changing at the rate $\delta F_x / \delta x$, we have,

$$F_x = (P_2) = F_x(P_1) + \frac{\delta F_x}{\delta x} dx$$

so that, the flux through right faces is cube

$$= \left[F_x(P_1) + \frac{\delta F_x}{\delta x} dx \right] dy\, dz$$

Hence flux through the left and right faces is

$$= \frac{\delta F_x}{\delta x} dx\, dy\, dz$$

Similarly flux through top and bottom faces of the cube

$$= \frac{\delta F_x}{\delta y} dx\ dy\ dz$$

and flux through remaining faces (correspoding to z-axes) of the cube

$$= \frac{\delta F_y}{\delta_z} dx\ dy\ dz$$

Hence the total flux through the cube is sum of these and using Eq.(3.31) for the definition of flux, we get

$$\int \int_{\text{Surface of cube}} \vec{F}.\hat{n}dS = \left[\frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}\right] dx\ dy\ dz$$

in which $d\vec{S} = \hat{n}dS$ is the area of the surface of the cube and $\hat{n}$ is unit vector normal to this surface, or

$$\int \int_{\text{Surface of cube}} \vec{F}.dS = (\text{div } \vec{F})dv \text{ , (since div } \vec{F} = \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z} \text{ and } dv = dx\ dy\ dz)$$

or $$\int \int_{\text{Surface of cube}} \vec{F}.dS = (\vec{\nabla}.\vec{F})dv$$

Taking the sum of fluxes of all the elementary cubes constituting the surface, we get,

$$\iint_S \vec{F}.d\vec{S} = \iiint_v (\nabla.\vec{F})dv = \iiint_v (\text{div } .\vec{F})dv \qquad ...(3.41)$$

which is Gauss's theorem of divergence.

In terms of cartesian components, Gauss's theorem is

$$\iint_S (Fx\ dS_x + F_y\ dS_y + F_z\ dS_z)$$

$$= \iiint_v \left(\frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}\right) dx\,dy\ dz \qquad ...(3.42)$$

3.14 Stoke's Therem

This theorem states that the line integral of a vector $\vec{F}$ around any closed curve C is equal to the surface integral of Curl $\vec{F}$ taken over any surface S of which the curve C is boundary edge, *i.e.*,

$$\oint_C \vec{F}.\vec{dl} = f\int_S \text{ourl } \vec{F}.d\vec{S} = f\int_S (\vec{\nabla} \times \vec{F}).d\vec{S} \qquad ...(3\ 43)$$

Proof : Let a vector | $\vec{F}$ be a function of position. Then its line intergral along a closed curve C is given by.

$$L = \oint_C \vec{F}\,\vec{dl}$$

where $\vec{dl}$ is a small element of path as shown in Fig. 3.10.

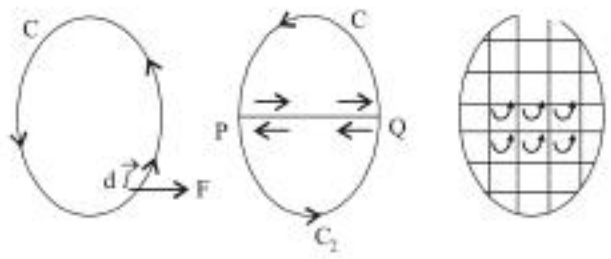

Fig. 3.10

Let us draw a line PQ, so that the enclosed ares is divided into two parts as to from two separate closed curves C_1 and C_2. If L_1 and L_2 are the line integrals for the two parts C_1 and C_2 then,

$$L_1 = L_1 + L_2$$

or $$\oint_C \bar{F}.d\bar{l} = \oint_{C_1} \bar{F}.d\bar{l} + \oint_{C_2} \bar{F}.d\bar{l}$$

Because of the fact that the part PQ is traversed from P to Q for the curve C_1 and from Q to P for the path C_2, so that its contribution is cancelled. Thus if we divide the area enclosed by the curve C into number of smaller areas such as dS_1, dS_2,.... dS_n bounded by curves C_1, C_2,....., C_n then

$$\oint_C \bar{F}.d\bar{l} = \sum \oint_{C_n} \bar{F}.d\bar{l}$$

But from the defination of curl $\bar{F}$ given in Eq. (3.36) *i.e.*, $\oint_C \bar{F}.d\bar{l}$ = curl $\bar{F}.\bar{d}S$ here the surface area d S_n, so we have curl $\bar{F}.\bar{d}S_n = \oint_{C_n} \bar{F}.\bar{d}l$, putting this value, we get

$$\oint_C \bar{F}.d\bar{l} = \sum \oint_{C_n} \bar{F}.d\bar{l} = \sum \text{Curl } \bar{F}.d\bar{S}_n$$

The summation on the right hand side is clearly the surface integral of curl $\bar{F}$ over the entire surface S enclosed by the curve so that, we have,

$$\oint_C \bar{F}.d\bar{l} = \iint_S \text{Curl } \bar{F}.d\bar{S} \qquad ...(3.44)$$

This is Stoka's theorem.

3.15 Some useful Results from Gause's and Stoke's Theorem

1. The curl of the gradient of a scalar ϕ is zero:

$\bar{\nabla} \times \bar{\nabla}\phi = 0$. As we have discussed that the line integral of the gradient of a scalar function along a closed path is zero, i.e. if

$$\bar{F} = \bar{\nabla}\phi$$

$$\oint \bar{\nabla}\phi.d\bar{l} = 0$$

Hence from Stoke's theorem,

$$\oint_C \bar{\nabla}\phi.d\bar{l} = \int_S^{\prime} \text{curl}\,\bar{\nabla}\phi d\bar{S} = 0$$

Since the integral $\iint$ curl $\bar{\nabla}\phi.\bar{d}$ S is zero, the integrand itself should be zero i.e.,

$$\text{curl } \bar{\nabla}\phi = 0 \text{ or } \nabla \times \nabla\phi = 0$$

which is same as Eq. (3.30).

2. The divergence of the Curl of a Vector Function $\bar{F}$ is zero:

According to the Stoke's theorem the line integral around a curve is equal to the surface integral of the curl $\bar{F}$ taken over any surface S of which the curve C is a boundary edge.

It means, that as the boundary or the loop decreases which corresponds as the surface tends to close as shown in Fig. 3.11, and the value of integral on both side of Eq. (3.44) decreases, when the boundary or loop decreases to a mere point, the surface is closed and the value of two integrals is zero, so that for a closed surface,

$$\iint_S (\bar{\nabla} \times \bar{F}).d\bar{S} = 0$$

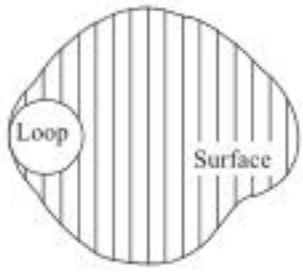

Fig. 3.11

Now, according to Gauss's theorem, we can transform surface integral to volume integral,

$$\iint_S (\bar{\nabla} \times \bar{F}).d\bar{S} = \iiint_V \bar{\nabla}.(\bar{\nabla} \times \bar{F})dv$$

Clearly,

$$\iiint_V \bar{\nabla}.(\bar{\nabla} \times \bar{F})dv = 0$$

i.e., $\bar{\nabla}.(\bar{\nabla} \times \bar{F}) = 0$, at every point ...(3.45)

which is same a Equation 3.25

3.16 The Classification of Vector Fields

So far, we have discussed only two kinds of vector fields. viz.

1. Lamellar or scalar potential field: It is characterised by zero curl and can be expressed as the gradient of a scalar potential (*see* Eq. 3.26) and it is, therefore, called a lamellar of scalar potential field.

2. Solenoidal Field: It is characterised by zero divergence which means there are no sources of vector flux in the field. The flux lines are closed curves and the field is called solenoidal. [*see* Eq. (3.27).]

Since the div. courl of a vector $\{\bar{\nabla}.(\bar{\nabla} \times \bar{F}) = 0\}$ is zero, therefore, *a solenoidal vector can always be expressed as the curl of another vector as given below:*

On the basis of different association of curl and divergence the vector fields are of four kinds given below.

1. The Laplace Equation

Curl $\bar{F} = 0$ and div $\bar{F} = 0$. Acoording to first condition curl $\bar{F} = 0$ the field is lamellar or irrotational. The second condition makes the field solenoidal or incompressible.

So from first condition $\bar{F}$ = grad ϕ and from second condition div grad

$$\phi = 0 \text{ or } \bar{\nabla}.\bar{\nabla}\phi = \nabla^2\phi = 0 \quad ...(3.46)$$

which is *Laplace equation.*

The *examples* of this type of field are electric intensity in free space due to static charges on boundary of conductors, or the irrotational motion of an incompressible fluid.

2. The Poisson's Equation

Curl $\bar{F} = 0$ but div $\bar{F} \neq 0$. Here the field is irrotational but not solenoidal. From first condition, we have,

$$\bar{F} = \text{grade } \phi$$

Applying second condition, we have,

div grad $\phi \neq 0$ or $\nabla^2\phi \neq 0$

which is Poisson's equation. The field must contain a spatial distribution of sources of flux. The *examples* are the *electric field of a volume distribution of charge, e.g. electrons in a thermionic tube,* or the gravitational force inside mass, or the irrotational motion of compressible fluid.

3. The Idea of Vector Potential

Curl $\bar{F} \neq 0$ but div $\bar{F} \neq 0$. The field is rotational and cannot have a scalar potential, but it is solenoidal. Let $\bar{A}$ be a vector such that $\bar{F}$ = curl $\bar{A}$ then div $\bar{F}$ = div curl $\bar{A}$ = 0 (According to relation 3.45) as required by second condition and curl $\bar{F}$ = curl curl $\bar{A}$ = grad div $\bar{A}$ $- \nabla^2$ $\bar{A}$ (*see* Eq. 3.29). The nature of the vector $\bar{A}$ is a matter of choice, let $\bar{A}$ to be solenoidal, then div $\bar{A}$ = 0 so that

$$\text{curl } \vec{F} = -\nabla^2 \vec{A} \qquad ...(3.47)$$

This relation is identical to Poisson's equation but contains a vector instead of scalar operand. That is why, $\vec{A}$ is called the vector potential of $\vec{F}$

Examples: The important examples are the magnetic field within a conductor carrying a steady current and the rotational motion of an incompressible fluid.

4. The Electro-magnetic Field

Curl $\vec{F} \neq 0$ and div $\vec{F} \neq 0$. This is the most general type of vector field as rotational motion of compressible fluid and in electro-magnetic theory. *Here the field can be expressed as the sum of two fields.* The first is a *lamellar vector,* which has no curl but can have divergence, and the second. is a *solenoidal vector,* which has no divergence but can have curl.

Lamellar vector can eb taken as grad ϕ and solenoidal vector as curl $\vec{A}$. Since the curl of a scalar function is zero and divergence of a curl of a vector is also zero, hence the required condition is satisfied, So,

$$\vec{F} = \text{grad } \phi + \text{curl } \vec{A} \qquad ...(3.48)$$

Hence $\quad \text{div } \vec{F} = \text{div grad } \phi + \text{div curl } \vec{A}$

or $\quad \text{div } \vec{F} = \vec{\nabla}.\vec{\nabla}\,\phi$, since div curl $\vec{A} = 0$ (*see* Eq. 3.26)

$$= \nabla^2 \phi = 0$$

This is Poisson's equation and solution of it gives ϕ, and curl $\vec{F}$ = curl grad ϕ + curl curl $\vec{A}$

or curl $\vec{F}$ = curl curl A. (Since curl grad $\phi = 0$ *see* Eq. 3.24)

Since,

curl curl $\vec{A}$ = grad div $\vec{A} - \nabla^2$ **A,** (*see* Eq. 3.29)

But for solonoidal field $\vec{A}$ = curl A. so first term on right hand side is zero, hence,

$$\text{curl } \vec{F} = -\nabla^2 \vec{A} \qquad ..(3.49)$$

The solution of this equation gives $\vec{A}$. *Thus, the general vector field may be decomposed into two fields, one being lamellar, with scalar potential ϕ and the other solenoidal, with vector $\vec{A}$.* This decomposition is know as *Helmholtz's Theorem.*

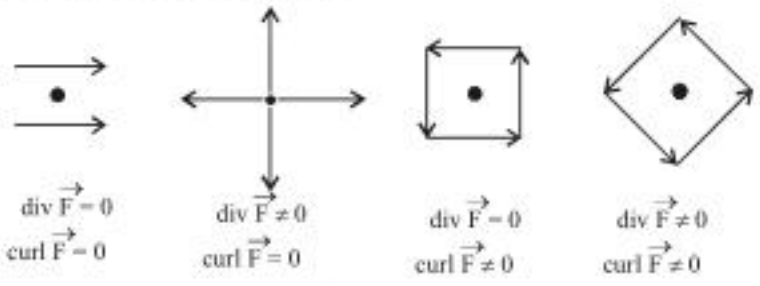

Fig. 3.12

Fig. 3.12 illustrate diagrammatically the characteristics of vector fields near a point.

3.17 Reciprocal Vectors

If $\vec{a}, \vec{b}$ and $\vec{c}$ are the three orthogonal unit vectors along the three axes, then the *Reciprocal Vectors* $\vec{a^x}$, $\vec{b^x}$ and $\vec{c^x}$ are defined as,

$$\vec{a^x} = \frac{\vec{a} \times \vec{c}}{\vec{a}.(\vec{b} \times \vec{c})}, \quad \vec{b^x} = \frac{\vec{c} \times \vec{a}}{\vec{b}.(\vec{c} \times \vec{a})}, \quad \vec{c^x} = \frac{\vec{a} \times \vec{b}}{\vec{c}.(\vec{a} \times \vec{b})}$$

the vectors $\vec{a},\vec{b}$ and $\vec{c}$ are assumed to be non-coplanar so that product $\vec{a}.(\vec{b}\times\vec{c})$ is not zero. **The** vectors $a^{\vec{x}}, b^{\vec{x}}$ and $c^{\vec{x}}$ are perpendicular to the planes of $\vec{b},\vec{c}$; $\vec{c},\vec{a}$ and $\vec{a},\vec{b}$ respectively. As the dot product which can be easily seen that,

$$\vec{a}\,.\,\vec{a^x} = \vec{b}\,.\,\vec{b^x}\times\vec{c}\,.\,\vec{c^x} = 1$$

Hence they are called reciprocal vectors and the dot product of any other pair, on from each system, is zero. Thus,

$$\vec{a}\cdot\vec{b^x} = 0 \text{ because } \vec{a}\cdot\vec{b^x} = \frac{\vec{a}.(\vec{c}\times\vec{a})}{\vec{a}.(\vec{b}\times\vec{c})}$$

Here the numerator being the triple product containing two equal vectors *i.e.*, ($\vec{a}$ and $\vec{a}$) is equal to zero.

Similarly, $$\vec{b}\cdot\vec{a^x} = \vec{b}\cdot\vec{c^x} = \vec{c}\cdot\vec{a^x} = \vec{c}\cdot\vec{b^x} = 0 \qquad ...(3.51)$$

Further, it is true that,

$$\vec{a} = \frac{\vec{b^x}\times\vec{c^x}}{\vec{a^x}.(\vec{b}\times\vec{c^x})},\ \vec{b} = \frac{\vec{c^x}\times\vec{a^x}}{\vec{b^x}\times(\vec{c^x}\times\vec{a^x})}$$

$$\vec{c} = \frac{\vec{a^x}\times\vec{b^x}}{\vec{a^x}.(\vec{a}\times\vec{b^x})} \qquad ...(3.52)$$

Hence the two sets of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a^x}$, $\vec{b^x}$, $\vec{c^x}$ are said to be reciprocal to each other.

3.18 Green's Theorem

This theorem enables us to express an integral taken over the surfaces of a number of bodies as an integral taken through the space between them.

Thus, it u, v w are continuous functions of the cartesian coordinates x, y, z than

$$\sum\iint (lu+mv+nw)\,ds = -\iiint\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dx\,dy\,dz \qquad ...(3.53)$$

Here Σ denotes that the surface integrals are summed over any number of closed surface and l,m,n are the direction -cosines of the normal drawn in every case from the element ds into the space between the surfaces. The volume integral is taken throughout the space between the surfaces.

Proof. Let us consider any small prism with its axis parallel to that of x-axis and of cross-section dydz as shown in figure 3.13. Let this prism meet the surfaces at P, Q, R, S, T, U, V, W cutting of areas $\mathrm{d}S_P$, S_Q, $\mathrm{d}S_R$ ..... .

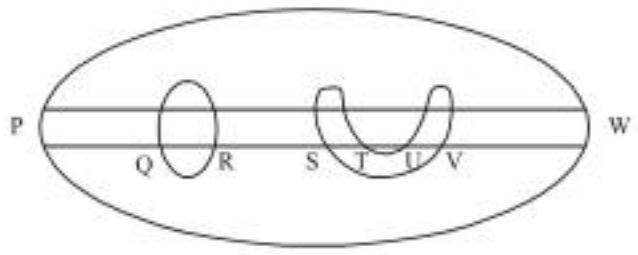

Fig. 3.13

Consider the first term of R. H. S eqn 13.53 *i.e.*, $\iiint\frac{\partial u}{\partial x}\,dx\,dy\,dz$

Since we have considered prism with its axis parallel to that of x-axis hence

$$\iiint\frac{\partial u}{\partial x}\,dx\,dy\,dz = dy\,dz\int\frac{\partial u}{\partial x}\,dx$$

Thus $$\int \frac{\partial u}{\partial x}\, dx = \int_{P}^{Q} \frac{\partial u}{\partial x}\, dx + \int \frac{\partial u}{\partial x}\, \partial x + \ldots\ldots$$

$$= - u_P + u_Q - u_R + u_S$$

where the integral is taken over those parts of the prism which are between the surfaces. Where $u_P, u_Q, u_R \ldots$ are the values of u at P, Q, R ..... . Now the projection of each of the areas dS_P, S_Q on the plane of yz is $dy\, dz$ so we have

where l_P, l_Q, l_R are the the values of l at P, Q, R on substitution we get

$$dy\, dz \int \frac{\partial u}{\partial x}\, dx = - l_P\, u_P\, d S_P - l_Q\, u_Q\, dS_Q - l_R u_R\, dS_R \ldots\ldots$$

If we take all prisms and addiing equations of all prisms along x-axis we get

$$\iiint \frac{\partial u}{\partial x}\, dx\, dy\, dz = -\sum \int l u\, dS$$

Similarly we have the value for y and z axis

$$\iiint \frac{\partial v}{\partial y}\, dx\, dy\, dz = -\sum \int m v\, dS$$

$$\iiint \frac{\partial w}{\partial z}\, dx\, dy\, dz = -\sum \int n w\, dS$$

on addition we get

$$\iiint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dx\, dy\, dz = - \sum (lu + mv + nw)\, dS \qquad \ldots(3.54)$$

which is Green's theorem.

Another form of Green's theorem

In equation 3.53 if substitute

$$u = \phi \frac{\partial \psi}{\partial x}\,,\ v = \phi \frac{\partial \psi}{\partial y} \text{ and } w = \phi \frac{\partial \psi}{\partial z}$$

where ϕ and ψ are any functions of x, y, z then

$$lu + mv + nw = \phi \left(l \frac{\partial \psi}{\partial x} + m \frac{\partial \psi}{\partial y} + n \frac{\partial \psi}{\partial z} \right)$$

Suppose we have a normal k direction cosines of which are l, m and n along x, y and z axis. Then we have

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial k} \cos(k, x), \qquad \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial k} \cos(k, y),$$

$$\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial k} \cos(k, z),$$

where (k, x), (k, y) and (k, z) are the angels which k makes with x, y and z directions respectively. Then

$$(lu + mv + nw) + \phi \frac{\partial \psi}{\partial k} [l \cos(k, x) + m \cos(k, y) + n \cos(k, z)]$$

But from definition of direction cosines we have

$l = \cos(k, x)$, $m = \cos(k, y)$, $n = \cos(k, z)$ then

$$(lu + mv + mv) = \phi\frac{\partial\psi}{\partial k}\left[\cos^2(k,y) + \cos^2(k,y) + \cos^2(k,z)\right]$$

$$= \phi\frac{\partial\psi}{\partial k},$$

Since $\cos^2(k, x) + \cos^2(k, y) + \cos^2(k, z) = 1$

Putting the values of u, v and w we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z} + \phi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

Putting the above values in eq. 3.54 we get

$$\iiint\left[\phi\nabla^2\psi + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}\right]dx\,dy\,dz$$

$$= -\sum\phi\iint\frac{\partial\psi}{\partial k}dS \qquad ...(3.55)$$

However this equation is true for all values of ϕ and ψ . So that if they are interchanged equation remains true. Thus

$$\iiint\left[\psi\nabla^2\phi + \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial z}\right]dx\,dy\,dz$$

$$= -\sum\iint\psi\frac{\partial\phi}{\partial k}dS \qquad ...(3.56)$$

Substracting eq. 3.56 from eq. 3.55 we get.

$$\iiint\left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]dx\,dy\,dz = -\sum\iint\left[\phi\frac{\partial\psi}{\partial k} - \psi\frac{\partial\phi}{\partial k}\right]dS$$

$$\sum\iint\left[\phi grad\,\psi - \psi\,\mathrm{grad}\,\phi\right]dS \qquad ...(3.57)$$

To derive Green's theorem from Gauss Divergence theorem

As derived in eq. 3.41 Gauss divergence theorem is expressed as

$$\iiint_a(div\vec{F})dv = \iint_s\vec{F}.d\vec{s}$$

If we substitute in this equation

$$\vec{F} = \phi\vec{\nabla}\psi \text{ then}$$

$$\vec{\nabla}.\vec{F} = \frac{\partial Fx}{\partial x} + \frac{\partial Fx}{\partial y} + \frac{\partial Fx}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z}\right)$$

$$= \left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right]\phi + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}$$

Putting the value in expression for Gauss theorem stated above we get

$$\iiint_V \left[\phi\nabla^2\psi + \nabla\phi\nabla\psi\right]dv = \iint_S (\phi\nabla\psi)d\mathrm{S} \qquad ...(3.58)$$

Interchanging ϕ and ψ we get

$$\iiint_V \left[\psi\nabla^2\phi + \nabla\psi\nabla\phi\right]dv = \iint_S (\psi\nabla\phi)d\mathrm{S} \qquad ...(3.59)$$

$$\iiint_V \left[\phi\nabla^2\psi + \psi\nabla^2\phi\right]dv = \iint_S \left[\phi\nabla\psi - \psi\nabla\phi\right]d\mathrm{S} \qquad ...(3.60)$$

which is Green's theorem

To derive Gauss divergence theorem from Green's theorem

If we put $\phi = 1$ then Green's theorem reduces

$$\iiint_V \nabla^2\psi dv = \iint_S \text{grad}\,\psi d\mathrm{S}$$

or $$\iiint_V div\ (\text{grad}\ \psi)\ dv = \iint_S grad\psi\, d\mathrm{S}$$

if we have a vector field

$$\vec{\mathrm{F}} = \text{grad}\,\psi \text{ then}$$

$$\iiint_V div\,\vec{\mathrm{F}} = \iint_S \vec{\mathrm{F}}.d\vec{\mathrm{S}}$$

$$\iint_S \vec{\mathrm{F}}.\hat{n}.\mathrm{dS}$$

which is Gauss theorem

REVIEW QUESTIONS

3.1 What do you understand by the field of any physical quantity ? Give examples of scalar and vector fields.

3.2 What do you understand by gradient of scalar field ? If $\vec{r}$ be the position vector of any particle find the value of grad (1/r).

3.3 Explain clearly what do you understand by divergence and curl of a vector field $\vec{\mathrm{F}}$. Obtain expressions in general orthogonal coordinates for

(a) div $\vec{\mathrm{F}}$ and (d) curl $\vec{\mathrm{F}}$

3.4 What do you mean by the flux of vector field ? Show that flux of a vector $\vec{\mathrm{F}}$ through the surface is given by

$\iint_S \vec{\mathrm{F}}.\hat{n}.d\mathrm{S}$ $\int_S \vec{\mathrm{F}}.\hat{n}.d\mathrm{S}$. where $\hat{n}$ is unit vector pointing outward at right angle to the surface.

3.5 What are line and surface integrals? Clearly explain what they represent by giving examples.

3.6 What a field is called as conservative field? Give examples of such field?

3.7 State and prove Gauss's divergence theorem. What is its importance?

3.8 State and prove Stoke's theorem.

3.9 What are Solonoidal and Lamellar or scalar potential fields ? Give examples.

3.10 Why the vector fields are classified on the basis of different association of curl and divergence? What are these combinations ? Illustrate by giving examples.

3.11 Obtain the expressions for Laplace and Poisson's equations.

3.12 Show that curl of a gradient of scalar is zero and also that the div of a curl of vector is zero i.e.,
curl grad $\phi = 0$, where ϕ is scalar

div curl $\vec{F} = 0$ where $\vec{F}$ is vector

What you conclude from these relations ? How the idea of vector potential is introduced. ?

PROBLEMS

3.1 A proton of 1.6×10^{-19} coul charge and 1.7×10^{-27} kg mass has velocity of 10^7 met/sec in a direction inclinod at 30^0 to the direction of a magnetic field of strength I weber/metre^2. Calculate Lorentz force.

(Ans. 8.00×10^{-13} Newton perpendicular to $\vec{v}$ and $\vec{B}$)

3.2. If potential function $\vec{v} = -axy$, obtain the value of electric field. where α is constant.

[Ans. $\alpha (\hat{i}\, y + \hat{j}\, x)$]

3.3 If $\phi (x, y, z) = 3x^2 - y^3x^2$; calculate gradient ϕ at the point (1, –2, –1).

[Ans. $(-12\hat{i} - 9\hat{j} - 16\hat{k})$]

3.4 If $\vec{A} = x^2 s\hat{i} - 2y^3s^2\hat{j} + x y^2 s\hat{k}$; calculate div $\vec{A}$ (or $\nabla.\vec{A}$) at the point (1, –1, 1).

[Ans. $(\nabla.\vec{A}) = -3$]

3.5 If $\vec{r}$ is the position vector, show that $\iint \vec{r}.\hat{n}\, dS = 3v$, where v is the volume enclosed by the surface S.

3.6 Verify Stoke's theorem for $\vec{F} = (2x - y)\,\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

3.7 Evaluate $\iint_S \vec{F}\, d\vec{S}$ where $\vec{F} = 4xz\,\hat{i}_1 - y^2\hat{i}_2 + yz\,\hat{i}_3$

S is the surface bounded by x = 0, x = 1, y = 1, z = 0, z = 1, and $\hat{i}_1, \hat{i}_2, \hat{i}_3$ are unit vectors along x, y, z axes.

[Hint. $\iint_S \vec{F}.d\vec{S} = \iiint_V div\, \vec{F}\, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} div\vec{F}\, dx\, dy\, dz$]

SELF TEST OBJECTIVE QUESTIONS

3.1 Which of the following is a vector field ?
(a) distribution of temperature
(b) magetic and electrostatics potentials
(c) density of any non-directed quantity in a given region of space
(d) the distribution of electric or magnetic field intensity

3.2 Which of the following is a scalar field ?
(a) The distribution of velocity in liquid
(b) The distribution of velocity in gas
(c) The distribution of temperature
(a) The distribution of magnetic or electrostatics field intensity.

3.3 In any scalar field, we can draw surfaces corresponding to equal values of the scalar. They are known as
(a) isothermal, equidensity or equipotential surfaces
(b) vector line, line of flow or flux line

3.4 The gratient of scalar field grad ϕ or $\vec{\nabla}\phi$ is

(a) a vector (b) a scalar

3.5 If $\vec{\nabla} \times \vec{A} = 0$, there is a f such that

(a) $\vec{A} = \vec{\nabla}\phi$ (b) $\vec{A} = \vec{\nabla} \times \phi$

3.6 If $\vec{\nabla} \cdot \vec{D} = 0$

there is a $\vec{C}$ such that

(a) $\vec{D} = \vec{\nabla} \cdot \vec{C}$ (b) $\vec{D} = \vec{\nabla} \times \vec{C}$

3.7 If $\vec{B}$ is a magnetic field and its divergence $\vec{B} = 0$, everywhere, it means

(a) the lines of $\vec{B}$ could have a source in the field $\vec{B}$
(b) an isolated magnetic pole is possible
(c) an isolated magnetic pole is not possible

3.8 Gauss's divergence theorem is used to transfrom:
(a) a volume integral into is used surface integral
(b) a line integral into surface integral

3.9 Fill in the blanks by inserting appropriate words:
(a) The curl of the gradient of a scalar function is .....
(b) The divergence of the curl of a vector function is .....
(c) Solenoidal field is characterised by zero .....
(d) a solenoidal vector can always be expressed as the ..... of another vector.

3.10 $\vec{F}$ represents a vector field and curl $\vec{F} \neq 0$ but div $\vec{F} = 0$, it means
(a) the field has a scalar potential
(b) the field has a vector potential

3.11 Electric field created by a electrostatics charge Q is
(a) continuous
(b) Irrotational
(c) Solenoidal
(d) Isotropic

ANSWERS: 1. (d) **2.** (c) **3.** (a) **4.** (a) **5.** (a) **6.** (d) **7.** (c) **8.** (a) **9.** [(a) zero, (b) zero, (c) divergence, (d) curl]. **10.** (b) **11.** (d).

CHAPTER 4

ELECTROSTATICS : ELECTRIC FIELD

41. Electrostatics

We hope that the reader is familiar with the elementary facts of electricity, such as the existence of electric charge and the electrical constitution of matter. Here, we want to study the basic laws on which the whole electromagnetic theory is based. *We begin with the Physics of stationary electric charges, i.e. charges at rest called electrostatics.*

4.2 Electric Charge

Let us perform a simple experiment with two glass rods and a piece of silk. First, glass rod is rubbed with silk and is hanged by a thread. Now, the second rod is also rubbed with silk and is brought near the first. The rods will repel each other, on the other hand, a hard-rubber rod rubbed with fur will attract the glass rod rubbed with silk and the two rubber rod rubbed with fur will repel each other. The repulsion and attraction of rods is due to, the two facts given below :

(*i*) That when a rod is rubbed, an electric charge appears on it.

(*ii*) That the charges on the glass and hard-rubber rods are different.

4-3. Duality of Charge

Certainly, one fundamental property of electric charge is its *existence in the two varieties, that were long ago named positive and negative.* All charged particles are divided into two classes such that all members of one class repel each other, while attracting members of the other class. Even today, we cannot say certainly, why this universal law prevails. However, *today physicist regard positive and negative charge as, fundamentally, opposite manifestation of one quality.* Such as "right and left" are opposite manifestation of "handedness".

Indeed, the question of symmetry involved in right and left seems to be intimately related to this duality of electric charge. And to another fundamental symmetry, the two directions of time. However, elementary particle physics which is beyond the scope of this book is throwing some light on the duality of electric charge.

What we call negative charge could just as well have been called positive and vice versa. *The choice was a historical accident. The charge of the ordinary electron has nothing intrinsically negative about it.* Benjamin Franklin, a pioneer of electrostatics gave the name of "positive charge'" to the charge on a glass rod rubbed with silk, and "negative charge" to that on an ebonite rod rubbed with fur.

4.4 Quantisation of Charge

Today, the electric charge, like other physical quantities such as energy and angular momentum in the orbit, is not regarded as a continuous fluid. Millikan's oil drop experiment discussed in art (4.27) have shown that the electric fluid is not continuous but it is made up of integral multiples of a certain

minimum electric charge. This minimum electric charge is denoted by e, the electronic charge, has the magnitude 1.6×10^{-19} *coul.* Any physically existing charge in the universe, is an *integer multiple* of e, i.e. ne,* irrespective of its origin. So that the *charge exists in discrete packets rather than in continuous amount. This* is referred as charge quantisation.

However, the fact of charge quantisation lies outside the scope of classical electromagnetism. The question that how this electronic charge is constituted *i.e.* what is the *force which holds the electron together* or *how the precise value of its charge is fixed. Something more than electrical forces must be involved because the electrostatics forces between the different parts of the electron would be repulsive.*

In our present study of electricity, *we shall treat charged particle simply as carrier of charge with dimensions so small that their extension and structure is almost insignificant.*

4 5. Conservation of Charge

Let us look carefully, the experiment by which the existence of electric charge was demonstrated. When a glass rod is rubbed with silk, a positive charge appears on the rod. And a negative charge of equal magnitude appears on the silk. So that, *rubbing does not create charge but merely transfers charge from one object to other, disturbing slightly the electrical neutrality of each.* In other words, actually no new charge is *created* but simply charge is *transferred.* The net charge before and after rubbing is same. So the conservation rule is satisfied and we say that *charge is conserved quantity without any exceptions provided the system is isolated* (in which no charge is allowed to cross the boundary).

The total charge in an isolated system never changes. For instance a *"pair creation"* event in which a high energy photon ends its existence with the creation of a negative electron and a positive electron (or positron). Thus, *the two electrically charged particles have been newly created but the net change in total charge is zero.* The net charge is zero before and after the event, as shown in Fig. 4.1.

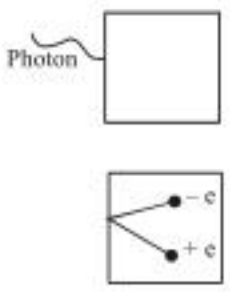

Fig. 4.1

The creation of positively charged particle without the simultaneous creation of negatively *charged particle has never been observed.* However, we have a *reverse process* when an electron and a positron *combine* to form a photon, known as *"annihilation"* process, here again the total charge before and after annihilation remains constant, *i.e.,* zero, which confirms the law of conservation of charge.

Another example **of charge conservation is** ***"radioactive decay"*****, such** as

$$_{92}U^{238} \longrightarrow {}_{90}Th^{234} + {}_{2}He^{4}$$

Thus, we see that amount of charge (92 *e)* present before disintegration is the same as that present after the disintegration.

The *final example* of charge conservation is the *"nuclear reactions"*. Such as, proton and α particle reaction given below :

$$_{80}Hg^{200} + {}_{1}H^{1} = {}_{79}Au^{197} + {}_{2}He^{4}$$

Mercury Proton Gold α–particle

We again see that amount of charge (81 e) present before disintegration is the same as that present after the disintegration.

So, ***we can conclude that the electric charges can be only separated or combined when positive and negative charges always appear or disappear in equal quantities.*** **Therefore, the** *charge*

*In some theoretical speculations about elementary particles, the possible existence of fractionally charged particles called "quarks" was suggested. Quarks have not boon detected experimentally, however; and even if they were, that would not alter the fact that charge is quantised. It woulfl merely reduce the size of the basic unit.

conservation law is that the total electric charge in an isolated system, that is, the algebraic sum of the positive and negative charge present at any time, never changes.

4.6. Conductors and Insulators

William Gilbert classified the materials in two main classes. The materials behaving like, amber, were called *electrics,* deriving the name from the word elektron for amber in Greek. The materials those did not behave like amber were called by Gilbert *non-electrics,* But today, these two kinds of materials are known as *conductors* and *insulators* (or dielectrics) respectively.

The Gilbert classification of materials is based on their ability to acquire electrification (or charge). But according to modern concept, *i.e.* the electron theory, *the classification is based on the ability of the materials to conduct electric charges.* The substances and bodies *capable of conducting electric charges well are called conductors,* whereas, the substance and bodies that *do not conduct electric charges well are caused insulators* (or dielectrics). In other words, in conductors electric charges are free to move through the material, whereas in insulator they are not.

4.7. Coulomb's Law of Force

In 1785, Coulomb reported the results of a series of measurements on the dependence of the electrical force between the charged bodies at rest upon the distance between them and magnitude of charges. Coulomb's experimental results can he expressed as,

$$F \propto \frac{q_1 q_2}{r^2} \qquad ...(4.1)$$

where F is the magnitude of the force that acts on each of the two stationary charges of magnitude q_1 and q_2 in a homogeneous and isotropic medium and r is distance between them as shown in Fig. 4.2. The equation (4.1) holds only for charges whose sizes are much smaller than the distance between them. *i.e.* only for point charges.

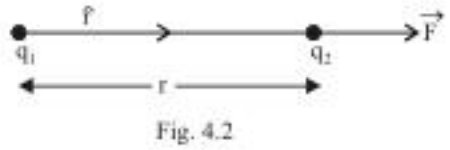

Fig. 4.2

The equation (4.1) is known as the *Coulomb's law of electrostatics forces,* which states that *the force between two point charges at rest is directly proportional to the product of the magnitude of the charge, i.e,* $q_1 q_2$ *and is inversely proportional to the square of the distance between them i.e.,* $\frac{1}{r^2}$. This is also called *inverse square law.*

This law is analogous to the inverse square law of gravitation simply charge *(q)* plays the role of mass *(m)* in that law. However, in gravitation the forces are always attractive because there is one kind of mass whereas in electricity, since there are two opposite types of charges (positive and negative), hence the forces are *attractive as well as repulsive.* The equation (41) gives after inserting appropriate constant, the magnitude of the force. As the force and distance are vector quantities, the Coulomb's law in vector form becomes,

$$\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r} \qquad ...(4.2)$$

where k is a constant of proportionality and $\hat{r}$ is a unit vector in the direction from q_1 to q_2. $\vec{F}$ is the force on q_2 due to q_1. The force on q_1 due to d_2 is equal and opposite to $\vec{F}$ *(i.e.,* equal to $-\vec{F}$).

The value of *constant k depends on the choice of units of F, r and q and the nature of the medium in which the charges are situated.*

CGS ELECTROSTATICS UNITS OR E.S.U

In electrostatics system of unit the constant k in vacuum, is unity. In air k =1.00059 or very nearly one.

When F=l, $q_1 = q_2 = q$, r =1 and k =l, we get from Eq. (4.2),

$$q = \pm 1$$

Hence in CGS or esu, *an unit charge is that charge, which when placed at a distance of one cm in air or vacuum from an equal and similar charge, repels it with a force of one dyne.*

Thus in electrostatics system units the Coulomb's law reduces,

$$\vec{F} = \frac{q_1 q_2}{r^2} \hat{r} \text{ dynes} \qquad ...(4.3)$$

THE RATIONALISED MKS SYSTEM OR SI UNITS

This is most widely used system today which *we shall adopt throughout in this book,* and write simply as RMKS. In rationalised MKS units the unit of charge is coulomb (abbr. coul). The reason for introducing the coulomb is its simple relation to the common electrical units (ampere, volt, ohm and watt) which we use in engineering, in the laboratory and in our everyday life.

A coulomb is defined as the amount of charge that flows through a given cross-section of a wire in one second if there is a steady current of one ampere in the wire, i.e.,

$q = I\,t$, q = 1 coul, if I = 1 amp and t= 1 sec.

In rationalised MKS units the constant k is chosen as $1/4\pi\varepsilon_0$ when the charges are situated in vacuum. Then Coulomb's law given in Eq. (4.2) can be written as.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r} \text{ newton} \qquad ...(4.4)$$

where ε_0 (epsilon zero) is known as the *permittivity of free space* (or vacuum). If the charges are situated in *medium having the permittivity* ε* (epsilon), then the force between the charges is reduced and Eq. 4.2 reduces

$$\vec{F}_m = \frac{1}{4\pi\varepsilon} \frac{q_1 q_2}{r^2} \hat{r} \text{ newton} \qquad ... (4.5)$$

Here $\vec{F}_m$ denotes the force between the two charges and ε as the permittivity of the medium. The degree to which the electric charge can influence the medium is known as its permittivity. How is the medium affected and what is new phenomena ? How do we describe what happens to the material ? What new phenomena arise that can be measured and used to learn something about the material's microscopic structure ? We shall discuss these questions in chapter six.

Relative permittivity. If we divide Eq. (4.4) by Eq. (4.5). we get,

$$\frac{F}{F_m} = \frac{\varepsilon}{\varepsilon_0} = k \qquad ...(4.6)$$

where *k* is known as the *relative permittivity* of the medium or the dielectric constant (or coefficient).* It is *defined* as the *ratio* of the permittivity of the medium (or absolute permittivity of the medium) to that of free space. Clearly, it is a *dimensionless quantity,* hence it is sometimes called ***specific inductive capacity.***

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* The permittivity of air is substantially the same as for vacuum. The permittivity of air at normal pressure is only about 1.005 times that, ε_0 of a vacuum. The permittivity of water is about eighty times that of a vacuum.

Why $\frac{1}{4\pi\varepsilon_0}$ is chosen ?

In Eqn. (4.4) the constant of proportionality in rationalised MKS system is chosen as $\frac{1}{4\pi\varepsilon_0}$. As we know that in CGS system of units, the factors 4π and 2π occur very frequently in an irrotational manner such as expression for the capacitance of a parallel plate capacitor, is $C = \frac{A}{4\pi t}$, field inside a solenoid is $H = 4\pi ni$, in which the lines of force are parallel. In fact these factors (4π and 2π) should generally occur when we deal quantities possessing spherical, circular, radial or cylindrical symmetry. But in CGS the factors π does not appear in some systems which possess spherical or circular symmetry, for example the circular magnetic field due to a straight conductor carrying current is $H = \frac{2i}{r}$ and the capacity of a spherical capacitor is $C = \frac{ab}{a-b}$ etc.

Hence the electric and magnetic units were *redefined such that the factor π should appear* in equations concerning spherical, circular or radial systems. That is possible if we introduce $\frac{1}{4\pi\varepsilon_0}$ in place of k in equation (4.2). The certain equations that are derived from Eq. (4.2) but are used frequently in electro-magnetic theory will be simpler in form.

Value of $\frac{1}{4\pi\varepsilon_0}$

In the RMKS system we can measure q_1 q_2, r and F in equation (4.4) in ways that do not depend on Coulomb's law. Numbers with units can be assigned to them. There is no choice about the permittivity constant ε_0, it must have that value which makes the right hand side of equation (4.4) equal to the left hand side (*i.e.* by dimensional analysis).

If q_1 and q_2 are measured in coul., distance r in metres and force in newton, then,

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{Coul}^2}{\text{Nt. met}^2} \text{ or } \frac{\text{Farad}}{\text{Metre}}$$

and

$$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \frac{\text{Nt. met}^2}{\text{coul}^2}$$

Thus, *toe see that ϵ_0 is not an arbitrary constant, but an experimentally determined constant.* In Eq. (4.4), if $F = 9 \times 10^9$ newtons, $q_1 = q_2 = q$ coul, $r = 1$ metre, then $q = \pm 1$. Hence, *in RMKS or the practical unit of charge is a coulomb. One coulomb is that charge which when placed at a distance of one metre in vacuum, from an equal and similar charge repels it with a force of 9×10^9 newtons.*

1 coul= 3×10^9 e.su. of charge

1 newton = 10^5 dynes.

Ex. 4.1. ***A negative point charge of 10^{-6} coul is situated in air at the origin of a rectangular coordinate system. A second negative point charge of 10^{-4} coul is situated on the positive x-axis at a distance of 50 cm from the origin. What is the force on the second charge ?***

---

*The **term** capacitivity is also used for permittivity. The term dielectric constant is used synonymously with permittivity. However, the permittivity is not always a constant as might be inferred from the term dielectric constant, but may depend on the temperature and the frequency.

Sol. By Coulomb's law the force F is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

Here according to given values,

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{(-10^{-6})(-10^{-4})\ \text{coul}^2}{(0.5)^2\ \text{metre}^2}\ \hat{i}$$

On substituting the value of $\frac{1}{4\pi\varepsilon_0}$ we get,

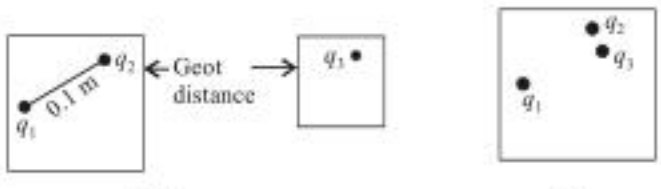
$$\vec{F} = 9\times 10^9\ \frac{\text{Nt. met}^2}{\text{coul}^2}\times\frac{10^{-10}\ \text{coul}^2}{0.25\ \text{met}^2}\hat{i}$$

$$= (3.6)\,\hat{i}\ \text{newton}$$

Thus, there is a force of 3.6 newton in the positive x-direction on the second charge.

4 8. The Principle of Linear Superposition of Forces

Suppose we have only two charges q_1 and q_2, then we could verify only that $\vec{F}$ is proportional to $1/r^2$. One can ask what about the proportionality of $\vec{F}$ to product $q_1 q_2$? *No doubt, the significant* ***part in Coulomb's law is the statement of inverse-square dependence, and product $q_1 q_2$ is additive in its effect.*** **To explain the latter point,** we have to consider more than two charges. Let there be three charges q_1, q_2 and q_3. We can measure the force $\vec{F}_1$ on q_1 when q_2 is 10cm away from q_1 and q_3 is very far away, as shown in Fig. 4.3(a). Then inter change q_2 and q_3, *i.e.* now q_2 is away from q_1 and again measure the force F_2 on q_1. Finally, we bring q_2 and q_3 very close [Fig. 4.3(b)] together and locate the combination at 0.1 met. We



(a) (b)

Fig. 4.3

find **a** very interesting result by measurement that the force $\vec{F}_3$ on q_1 is equal to the sum of the forces previously measured.

$$\vec{F}_3 = \vec{F}_1 + \vec{F}_2$$

Thus, *we can conclude that the force with which two charges inter-act is not changed by the presence of third charge.* Suppose we have three charges as shown in Fig. 4.4, the force on q_3 is given by,

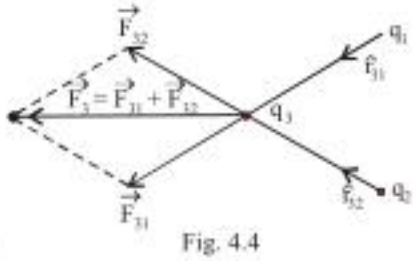


Fig. 4.4

$$\vec{F}_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_3}{r_{31}^2}-\hat{r}_{31}+\frac{q_2 q_3}{r_{32}^2}\hat{r}_{32}\right]$$

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32} \qquad ...(4.7)$$

The principle of superposition of electrical forces is that the total or resultant force at a point is the vector sum of the individual component forces at that point.

So the Coulomb's law holds for every pair of charges irrespective of the number of charges in a system. This is the basis for principle of superposition. *Superposition means combining two sets of sources into one system by adding the second system on top of the first without affecting the configuration of any one.*

1. The force on the point charge due to group of point charges

The force on a charge q at p due to number of charges $q_1, q_2, q_3 \ldots\ldots q_1$, is equal to vector sure of

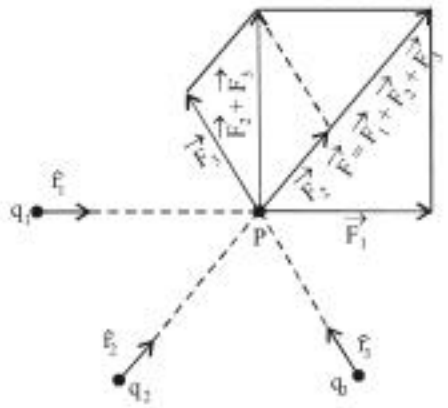


Fig. 4.5

the forces due to the individual charges. This fact is called the principle of superposition. .Thus .we write,

$$\mathrm{F}_p = \sum_{i=1}^{i=n} \frac{1}{4\pi\varepsilon_0} \frac{qq_i}{r_i^2} \hat{r}_i \qquad \ldots(4.8a)$$

The situation is illustrated in Fig. 4·5 for charges q, q_1, q_2 and q_3. The force on q is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

2. The Force on a charge q due to a linear charge distribution

If the charge is not concentrated at a point but is distributed along a line as shown in Fig. 4.6, then the force due to this linear

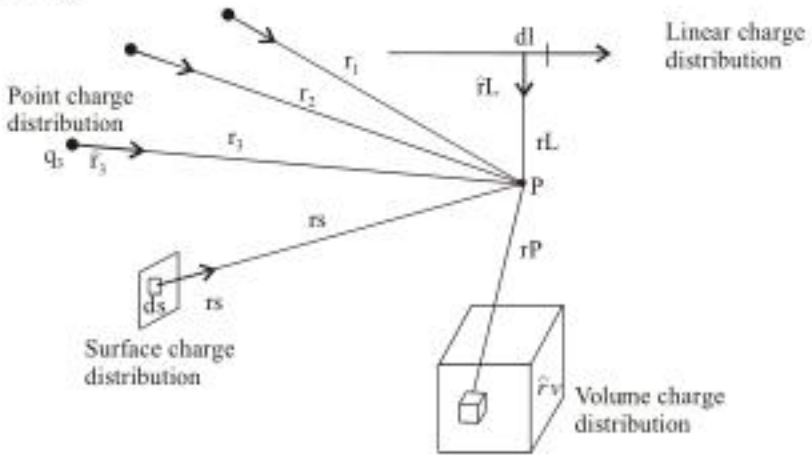


Fig. 4.6

charge distribution on the charge q at P is

$$\vec{F}_L = \frac{1}{4\pi\varepsilon_0} \int_L \frac{q\rho dl'}{rL^2} \hat{r}L \qquad \ldots(4.8b)$$

where $\hat{r}_L$ is unit vector towards **P**

ρ_L = linear charge density (coul/metre)

dl' = element of length of line (metre) The integration is carried out over the entire length of charge.

3. The force on a charge q due to a surface charge distribution

When the charge is distributed over a surface as shown in Fig. 4.6, the force on q at P is

$$\vec{F}_S = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{q\sigma dS'}{rs^2} \hat{r}s \qquad ...(4.8c)$$

where,

$\hat{r}$S is a unit vector towards P and

σ = surface density of charge (coul/met^2)

dS′ =the element of the surface met^2

and the integration is carried out over the entire surface of the charge.

4. The force on charge q doe to a volume charge distribution

When the charge is distributed as shown in Fig. 4.6, the force on charge q at P is

$$\vec{F}_v = \frac{1}{4\pi\varepsilon_0} \iiint \frac{q\rho dv'}{r_v^2} \hat{r}_v \qquad ....(4.8d)$$

where,

p = volume charge density (coul/metre^3)

dv' = element of volume (metre^3)

The integration is carried out over the entire volume containing charge.

where $\hat{r}_v$ is a units vector towards point P, where charge q is situated.

$$r = [(x_1 - x'_1)^2 + (y_1 - y'_1)^2 + (z_1 - z'_1)^2]^{1/2}$$

where, P has coordinate (x, y, z) and dv' is situated at $(x', y' z')$

If all the distributions are present simultaneously, then total force on charge q by supper position principle is the algebraic sum of the individual components, *i.e.*,

$$\vec{F} = \vec{F}_P + \vec{F}_L + \vec{F}_S + \vec{F}_V \qquad ...(4.9)$$

If all the charge distributions are such that they occupy finite volume instead of being distributed to points, lines, and surface, then Eq. (4.9) reduces to (4.8*d*).

Ex. 42. ***Fig. 4.7 shows three charges q_1 q_2, q_3 having the values $q_1 = -1.0 \times 10^{-9}$ coul, $q_2 = 3.0 \times 10^{-6}$ coul, $q_3 = -2.0 \times 10^{-6}$ coul, $r_{12} = 15$ cm, $r_{13} = 10$ cm and $\theta = 30°$, calculate the force on q_1.***

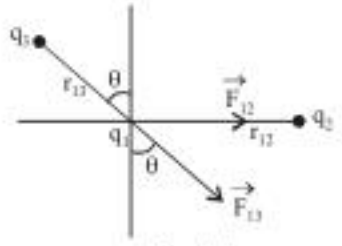


Fig. 4.7

Sol. Using Coulomb's law, we get the magnitude of the force,

$$F_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}^2}$$

on substituting values,

$$F_{12} = 9 \times 10^9 \frac{\text{Nt. m}^2}{\text{coul}^2} \cdot \frac{(10^{-6}\ \text{coul})\,(3 \times 10^{-6}\ \text{coul})}{(15 \times 10^{-2}\ \text{met})^2}$$

Thus, $F_{12} = 1.2$ Nt

Similarly, $F_{13} = 1.8$ Nt

The directions are shown in Fig. 4.7. The component of the resultant force F_1 acting on q_1 along x-direction,

$$F_{1x} = F_{12x} + F_{13x} = F_{12} + F_{13} \sin\theta$$
$$= 1.2 \text{ Nt} + (1.8 \text{ Nt}) \sin 30 = 2.1 \text{ Nt}$$

and component of F_1 along y-direction,

$$F_{1y} = F_{12y} + F_{13y} = 0 - F_{13} \cos\theta$$
$$= (1.8 \text{ Nt}) \cos 30 = -1.6 \text{ Nt}$$

Hence the magnitude of $F_1 = [F^2_{1x} + F^2_{1y}]^{1/2}$

or $F_1 = \sqrt{6.77}$

and $\vec{F}_1 = (2.1)\hat{i} - 1.6\hat{j}$

The angle between $\vec{F}_1$ and the x-axis is i.e. direction cosine,

$$\cos\theta = \frac{2.1}{\sqrt{6.77}}, \theta = \cos^{-1}\left(\frac{2.1}{\sqrt{6.77}}\right)$$

4.9. Scope of the Coulombs Law

Let us now discuss the range of distances over which Coulomb's law is valid. In 1785 Coulomb measured with a torsion balance the force between small charged spheres. Cavendish had checked the Coulomb's law over distances of the order of inches or feet. How-ever, we want to know, at what range of distances, the Coulomb's law breaks down. Let us consider two cases. The first is the domain of very small distances less than 10^{-14} cm. The second is the domain of very large distances greater than feet. We have no experimental test of Coulomb's law for these two domains. In fact, the stability of every nucleus gives a sufficient basis for the breakdown of Coulomb's law at very small distances. But we have no particular reason to expect a breakdown at large distances. In fact the modern quantum theory of the electromagnetic field gives some basis for arguing that Coulomb's law holds over distances much greater than those used in the Cavendish experiment. The reason is that a breakdown, at large distances, of the Coulomb's force would imply a small but finite rest mass for the light quantum, or photon, and consequently a slight variation with wavelength in the speed of electromagnetic wave in vacuum. It is known from direct observations that short radio waves travel through vacuum with the same speed as visible light, which contradicts the breakdown of Coulomb's law. Thus it is possible to argue theoretically, that Coulomb's law is valid upto distances of several kilometers.'

To summaries, we have every reason for confidence in Coulomb's law, over the wide range from 10^{-13} cm to many kilometres. This law when incorporated into the structure of Quantum Physics, correctly describes,

1. The electric forces that bind the electrons of an atom to its nucleus,
2. The forces that bind atoms together to form molecules and
3. The forces that bind atoms or molecules together to form solids or liquids.

Hence, we can say that most of the forces, except gravitational, are electrical.

LIMITATIONS OF THE COULOMB'S LAW (OR ELECTRICAL FORCES AND NUCLEAR FORCES)

The Coulomb's law fails to explain the stability of the nucleus, since, in a nucleus, there are several protons, all having the positive charge. According to Coulomb's law, they should repel each other. But actually, they do not push themselves apart, because nucleus has a stable identity. Hence, here the Coulomb's law fails. However, *in nuclei, there are some non-electrical and non-gravitational forces, in addition to electrical forces, called nuclear forces.* Nuclear forces are stronger than electrical forces. Hence, they are able to hold the protons together, in spite of the electrical repulsion.

COULOMB'S LAW AND NUCLEAR FISSION

However, the nuclear forces are of short range. Their magnitude falls off much more rapidly than $1/r^2$, and this has an important consequence. If a nucleus has too many protons in it, it gets too big and it will not stay together. An example is uranium, with 92 protons. The nuclear forces act mainly between each proton or neutron and its nearest neighbour, while the electrical forces act over larger distances, giving a repulsion between each proton and all of the others in the nucleus. The greater is the number of protons in the nucleus, stronger is the electrical repulsion and so it is easy to break such nucleus.

4 10. Relative Strength of the Fundamental Forces

In the hydrogen atom the electron and proton are separated by 5.3×10^{-11} metre. The magnitude of the electrical force is given by Coulomb's law, *i.e.*

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

On substituting the value of charge and realising that $q_1 = q_2 = q = 1.6 \times 10^{-19}$ coul, and r given above, we get,

$$F_e = 8.1 \times 10^{-8}\ \text{N}$$

Similarly the gravitational force is given by,

$$F_e = G\frac{m_1 m_2}{r^2}$$

$$= 6.7 \times 10^{-11}\ \text{N. m}^2/\text{kg}^2\ \frac{(9.1\times10^{-31}\ \text{kg})(1.7\times10^{-27}\ \text{kg})}{(5.3\times10^{-11}\ \text{metre})^2}$$

$$= 3.7 \times 10^{-47}\ \text{N.}$$

Thus the electrical force is about 10^{39} times stronger than the 7-16 150 gravitational force.

Let us now calculate the force of repulsion between two protons in a nucleus of iron, assuming a separation of 4.0×10^{-16} m.

From Coulomb's law,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

$$= 9 \times 10^9\ \text{N.m}^2/\text{coul}^2.\ \frac{(1.6\times10^{-19}\ \text{coul})^2}{(4\times10^{-15}\ \text{m}^3)}$$

$$= 14\ \text{N}$$

The nuclear binding force is greater than this repulsive force. Thus we can conclude that the nuclear binding forces are much stronger than the electrical forces that bind the electrons with the nucleus in an atom. And in turn these atomic binding forces are much stronger than gravitational forces for the same particles separated by the same distance.

4.11. The Electric Field

When an electric charge is placed at some point in space, this establishes everywhere a state of electric stress, which is called an electric field. The space where charge influence can be felt, is called site of electric field.

MECHANISM OF ELECTRICAL INTERACTION

Since the charges are not in physical contact, the equations representing the coulombian forces may be considered to represent the existence of action-at-*a-distance point i.e.,* a direct and. instantaneous interaction between the two charges, same as that for magnetic and gravitational forces.

If the only problem in electromagnetism was that of the forces between stationary charges, the field and the action at-a distance point view is perfectly justified. The justification does not hold for charges in motion.

Suppose q_1 is suddenly accelerated towards q_2. How quickly does charge q_2 learn that q_1 has moved and that the force q_2 experienced must increase ? According to action at a distance view, the information about q_1*s* acceleration is communicated instantaneously to q_2. This is not in accord with experiment. Electromagnetic theory predicts that q_2 learns q_1*'s* motion by a *field disturbance that originates from* q_1 *travelling with the speed of light.* Therefore q_2 takes certain time, however small, before it knows the motion of q_1.

An electric field is a physical quantity which has different values at different points in space. The force between two charges is considered in terms of electric field as follows :

1. The charge q_1, in Fig. 4.2, sets up an electric field in the space itself.
2. The field of q_1 acts on q_2 and q_2 experiences a force $\vec{F}$.

Thus the force $\vec{F}$ on q_2 is due to the action of the field of q_1 on q_2. However, we can also imagine that q_2 sets up a field and that this field acts on q_1, producing a force $-\vec{F}$ on q_1. *The problem is symmetrical, each charge being immersed in a field-associated with the other charge.* The field plays an *intermediary role* in the forces between charges.

412. Intensity of Electric Field $\vec{E}$

To represent an electric field, we need to associate a vector, that is, a magnitude and direction with every point around a charge. Consider a positive charge q; it will set up a field around it; now bring some other small charge q_0, called test-charge (assume to be positive for convenience) into q's vicinity.

We can calculate the resulting force $\vec{F}$ on q_0 by Coulomb's law, *i.e.,*

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r}$$

where $\hat{r}$ is a unit vector in a direction from q to q_0. On dividing the above equation by q_0, we get,

$$\frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}$$

We obtain a vector quantity, $\frac{\vec{F}}{q_0}$, say $\vec{E}$

Thus,
$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad ...(4.10)$$

The vector quantity $\vec{E}$ depends only on charge q and on the position of the point where test charge

q_0 was placed, and is called *intensity of electric field* or *field strength at that point*. Thus the intensity of electric field due to a charge q is defined as the force experienced by a unit positive (for convenience) charge placed at the point at which the field is to be determined.

However, the test charge q_0 should be as small as possible because the large value of q_0 will disturb the primary charge q which is responsible for the field, thus changing the very quantity that we are trying to measure. Therefore the equation (4.10) should be strictly under $\lim q_0 \to 0$. Hence, we can write,

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0} = \lim_{q_0 \to 0} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

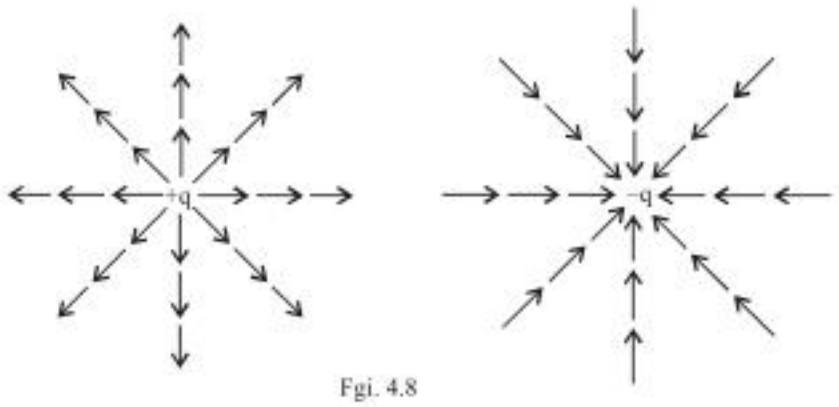
The direction of $\vec{E}$ is the same as of $\vec{F}$, hence along the direction of unit vector $\hat{r}$.

Unit of $\vec{E}$

From equation (4.10), which is the definition of $\vec{E}$, its RMKS unit is Newton/Coulomb. However, after the discussion of electric potential (See art. 5.2), an equivalent unit for the electric field intensity is the volt /metre.

413. Field Lines

Consider equation (4.10). Let q be a positive charge, surely greater than test-charge q_0, $\vec{F}$ will be repulsive, i.e., from q to q_0, hence direction of $\vec{E}$ is radially outward from q. The magnitude of $\vec{E}$ **at** any point is the magnitude of the force at the point where q_0 is situated which decreases as we choose points away from the charge q because r in equation (4.10) increases. However, if charge is negative *i.e.* (—q). The situation is quite reverse. That is $\vec{E}$ is directed inward or toward —q and magnitude increases as test-charge q_0 approaches — q. The electric field of two equal and opposite charges is shown in Fig. 48. The length of the vector represents the force at the point.



Fgi. 4.8

4 14. Relationship between Field Lines and Lines of Forces

Faraday, never appreciated the concept of the electric field as a vector, he always preferred lines of forces. We can visualise electric field patterns conveniently in terms of lines of forces.

The electric field E is related to lines of forces by the following two rules :

(*i*) The tangent to a line of force at any point gives the direction of $\vec{E}$ at that point.

(*ii*) The lines of force are drawn such that the number of lines per unit cross-sectional area is proportional to the magnitude $\vec{E}$. *The density of lines of force gives the magnitude of* $\vec{E}$.

Thus $\vec{E}$ *is strong where the lines are close and* $\vec{E}$ *is weak where the lines are far apart. And the density of lines of forces depends on the magnitude of charge.* The lines of forces for equal but opposite charges are shown in Fig. 4.9.

4 15. An Important Difference Between Field Lines and Lines of Force

The electric field $\vec{E}$ varies in a perfectly continuous way along any path in field, as shown in

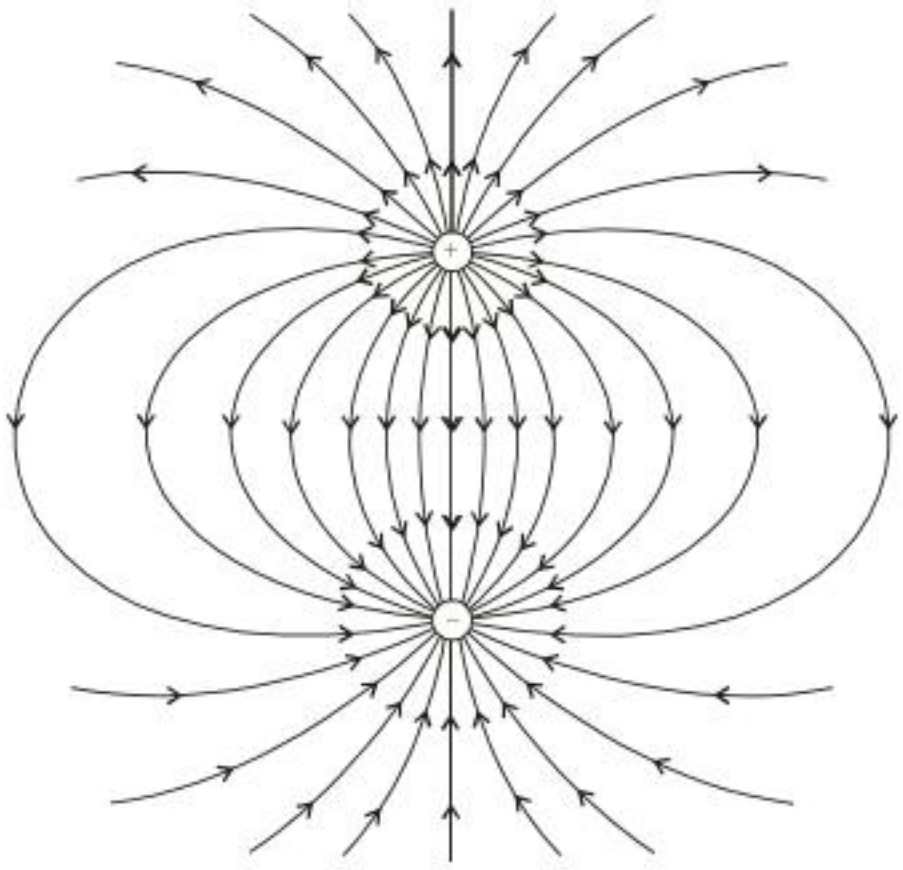


Fig. 4.9

Fig. 4.10(*a*). However, this type of physical continuity is different from the continuity of the lines of force, which has no real physical meaning. To explain this point, let us consider the lines of force as discontinuous, as shown in Fig. 4.10(*b*). These lines satisfy the two conditions mentioned above, but they are not continuous. Thus the lines of force need not be necessarily continuous. The physical meaning of the continuity of $\vec{E}$ means that the field is continuous and has different values at different points. But the continuity of lines of force does not represent such a thing.

4.16. The Electric Field of Several Point Charges (i.e. the Principle of Superposition of Fields)

We can generalise the equation (4.10) for number of charges, since the electric field $\vec{E}$ at any point is the force experienced by test-charge at that point. The resultant force due to number of

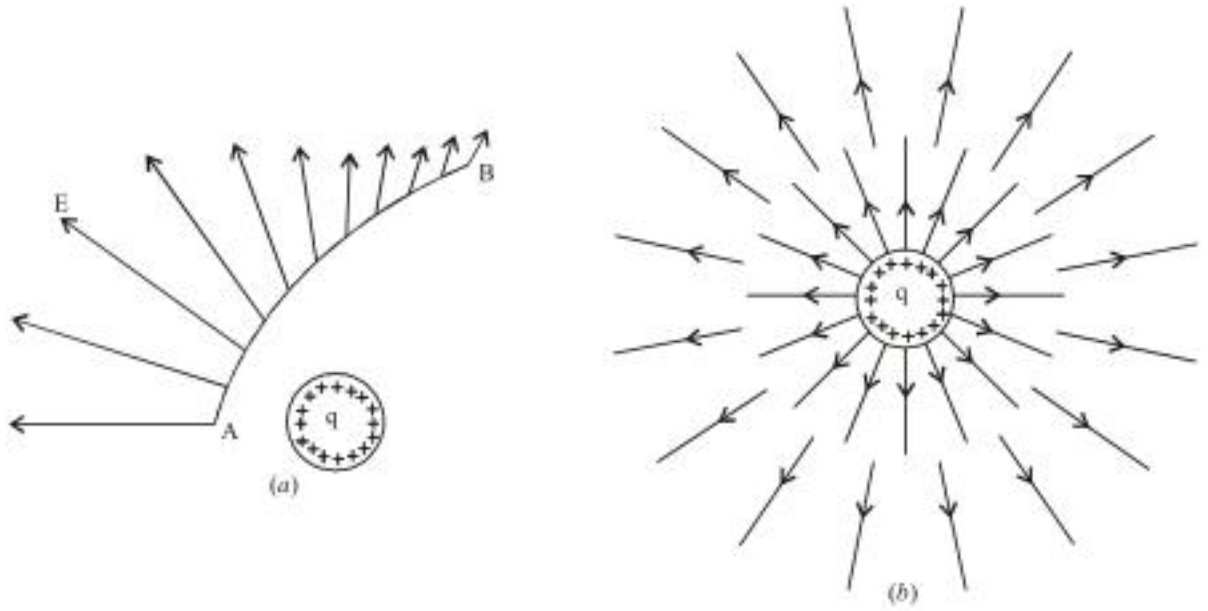


Fig. 4.10

charges is determined by principle of superposition. Therefore, the field $\vec{E}$ due to n charges is given by,

$$\vec{E}_P = \sum_{i=n}^{i=n} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i^2} \hat{r}_i \qquad ...(4.12a)$$

Thus, the principle of superposition of electric fields is that the total or resultant field at a point is the vector sum of the individual component fields at a point. The field intensity of two point charges q_1 and q_2 at the point P is shown in Fig. 411.

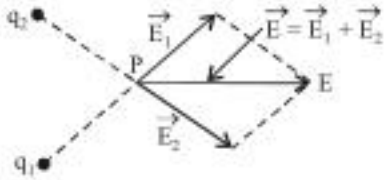


Fig. 4.11

As we have discussed for the force, the same is applicable to field and the field due to :

1. Linear charge.
2. Surface charge, and
3. Volume charge distribution, as shown in Fig. 4.6, at any point P when a test-charge q_0 is placed, is given by the following relations.

1. Field at a point due to a Linear Charge Distribution is

$$\vec{E}_L = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho dl'}{rL^2} \hat{r}_L \qquad ...(4.12b)$$

2. Field at a point due to a surface charge distribution is,

$$\vec{E}_S = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\rho dS'}{rS^2} \hat{r}_S \qquad ...(4.12c)$$

3. Field at a point due to a volume charge distribution is where symbols have their usual meaning.

$$\vec{E}_V = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\rho dv'}{r_v^2} \hat{r}_v \qquad ...(4.12d)$$

And if all distributions are present simultaneously then total electric field at P by the super-position principle,

$$\vec{E} = \vec{E}_P + \vec{E}_L + \vec{E}_S + \vec{E}_V \qquad ....(4.13)$$

If all of the charge distributions are such that they occupy finite volumes instead of being distributed to points, lines and surface, then Eq. (4.13) reduces to equation. (4.12d).

Ex. 43. *Find the Electric field intensity due to a very long straight, uniformly charged wine.*

Sol. Fig. 4.12 shows an uniformly charged a long wire. Let us calculate the electric field at a point P at distance r from the wire. Let λ be the density of charge per unit length of the wire.

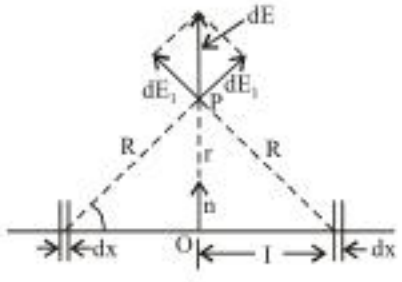


Fig. 4.12

Consider two symmetrical elements of the wire dx at a distance x from the perpendicular, drawn on the wire from the point P. Due to symmetry, the horizontal components of the field intensity will cancel each other, and only the vertical components remain. Because of this, the resulting electric field due to two elements of the charged wire, dx, will be along the y-axis *i.e.* dE_y.

$$d'E_y = 2dE_1 \sin\theta$$

the field due to two elements is equal and given by,

$$dE_1 = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{R^2}$$

where $\lambda\, dx$ is charge on the length dx.

Therefore,

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{R^2}2\sin\theta$$

the total field is given by the relation,

$$E_y = \int_0^{\infty}\frac{1}{2\pi\varepsilon_0}\frac{\lambda dx}{R^2}\sin\theta$$

From the diagram,

$$x = r\cot\theta, \frac{r}{R} = \sin\theta$$

$$dx = -r\,\mathrm{cosec}^2\,\theta\, d\theta$$

When $x = 0$, $\theta = \pi/2$, $x = \infty$ $\theta = 0$

Therefore,

$$E_y = -\int_{\pi/2}^{0}\frac{1}{2\pi\varepsilon_0}\frac{\lambda r\,\mathrm{cosec}^2\theta\sin\theta\, d\theta}{R^2}$$

$$= \frac{1}{2\pi\varepsilon_0}\int_{\pi/2}^{0}\frac{\sin\theta\, d\theta}{r}$$

$$= -\frac{\lambda}{2\pi\varepsilon_0 r}|-\cos\theta|_{\pi/2}^{0}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 r}$$

Since the field is normal to the wire it can be written as,

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{n}$$

where' $\hat{n}$ is unit vector along normal *i.e.* r.

Ex. 4 4. *Trace the Electric field due to a finite line of charge.*

Sol. Fig. 4.13 shows an uniformly charged wire of length L. Let λ be the density of charge per unit length of the wire. Further, let the distance of P from the centre O of the rod be r.

Consider two symmetrical elements dx of the wire at a distance x from the centre O. Due to symmetry, the horizontal components of the electric field intensity will cancel each other, and only the vertical components remain. Therefore the resultant electric field due to two elements dx will be along the y-axis, *i.e.*,

$$d\,\mathrm{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{R^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(r^2 + x^2)}$$

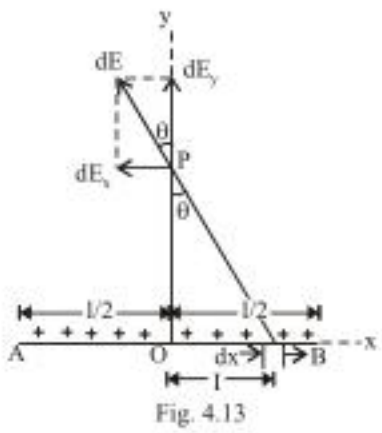


Fig. 4.13

But $d\,\mathrm{E}_y = d\mathrm{E}\cos\theta$, therefore,

Total field due to wire of length L is given by

$$\mathrm{E} = \int_{-L/2}^{L/2} d\,\mathrm{E}_y$$

$$= \int_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(r^2 + x^2)}\cos\theta$$

$$= \int_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(r^2 + x^2)}\cdot\frac{r}{(r^2 + x^2)^{1/2}}$$

Since $\cos\theta = \dfrac{r}{(r^2 + x^2)^{1/2}}$

$\therefore$ $$\mathrm{E} = \frac{2\lambda r}{4\pi\varepsilon_0}\int_0^{L/2}\frac{dx}{(r^2 + x^2)^{3/2}}$$

$$= \frac{\lambda r}{2\pi\varepsilon_0}\left|\frac{1}{r^2}\frac{x}{(r^2 + x^2)^{1/2}}\right|_0^{L/2}$$

(To evaluate integral, put $x = r\tan\theta$, then proceed).

or
$$E = \frac{\lambda r}{2\pi\varepsilon_0}\cdot\frac{1}{r^2}\frac{\frac{L}{2}}{\left(r^2+\frac{L^2}{2}\right)^{1/2}}$$

$$E = \frac{\lambda r}{2\pi\varepsilon_0 r}\cdot\frac{L}{\left(L^2+4r^2\right)^{1/2}}$$

This shows that if $L \to \infty$, $4r^2$ can be neglected,

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

The above relation is same as derived in previous example.

Ex. 4-5. *As an example of the procedure when symmetry does not exist, it is required to find the electric field intensity at the point F, as shown in Fig. 414, due to a uniformly charged wire of length L.*

Sol. Here Field $\vec{E}$ is due to a uniformly charged wire of length L, when symmetry does not exist *i.e.* at any arbitrary point P.

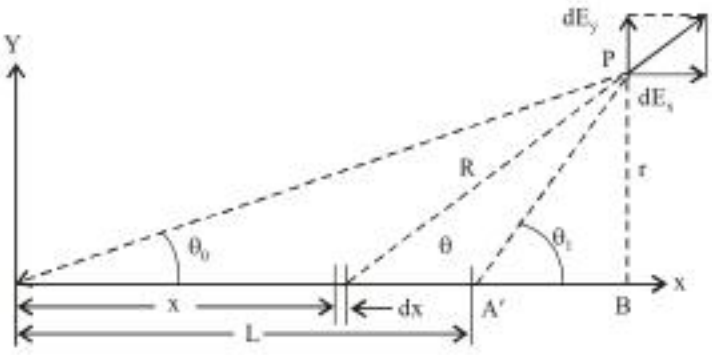


Fig. 4.14 A straight, uniformly charged wire of Length L.

For the point P shown in the Fig. 4.14, there is no symmetry, consider an element of wire of length *dx*. If λ be the density of charge per unit length, then charge on *dx* is λ *dx*. Suppose point P is at a distance R from this element, and r is the perpendicular distance of the point P from the line joining the wire, then the electric field *d*E due to this element of wire *dx* is given by,

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{R^2}$$

Due to lackiness of symmetry, here separate components are considered. Thus, *dE* can be resolved into two components—horizontal along x-axis and vertical along *y*-axis, *i.e.*,

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx\cos\theta}{R^2}, \text{ in the direction of wire}$$

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx\sin\theta}{R^2}, \text{ perpendicular to the wire}$$

But from the triangle APB,

$$AB = AA' + A'B,$$

and
$$AA' = L - x,\ A'B = r\cot\theta_1$$

Hence, $AB = L - x - r\cot\theta_1 = r\cot\theta$

On differentiation, the above expression for AB becomes

$$dx = r\,\text{cosec}^2\,\theta\,d\theta$$

Therefore, dE_x, after substituting dx from the above expression, becomes,

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda r\cos\theta\,\text{cosec}^2\theta\,d\theta}{R^2}$$

From Fig. 4.14, $r/R = \sin\theta$.

Hence

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\cos\theta\,d\theta$$

And similarly,

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\sin\theta\,d\theta$$

The total field components are obtained by integration of these two expressions. Thus,

$$E_x = \int_{\theta_0}^{\theta_1} dE_x = \int_{\theta_0}^{\theta_1}\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\cos\theta\,d\theta$$

or
$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}(\sin\theta_1 - \sin\theta_0)$$

And
$$E_y = \int_{\theta_0}^{\theta_1} dE_y = \int_{\theta_0}^{\theta_1}\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\sin\theta\,d\theta$$

or
$$E_y = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}(\cos\theta_1 - \cos\theta_0)$$

The total field is given by vector combination of the E_x and E_y and is,

$$E = \sqrt{E_x^2 + E_y^2}$$

Putting E_x and E_y from the above, we get,

$$E = \frac{1}{4\pi\varepsilon_0 r}\frac{\lambda}{R}[(\sin\theta_1 - \sin\theta_0)^2 + (\cos\theta_1 - \cos\theta_0)^2]^{1/2}$$

The direction of E from Fig. 4.14,

$$\theta = \tan^{-1}\frac{\vec{E}_y}{\vec{E}_x},$$

$$\theta = \tan^{-1}\frac{\cos\theta_1 - \cos\theta_0}{\sin\theta_1 - \sin\theta_0}$$

On putting the values, we get

If the point P is directly above the end of the wire, then $\theta_1 \to \pi/2$ and,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}[(1 - \sin\theta_0)^2 + \cos^2\theta_0]^{1/2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\sqrt{2}\,(1 - \sin\theta_0)^{1/2}$$

$$= \frac{1}{2\sqrt{2}\pi\varepsilon_0}\frac{\lambda}{r}(1-\sin\theta_0)^{1/2}$$

Since $\theta_1 = \pi/2$

$$\theta = \tan^{-1}\frac{\cos\theta_0}{1-\sin\theta_0}$$

In the limiting case when the charged wire is very long, $\theta_0 \to 0$, then,

$$E = \frac{1}{2\sqrt{2}2\pi\varepsilon_0}\frac{\lambda}{r}$$

and $\theta = 45°$

Ex. 4.6. *A negative point charge 10^{-8} coul is situated in air at the origin of a rectangular coordinates system. What is the electric field intensity at a point on the positive x direction 3 metres from the origin ?*

Sol. By Eq. (410), the field intensity is given by,

$$\vec{E} = \frac{-\hat{i}9\times10^9 \text{ N.m}^2/\text{coul}^2 \times 10^{-8} \text{ coul.}}{3^2 \text{ met}^2}$$

$$= -\hat{i}10 \text{ N/coul.}$$

that is, the field E is directed in the negative *x* axis and its magnitude is 10 N per coul (or volt per metre).

Ex. 47. *A positive point charge of 10^{-9} coul is situated in air at the origin (x = 0, y= 0), and a negative point charge of -2×10^{-9} coul is situated on the y-axis 1 metre from the origin (x= 0, y= l)*

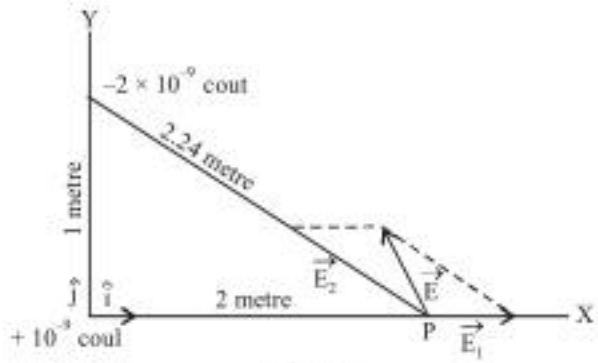


Fig. 4.15

as shown in Fig. 4.15. Find the total electric field intensity at P on the x-axis 2 metres from the origin (x=2, y = 0).

Sol. The electric field E_x due to the charge at (0, 0) is from Eqn. (4.10).

$$E_1 = \frac{-\hat{i}9\times10^9 \text{ Nm}^2/\text{coul}^2 \times 10^{-9} \text{ coul.}}{2^2 \text{ met}^2}$$

$$= (2.25)\,\hat{i} \text{ newton/coul.} \qquad ...(1)$$

Similarly, the magnitude of $\vec{E}_2$ due to charge at (0, 1) is

$$E_2 = \frac{9\times10^9 \times(-2\times10^{-9})}{(2.24)^2} \text{ newton/coul}$$

$$= -3.58 \text{ newton/coul}$$

and vector value of $\vec{E}_2$ is given by,

$$\vec{E}_2 = \hat{i}3.58\cos\alpha + \hat{j}\,3.58\sin\alpha$$

$$= \hat{i}3.58\frac{2}{2.24} + \hat{j}\,3.28\frac{1}{2.24} \qquad ...(2)$$

$$= -\hat{i}3.2 + \hat{j}\,1.6 \text{ newton/coul}$$

The total field $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$= \hat{i}(2.25 - 3.2)\hat{j}1.6$$

$$= -\hat{i}0.95 + \hat{j}1.6$$

Now it is easy to calculate the magnitude of $\vec{E}$ and its direction from axes.

4 17. Flux of the Electric field $\vec{E}$

As we discussed in chapter III there are two main properties of a vector field, namely *flux* and *circulation* (to be more specific *flow* and *rotation* respectively). We shall consider flux in this section and circulation latter on.

The *relation between* $\vec{E}$ *and its source can be expressed* in a simple way in terms of electric flux which is of very great use. We have discussed the flux in art (3.8). In the case of fluid, *we call the net amount of fluid going out through the surface the "flux of velocity" through the surface.*

The flow through an element of surface is just equal to the component of the velocity perpendicular to the surface, times the area of surface. *For an arbitrary closed surface, the net outward flow or flux is the average outdrawn normal component of the vector, times the area of surface.*

$$\text{Flux } \phi = (\text{average normal component}).\ (\text{surface area}) \qquad ...(4.14)$$

In the case of electric field, we can mathematically *define something analogous to an outflow and we again call it the flux, but of source it is not the flow of any substance, because the electric field is not the velocity of anything.* However, it turns out that the mathematical quantity, which is the average normal component of the field still has a useful *significance.* Hence electric flux is also defined by Eq. (4" 14).

4 18. Electric Flux Over a Closed Surface

Consider an infinitesimal surface element *dS* of surface S as shown in Fig. 4.16. Let us calculate the electric flux $d\phi$E due to this element According to equation (4.14), we have,

$$d\phi_E = E\cos\alpha\, dS$$

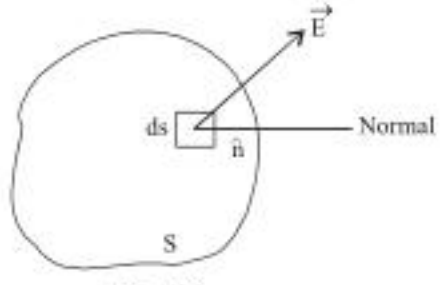


Fig. 4.16

Where α is angle between $\vec{E}$ and unit vector *n* drawn along a normal to the surface. In vector,

$$d\phi_E = \vec{E}.\hat{n}dS = \vec{E}.d\vec{S} \qquad ...(4.15)$$

Since $\hat{n}dS = d\vec{S}$, a vector area

The flux due to entire closed surface is obtained by integrating the above equation around the surface, *i.e.*

$$\phi_E = \oint_S d\phi_E = \oint_S \vec{E}.d\vec{S} \qquad ...(4.16)$$

Unit of flux. From Eqn. (4-16), the unit of flux is newton-metre^2/coul because $\vec{E}$ is expressed as newton/coul. and area in metre^2.

Ex. 48. ***Fig.*** *4.17 shows a hypothetical cylinder of radius R immerged in a uniform electric field* $\vec{E}$. *The cylinder axis is parallel to the field. Calculate* ϕ_E *for this closed surface.*

Sol. As is clear from the figure that there are three surfaces (i) the left cap *a. (ii)* the cylindrical surface, b and (*iii*) the right cap r, therefore,

$$\phi_E = \oint \vec{E}.d\vec{S}$$

$$= \oint_a \vec{E}.d\vec{S} + \oint_b \vec{E}.d\vec{S} + \oint_c \vec{E}.d\vec{S}$$

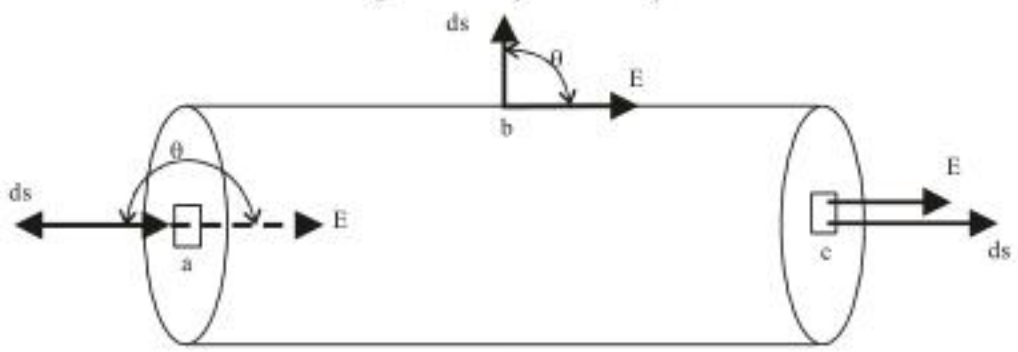


Fig. 4.17

For the left cap, angle θ between $\vec{E}$ and $d\vec{S}$, is 180° for all points, *i.e.*, $\vec{E}$ and $d\vec{S}$ are oppositely directed. Thus,

$$\oint_a \vec{E}.d\vec{S} = \oint_a E\cos 180°\, dS$$

$$= -E\oint_a dS = -ES$$

Where $S = \pi R^2$ is the cap area. Similarly for right cap,

$$\oint_c \vec{E}.d\vec{S} = ES.$$

Since the angle θ = 0. Finally for cylindrical wall θ = 90°. Hence we get,

$$\oint_b \vec{E}.d\vec{S} = \oint_c E\cos 90\, dS = 0$$

Thus the total flux,

$$\phi_E = -ES + ES + 0 = 0$$

Ex. 49. *Calculate the flux of a single positive charge q, as shown in Fig. 4.18.*

Sol. Let us take a simple case. The surface is a sphere of radius *r* centred on the positive charge *q*. The magnitude of $\vec{E}$ is same at every point on the surface and is given by,

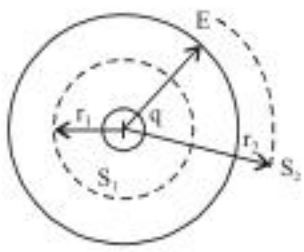


Fig. 4.18

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

and its direction is the same as that of out-drawn normal at that point. Now using Eqn.. (4.16) for ϕ_E, we get,

$$\phi_E = \int_S \vec{E}\, d\vec{S} = \oint_S \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\, dS \text{ , since angle } 0 = 0$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\oint_S dS = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}.4\pi r^2 = \frac{q}{\varepsilon_0}$$

We see that ϕ_E is independent of the size of the sphere, so long as the charge q is inside. Thus ϕ_E is the same for Si, S_2 etc. surfaces. As a *corollary we can say, as is evident from Ex. (4.8), that the total flux through a closed surface is zero, if the charge lies outside the surface.*

4 19. Electric Flux Density

A point charge is surrounded by an electric field. The electric field intensity $\vec{E}$ at a distance r from charge q is given by Eq. (4.10).

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

where $\hat{r}$ = unit vector in radial direction. The above relation, can also be expressed as,

$$\frac{q}{4\pi\varepsilon_0}\hat{r} = \varepsilon_0\vec{E} \qquad ...(4.17)$$

The dimensions of Eq. (4.17) are.

From L.H.S., coul/metre^2 is the unit, hence dimension

$$\frac{\text{charge}}{\text{area}} = \text{surface density of charge}$$

From the R.H.S. unit is

$$\frac{\text{newton}}{\text{coulomb}}\cdot\frac{\text{coul}^2}{\text{newton/metre}^2} = \frac{\text{coul}}{\text{metre}^2} \text{ that the same as L.H.S.}$$

Hence the product $\varepsilon_0\vec{E}$ *has the dimensions of surface charge density* ***(Q/L^2). This*** *product* $\varepsilon_0\vec{E}$ *denoted by symbol* $\vec{D}$ *is called the electric* ***flux density.*** *Thus,*

$$\vec{D} = \varepsilon_0\vec{E}\left(\frac{\text{coul}}{\text{m}^2}\right) \qquad ...(4.18)$$

According to Eqn. (4.18), the flux density and field intensity are vectors in the same direction. This is true for all isotropic mediums, that is media whose properties do not depend on direction.

Now the integral of the normal component of a vector over a surface is defined as the flux of the vector over that surface. Let us apply this definition to $\vec{D}$ by integrating it over a surface S at a. distance r from the point charge q as shown in Fig. 4.19.

Fgi. 4.19

$\vec{D}$ normal to the surface S, we have,

$$\text{Flux (of } \vec{D} \text{ over S)} = SD \qquad ...(4.19)$$

Since D has dimensions of charge/area and S has dimensions of area, the dimension of Eqn. (4.19) is charge.

The flux of $\vec{D}$ over a surface is called the electric flux; this is another way of defining electric flux. Thus,

$$\phi_E = SD$$

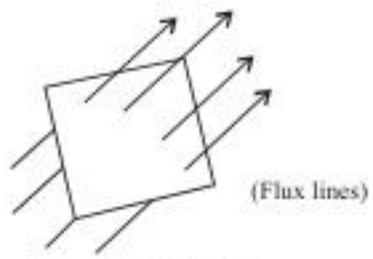
or
$$\vec{D} = \frac{\phi_E}{S} = \text{flux density}$$

Hence by comparing Eqn. (4.17) and (4.18), we can write

$$\vec{D} = \varepsilon_0 \vec{E} = \frac{q}{4\pi\varepsilon_0}\hat{r} \qquad ...(4.20)$$

4 20. Flux Lines

The electric field lines originate on positive charge and end on negative charge. The number of lines passing through a normal surface (Fig. 4.20) then indicates the flux density $\vec{D}$ sand *the individual lines are flux lines, each associated with a certain fraction of charge q.* The number of flux lines is arbitrary but it is always proportional to the charge *q*.


(Flux lines)

Fgi. 4.20

Ex. 4.10. *A point charge has a value $q=10^{-10}$ coul, taking 10^{20} flux lines per coul of charge; find the number of lines per square meter at a distance of 10 meters. Also calculate $\vec{D}$ at this point.*

Sol. Total number of lines of lines = $10^{-10} \times 10^{20} = 10^{10}$

Lines per square meters at 10 metres = $\frac{10^{10}}{4\pi 10^2} = 7.95 \times 10^6$

From Eqs. (4.20),

$$D = \frac{q}{4\pi r^2} = \frac{10^{-10}}{4\pi 10^2} = 7.95 \times 10^{-11} \text{ coul/metre}^2$$

4.21. Gauss's Law

In fact Gauss law is converse of Coulomb's law. By coulomb's law we can calculate $\vec{E}$ for a given charge, but the Gauss's law enables us to determine charge if $\vec{E}$ is known. The Gauss's law gives a connection between E and its source, not in terms of inverse square *i.e.* dependence of $\vec{E}$ on $1/r^2$ law, but something in a different way.

Gauss's law states *that the flux of the electric field $\vec{E}$ through any closed surface, that is, the integral $\oint \vec{E}\, d\vec{S}$ over the surface, is equal to $1/\epsilon_0$ times the total charge enclosed by the surface.* And

if surface does not enclose the charge, the flux of $\vec{E}$, *i.e.*, $\oint E.\,dS$ equals zero, Thus, we write,

$$\left.\begin{aligned} &\oint \vec{E}.d\vec{S} = \frac{q}{\varepsilon_0},\ q \text{ inside the surface or } \oint \vec{E}\,d\vec{S} = q \\ &\oint \vec{E}.d\vec{S} = 0,\ q \text{ outside the surface or } \varepsilon_0 \oint \vec{E}\,d\vec{S} = 0 \end{aligned}\right\} \quad ...(4.21)$$

where q is the net charge taking its algebraic sign into account. The *exact location* of inside charge does not affect the value, as we have shown in example 4.9 that electric flux is independent of the size of the surface.

To *understand Gauss's law, let* us consider a gun that shoots the bullets. Gauss's law say ? That the net flow of bullets through a surface is zero if the surface does not enclose the gun. If the Run is enclosed in a surface, whatever size and shape it has, the number of bullets passing through is the same, and it is given by the rate at which bullets are generated at the gun. But here we do not get any such thing as rate but a mathematical relation given in Eq. (4.21), which is one of the basic equation of e.m. theory.

Flux is an additive quantity. If we have number of charges $q_1, q_2, \ldots\ldots q_N$ with corresponding fields, $E_1, E_2, \ldots\ldots E_N$ enclosed in the surface S, then by the *principle of superposition,* we get,

$$\begin{aligned} \varepsilon_0 \oint \vec{E}.d\vec{S} &= \varepsilon_0 \oint (\vec{E}_1 + \vec{E}_2 + \ldots\ldots \vec{E}_N).d\vec{S} \\ &= (q_1 + q_2 + \ldots\ldots + q_N) \end{aligned}$$

or $$\varepsilon_0 \oint \vec{E}.d\vec{S} = Q \quad ...(4.22)$$

where Q is the total or net charge inside the surface.

Proof : Let us now consider the closed surface of shape shown in Fig. (4.21.).

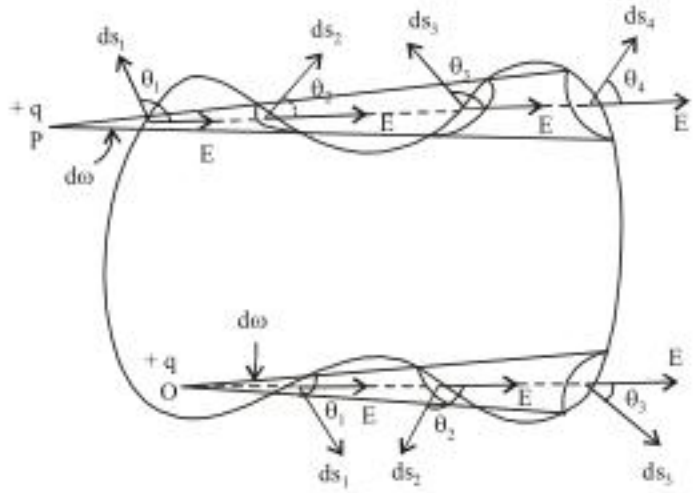


Fig. 4.21

If the point charge q is at 0, *i e.* inside the surface, then each infinitesimally small conical solid angle dw will cut the surface an odd number of times. Here, in this case with three intercepts dS_1, dS_2, dS_3, each surface is contributing to the flux. As it is evident from the direction of normal the contribution of dS_2 is opposite to that of dS_1 and dS_2 and dS_3. Let us evaluate,

$$\phi_E \oint \vec{E}_1\,d\vec{S}_1 = \oint \vec{E}_1\,d\vec{S}_1 + \oint \vec{E}_2\,d\vec{S}_2 + \oint \vec{E}_3\,d\vec{S}_3$$

Where $\vec{E}_1, \vec{E}_2$ and $\vec{E}_3$ are the fields at the points on the surfaces dS_1, dS_2, and dS_3 at a distance r_1, r_2 and r_3 respectively.

Thus we get,

$$\phi_E = \oint \vec{E}\, d\vec{S} = \oint \frac{q dS_1 \cos\theta_1}{4\pi\varepsilon_0 r_1^2} - \oint \frac{q dS_2 \cos\theta_2}{4\pi\varepsilon_0 r_2^2} + \oint \frac{q dS_2 \cos\theta_3}{4\pi\varepsilon_0 r_3^2}$$

or $$\oint \vec{E}\, d\vec{S} = \frac{q}{4\pi\varepsilon_0} \oint_s \left[\frac{dS_1 \cos\theta_1}{r_1^2} + \frac{dS_2 \cos\theta_2}{r_2^2} + \frac{dS_3 \cos\theta_3}{r_3^2} \right]$$

The quantity $\frac{dS\cos\theta}{r^2}$ is solid angle* dw, here with respective values we get $\oint \vec{E}\, d\vec{S}$ $= \frac{q}{4\pi\varepsilon_0} \oint_s [dw - dw + dw]$ because solid angle *is same for all surfaces.*

$$\oint \vec{E}\, d\vec{S} = \frac{q}{4\pi\varepsilon_0} \oint_s dw = \frac{q}{4\pi\varepsilon_0} \frac{4\pi r^2}{r^2} = \frac{q}{\varepsilon_0}$$

or $$\oint \vec{E}\, d\vec{S} = \frac{q}{\varepsilon_0}$$

or $$\varepsilon_0 \oint \vec{E}\, d\vec{S} = q$$

or $$\oint \vec{D}\, d\vec{S} = q \quad \text{since } \vec{D} = \varepsilon_0 \vec{E} \qquad \text{...(4.23)}$$

Proceeding in the same manner we get the value of surface integral for point P which is outside the surface. Hence,

$$\oint \vec{E}\, d\vec{S} = \frac{q}{4\pi\varepsilon_0} \oint_S (-dw + dw - dw + dw)$$

$= 0$, hence proved.

i.e. the inward flux equals the outward flue; (net flux zero); in other words; the number of flux tubes entering equals the number leaving. As was expected because surface does not enclose the charge.* *Looking over our method of proof, we see that it depends on the inverse square nature of the interaction and on principle of superposition.* Gauss's law is the basic theorem of electrostatics. It is a necessary consequence of the inverse square law (coulomb's law). Thus, if $\vec{D}$ for a point charge did not vary as $1/r^2$, the total flux over a surface enclosing it would not equal the charge.

*** Solid angle** : In three dimensions a given area subtends a solid angle at the centre of the sphere. The value is obtained by joining all the points on the periphery of the area dS to the centre to the sphere. The value of solid angle does not depend on the shape of tho area but depends only on the magnitude of the area. Solid angle is measured as normal area divided by square of the radius vector, just as the angle in one plane is measured as arc divided by radius vector. Hence,

$$dw = \frac{dS\cos\theta}{r^2}$$

The Gauss's law for electric field can be stated by equation 4.23. The surface integral of the normal component of the electric flux density $\vec{D}$ over any closed surface equals the charge enclosed. This statement of Gauss's law applies specifically to the rationalized Mks system. In general, gauss's law states that the surface integral of the normal component of the electric flux density over a closed surface is proportional to the charge enclosed (or equal to the charge times a constant, this constant being unity in the rationalized MKS system).

4.22 : Gauss's Law in Differential Form

When a charge is distributed over a volume such that ρ is the density of charge. Then the charge enclosed by surface enclosing the volume is given *by,*

$$q = \int_v \rho dv$$

substituting this in Eq. (4.21.),

$$\varepsilon_0 \oint_S \vec{E}\, d\vec{S} = \int_v \rho dv$$

Now, we have divergence theorem discussed in art (3.13) for converting the surface integral to volume integral *i.e.,*

$$\oint_S \vec{E}\, d\vec{S} = \int_v \text{div}\, \vec{E}\, dv$$

Hence $$\varepsilon_0 \int_v \text{div}\, \vec{E}\, dv = \int_v \rho dv$$

Since it is true for any orbitrary volume, integrands must be equal, therefore.

$$\left.\begin{aligned} \text{div}\, \vec{E} &= \frac{\rho}{\varepsilon_0} \\ \vec{\Delta}.\vec{E} &= \frac{\rho}{\varepsilon_0} \end{aligned}\right\} \quad \text{or} \quad \text{...(4.24)}$$

As in vacuum $$\vec{D} = \varepsilon_0 \vec{E}$$

$$\left.\begin{aligned} \text{div}\, \vec{D} &= \rho \\ \vec{\Delta}.\vec{D} &= \rho \end{aligned}\right\} \quad \text{...(4.25)}$$

The equation (4.24.) is the differential form of Gauss's law. According to which *divergence of electric field is proportional to the charge at that point.* This form of law Eq. (4.24.) is *fundamentally important in electromagnetic theory. Physically* we can interpret it that *flux of* $\vec{E}$ *directly depends on the magnitude of charge* or charge density ρ .

---

* If a volume contains no charge, the electric flux over the surface of the volume is always zero, even though the volume may be in an electric field.

4.23. Gauss's Law and Coulomb's Law

Let us apply Gauss's law to an isolated point charge q as shown in Fig (4.22). Construct a Gaussian surface taking r as radius, $\vec{E}$ is radial, on surface $\vec{E}.d\vec{S} = EdS$

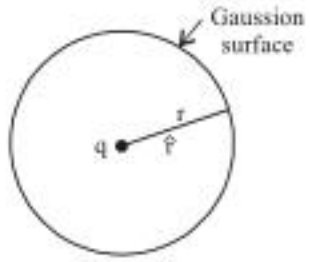


Fig. 4.22

$$\oint \vec{E}.d\vec{S} = \varepsilon_0 \oint E dS$$

$$= q \text{ from Gauss's law}$$

As E is constant for all points on the surface,

$$\varepsilon_0 E \oint ds = q$$

$$\varepsilon_0 E 4\pi r^2 = q$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

or

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Let us put a second point charge q_0 at the point where E is calculated. The force that acts on q_0 is,

$$\vec{E} = \vec{E}\, q_0 = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r}$$

which is precisely Coulomb's law. Gauss's law is one of the fundamental equations of the electromagnetic theory. So *for electromagnetic theory the basic law is Gauss's law which is equivalent to Coulomb's law of force. We have deduced Coulomb's law from Gauss's law and con-sideration of symmetry*. The simple form of Gauss's law *i.e.*, $\varepsilon_0 \oint \vec{E}\, d\vec{S} = q$ is due to the fact that we have introduced proportionality constant as $\frac{1}{4\pi\varepsilon_0}$ in Coulomb's law.

Ex. 4-11. *Five thousand lines of force enter a certain volume of space and three thousand lines leave it. Find the total charge contained in it.*

Sol. Let the volume of given space be v.

The total net outward flux from the volume v = 3000—5000= – 2000 lines.

Therefore, the total net outward flux from the given volume of space per unit volume = – 2000/v.

This is the divergence of Electric Field $\vec{E}$. Hence

$$\text{div. } \vec{E} = \frac{-2000}{v} \qquad (i)$$

Applying Gauss's law given in Eq. (4.24),

$$\text{div. } \vec{E} = \frac{\rho}{\varepsilon_0}$$

where ρ is volume charge density, let Q is total charge, then, $\rho = Q/v$

therefore,

$$\text{div. } \vec{E} = \frac{Q}{\varepsilon_0 v} \qquad ...(ii)$$

Equating (*i*) and (*ii*),

$$\frac{Q}{\varepsilon_0 v} = -\frac{2000}{v}$$

or $$Q = -2000\,\varepsilon_0$$
$$= -2000 \times 8.85 \times 10^{-12} \text{ couls}$$
$$= -1.77 \times 10^{-8} \text{ couls.}$$

4.24. Important Examples on Gauss's Law

When do we apply Gauss's law ? Gauss's law is *always true,* but it is *not always useful.* The usefulness of Gauss's law depends on our ability to evaluate $\oint \vec{E}\, d\vec{S}$. For this, we construct a closed surface called *Gaussian surface* near the charge-distribution, over which the magnitude of field and angle between $\vec{E}$ and normal to this surface have constant value, since then E cos θ can be taken outside the integral sign and then $\oint ds$ is easily evaluated. The magnitude of E is constant for uniform charge-distributions and θ is constant for symmetrical charge-distributions and symmetrical surfaces. *Thus, the necessory condition for applying Gauss's law is uniform, symmetrical charge-distribution and symmetrical Gaussian surface.*

If charge-distribution had not been uniform or, at any rate, not symmetrical, or if we had chosen some non-symmetrical shape for our Gaussian surface, it would still be true that $-\varepsilon_0 \oint \vec{E}\, d\vec{s} = q$ but it would not have been certain that $\vec{E}$ was radially directed or constant in magnitude and without this we cannot pull E cos 0 out of the integral sign. *Thus, symmetry is crucial for the application of Gauss's law in its integral form.* As far as we know, there are only three kinds of symmetry, which are sufficient :

1. Spherical symmetry : Make your Gaussian surface a concentric sphere.
2. Cylindrical symmetry : Make your Gaussian surface a coaxial cylinder.
3. Plane symmetry : Use a Gaussian "Pill box' which straddles the surface.

When Gauss's law applies, it's by far the fastest method; when it doesn't, we will have to fall back on the Coulomb's law.

1. Electric Field due to a Uniformly charged Sphere :

Suppose the charge q is uniformly distributed over a sphere of radius R, as shown in Fig. (4.23).

Case (*i*) At a point out- side the charged sphere. Let us find the electric field $\vec{E}$ at the point P outside the surface of the sphere. We construct a Gaussian surface, shown dotted at a concentric sphere of radius OP. Due to symmetry, the field at P and on every where on the Gaussian surface will be perpendicular to the surface, *i.e.* E is directed along the outdrawn normal. The flux through this surface is given by,

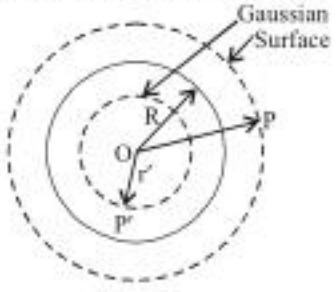


Fig. 4.23

$$\varepsilon_0 \oint \vec{E}\, d\vec{S} = \oint E dS \text{ , (angle between E and } dS \text{ is zero)}$$
$$E \oint dS = \frac{q}{\varepsilon_0} \text{ , (by Gauss's law)}$$

or $$E.\, 4\pi r^2 = q/\varepsilon_0$$

Or $$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \text{ Newton/Coul} \qquad ...(4.26)$$

Or $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \text{ newton/Coul.}$$

where $\hat{r}$ is unit vector along radius.

which is same as the electric field at a distance r from the point charge q. Hence the electric field at a external point due to a uniformly charged sphere is the same as if the charge on the sphere is concentrated at centre.

Case (*ii*) : At a point on the surface

Here $r = R$, hence,

$$E = \frac{q}{4\pi\varepsilon_0 R^2} \text{ Nt/Coul.} \qquad ...(4.27)$$

Case (*iii*) At a point inside the charged sphere.

Let us now find the electric field $\vec{E}$ at P′ which is inside the charged sphere. Again construct a Gaussian surface shown dotted as a concentric sphere of radius r', ($r' < R$). The electric field will be constant on this sphere and is directed along normal to the surface of this sphere. The outward flux through the surface of the sphere is given by,

$$\oint \vec{E}.d\vec{S} = E.\,4\pi r'^2$$

The total charge enclosed by the Gaussian surface

= Volume enclosed by it × charge-density

$$= \frac{4}{3}\pi r'^3.\rho \text{ , where } \rho \text{ is charge-density.}$$

Applying the Gauss's law

$$E.\,4\pi r'^2 = \frac{4}{3}\pi r'^3.\rho$$

or $$E = \frac{\rho}{3\varepsilon_0} r' \text{ Newton/Coul } (r' < R) \qquad ...(4.28)$$

At the surface of sphere, $r' = R$

$$E = \frac{\rho}{3\varepsilon_0} R$$

But $$\rho = \frac{\text{Charge}}{\frac{4}{3}\pi R^3} = \frac{q}{\frac{4}{3}\pi R^3}$$

hence, $$E = \frac{q}{3\varepsilon_0 \frac{4}{3}\pi R^3} R$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \text{ Which is same as given in Eq. (4.27)}$$

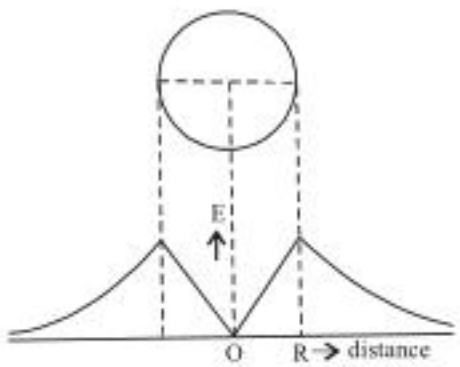


Fig. 4.24

Fig. 4.24 shows a charged sphere with graphs showing variation of electric field E as a function of radial distance *r*.

2. Electric field due to a charged spherical shell

A spherical shell is a hollow sphere of small thickness.

Case (*i*) At an External point

Let *q* be the charge, uniformly distributed over the shell. This case is same as that due to a uniformly charged sphere. Hence the field is given by Eqn. (4.26), *i.e.*

$$\oint \vec{E}\, d\vec{S} = \int E.dS = E\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{4\pi\varepsilon_0}\frac{q}{r^2}, \text{ outside } r \geq R, \text{ where R is the radius of shell.}$$

Case *(ii)* At a point on the surface of the Charged Spherical Shell :

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \text{ (Since } r = R)$$

Case (in) ***At** a point inside* the charged spherical shell :

Since the Gaussian surfaces lying inside the shell encloses no charge. The field according to Gauss's at this point will be zero, E = 0, inside $r < R$

Fig. (4.25) shows an uniformly charged spherical shell with graphs showing variation of electric field E as function of radial distance *r*.

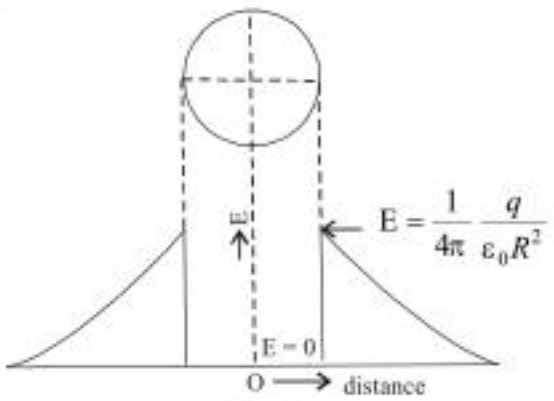


Fig. 4.25

3. Field Due to two Concentric Spherical Conductors

Let two imaginary concentric spherical conductors or shells have radii r_1 and r_2 with a charge q_1 uniformly distributed over the shell of radius r_2 and a charge q_2 uniformly distributed over the shell of radius r_2, as shown in Fig. (4.26-*a*). It is assumed that medium is air everywhere.

Applying Gauss's law in a similar manner to that used for the single spherical conductor.

(*i*) Field at a point inside first conductor of radius x_1

E is zero inside the conductor of radius r_1 because the Gaussian surface (1) does not enclose charge due to both conductors. E = 0, inside both the conductors ($r \le r_1$)

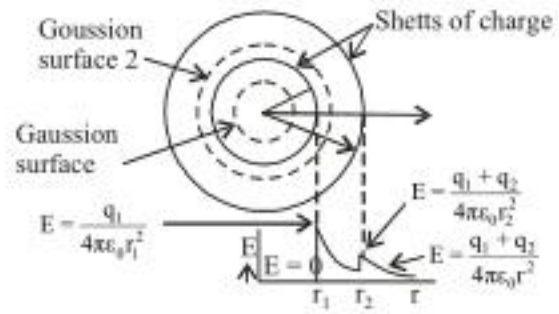


Fig. 26 (a) and (b)

(*ii*) Field at a point in between the two conductors

For this case, Gaussian surface is (2) and here the field is due to the charge on the first conductor having charge q_1 hence from Gauss's law,

$$\oint \vec{E}.d\vec{S} = E4\pi r^2 \frac{q_1}{\varepsilon_0}$$

or $$E = \frac{q_1}{4\pi\varepsilon_0 r^2}, \text{ between conductors } (r_1 \le r \le r_2)$$

(*iii*) Field at a point outside the second conductor

For such point contribution to the field is due to both charges q_1 and q_2 hence the field is sum of the two fields.

$$E = E_1 + E_2 = \frac{q_1}{4\pi\varepsilon_0 r^2} + \frac{q_2}{4\pi\varepsilon_0 r^2}$$

or $$E = \frac{1}{4\pi\varepsilon_0} \frac{q_1 + q_2}{r^2} \text{ outside both conductors } r \ge r_2$$

So we can summarise,

$$E = 0, \text{ inside both conductors } (r \le r_1)$$

$$E = \frac{q_1}{4\pi\varepsilon_0 r^2}, \text{ between conductors } (r_1 \le r \le r^2) \qquad ...(4.28a)$$

$$E = \frac{q_1 + q_2}{4\pi\varepsilon_0 r^2}, \text{ outside both conductors } (r \ge r_2)$$

The variation of E as a function of r is shown in fig. 4.26 (6). The sudden decrease at r_2 is due to the fact that contribution by q_2 is suddenly stopped, similarly for the point r_1.

4. Field of a Line Charge (or Uniformly Charged Wire)

Fig. (4.27) shows a section of an infinite long straight uniformly charged wire (or rod). If the thickness is neglected, then it can be characterised by the linear charge-density λ, *i.e.,* the charge per unit length measured in coul/meter, being constant for all points on the line.

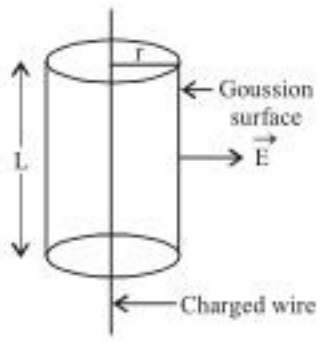


Fig. 4.27

From symmetry, the electric field will be directed radially outward from the wire. This can be understood by dividing the wire into two parts. If the electric field at any point due to these two parts make angles other than 90° *(i.e.,* not radial) with the wire then the components of the fields along the wire, i.e., the axial components due to two parts will be equal and directed opposite to each other. Hence they will cancell. The mean value of the two components may be taken to be equal due to infinite length of the wire. However, the components of the fields perpendicular to the wire (*i.e.*, radial components) will be in same directions, hence they will add. Thus, there will be only radial field.

Let us calculate E at a distance r from the line. In order to apply Gauss's law, we construct a Gaussian surface in the form of a circular cylinder of radius r and length l with the wire as its axis as shown in fig. (4.27.). E is constant over the cylindrical surface because it is at equi distance from the line (or axis). The flux of E through cylindrical surface is.

$$\oint \vec{E}.d\vec{S} = \oint E dS = E\oint dS = E2\pi rl$$

There is no flux through the two end faces (circular caps) because E is tangential to the faces and has no component perpendicular to them, *i.e.* $\vec{E}.d\vec{S} = 0$.

Hence, total outward flux = $E\ 2\pi rl$.

Charge enclosed by Gaussian surface = λl

Therefore by Gauss's law,

$$\oint \vec{E}.d\vec{S} = E\ 2\pi rl = \frac{\lambda l}{\varepsilon_0}$$

or
$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \qquad ...(4.29)$$

which is same as derived in example 4.3.

Clearly E is inversely proportional to the distance, r from the line. The direction of E is radially outward for a line of positive charge.

5. Field of a Uniformly Charged Hollow Cylinder

The field in this case can be calculated in the same way as discussed in the preceding example. The field at an outside point,

$$E.2\pi rl = \frac{\lambda l}{\varepsilon_0}$$

or
$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

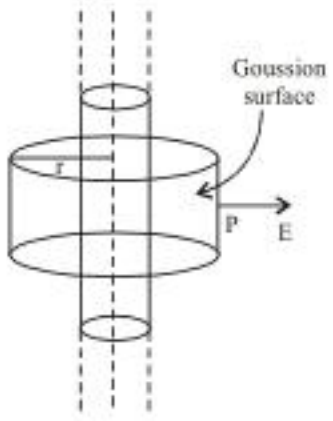


Fig. 4.28

where λ is the charge per unit length of the cylinder and r is the distance of the point from the axis of cylinder. But in this case charge is uniformly distributed over cylindrical surface. Let σ be the surface

density of charge on the cylinder. Then, if charge on unit length of cylinder is λ, we get,

$$\lambda = \sigma .2\pi r$$

or $$\lambda = 2\pi r\, \sigma$$

therefore,

$$E = \frac{2\pi r\sigma}{2\pi\varepsilon_0 r} = \frac{\sigma}{\varepsilon_0} \qquad ...(4.30)$$

The field at a point *inside the hollow cylinder* will be zero because then Gaussian surface does not enclose charge.

6. Field of Cylinder of Uniform Charge

If the charge is distributed throughout the volume, then we have field of an uniform cylinder of charge.

Field at Outside Point

For an outside point it behaves as line charge, hence we can treat it as above [example 4] and field is given by,

$$E = \frac{\lambda}{2\pi r\varepsilon_0}$$

where r is the radius of cylinder.

Field at Inside Point

Construct a Gaussian surface in the form of a cylinder of length *l* coaxial with the cylinder of charge and having radius equal to the distance of the point at which the field is to be determined as shown in Fig (4.29). The flux through the cylindrical Gaussian surface is,

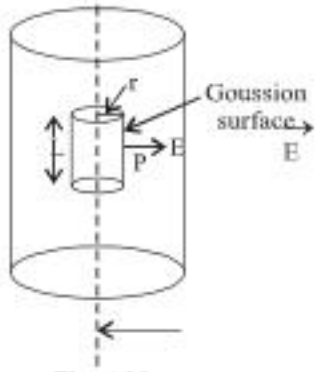


Fig. 4.29

$$\oint \vec{E}.d\vec{S} = E.2\pi rl$$

The total charge enclosed by the Gaussian surface is given by

Charge = volume × volume density.

If *f* is the volume density of charge, i.e., charge per unit volume.

Hence, charge enclosed in Gaussian surface = $\pi r^2 l\rho$

According to Gauss's law,

$$\oint \vec{E}.d\vec{S} = E.\ 2\pi\, rl = \frac{\pi r^2 l\rho}{\varepsilon_0}$$

or $$E = \frac{r\rho}{2\varepsilon_0} \qquad ...(4.31)$$

Clearly, E inside the cylinder of charge is directly proportional to the distance of the point from the axis of the cylinder.

7. Field of Sheet of a Charge

Fig. (4.30) shows a portion of thin, non-conducting, infinite sheet of charge. Let σ be the surface charge density, *i.e.*, charge per unit area.

From symmetry, E is everywhere perpendicular to the plane

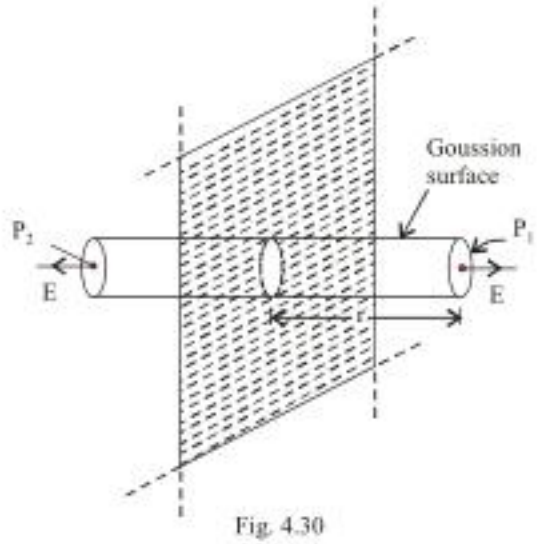


Fig. 4.30

and the field must have the same magnitude on both sides of the sheet. Let P_1 and P_2 be the two equidistant points on opposite sides of the sheet.

In order to use Gauss's law, we construct a cylindrical Gaussian surface of cross-sectional area A, that cuts the sheet, with P_1 and P_2 as shown in Fig. (4.30). E is normal to end faces and is away from the plane. $\vec{E}$ is parallel to the cylindrical surface, therefore the curved cylindrical surface does not contribute to the flux *i.e.* $\oint \vec{E}.d\vec{S} = 0$. Hence the total flux is equal to the sum of the contribution from the two end faces. Thus, we get,

$$\oint \vec{E}.d\vec{S} = EA + EA = 2EA$$

The total charge enclosed by the cylinder is σ A. Hence, according to Gauss's law,

$$\oint \vec{E}.d\vec{S} = 2EA = \frac{\sigma A}{\varepsilon_0}$$

or
$$E = \frac{\sigma}{2\varepsilon_0} \qquad ...(4.32)$$

Note that E is independent of the distance of the point from the sheet, so that E is same for all points on each side of the plane. Such sheet cannot exist physically but the Eqn. (4.32) yields correct results for real (not infinite) charge sheets, if we consider only points not near the edges, whose distance from the sheet is small compared to the dimensions of the sheet.

8. Field of Two Sheets of Charge

Let us consider two plane parallel infinite sheets of charge with equal and opposite charge densities + σ and – σ. As we have discussed above that the field due to a sheet is independent of the distance from the sheet of charge. Taking this fact into mind, we can calculate the field with the help of Gauss's law as given below.

Draw a Gaussian surface as shown by dotted line. Let the fields due to positive and negative charges on the sheets be E_2 and E_1 respectively, then E_2 and Ei are opposite in direction.

Field at an Outside Point

Let P_1 be situated outside the two sheets as shown in Fig. (4.31). Since E_1 and E_2 are independent of

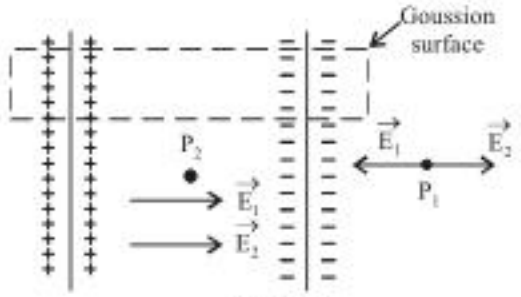


Fig. 4.31

the distance from the sheet. Their direction is opposite to each other and their magnitudes are equal hence the resultant field at P_1 is zero.

The same result is obtained if point P_1 is taken inside the Gaussame surface which includes both sheets. The net charge enclosed in the Gaussian surface is $\sigma A - \sigma A = 0$, where A is the area of cross-section of Gausian surface.

Field at point P_2 in between the Two Sheets

The field due to the two sheets will have the same magnitude *i.e*, $\sigma/2\varepsilon_0$ as given in Eq. (4.32) and in the same direction as shown in Fig. (4.31). Therefore the resultant field intensity at P_2 is equal to,

$$E = E_1 + E_2 = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \qquad ...(4.33)$$

However, E_1 and E_2 are calculated separately by constructing Gaussian surface. Equation 4.33 shows that field at point P_2 *is twice* ***of the field due to a single sheet of charge.***

9. Field of a Charged Conductor

Let us consider a conductor (or a metal) carrying on its surface a charge. In general *(r)* surface charge density will vary from point to point, assume it is constant for small surface *d*S, and have value σ.

The field inside the conductor must be zero. If for a while it is not, say have some value, then the free electrons will move and they arrange themselves to produce zero electric field inside the conductors. Infact charge resides on the outer surface of the conductor.

If the charge on the conductor is positive E is right angle to the surface pointing outward. If E were not normal to **the surface, it** would have a component lying in the surface, such components would set up surface currents and such currents are not assumed in electrostatic condition. E must be normal to the surface.

For calculating E draw a Gaussian cylindrical surface, as shown in Fig. (4·32). A is the area of cross-section of the surface.

The flux due to cylindrical surface is zero because angle-between E and normal to the surface is 90°.

Since E is zero everywhere inside the conductor, the only contribution to the flux is through the plane cap of area A lying outside the conductor. Therefore the total flux is given by,

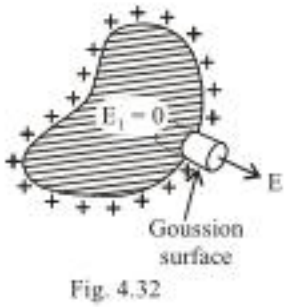


Fig. 4.32

$$\oint \vec{E}\, d\vec{S} = EA$$

Applying Gauss's law,

$$EA = \frac{\sigma}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

or $$E = \frac{\sigma}{\varepsilon_0} \qquad ...(4.34)$$

This relation is some time referred as Coulomb's law (or theorem).

Difference between Conductor and Non-conducting Sheet

1. Comparison of Eq. (4.34) with Eq. (4.32) shows that the electric field is twice at a near point for the same surface charge density in the case of a conductor than to a non-conducting sheet.
2. It is also to be noted that in the case of non-conducting sheet Fig. (4.30) half the flux emerges from one side of the surface and half from the other. Whereas in the case of conductor Fig. (4.32) all the flux emerges from the outside surface.
3. If we assume the same surface charge density and cross-sectional area A for the two Gaussian surface shown in Fig. (4.30) and (4.32), the enclosed charge will be the same *i.e.*, σ A. From Gauss's law the flux ϕ_E must be same in both cases. It means that ($E = \phi_E/A$) must be twice in the case of conductor than to non-conducting sheet.

4.25. Force on the Surface of a Charged Conductor

In the previous example we have calculated the magnitude of electric field E at a point which is very close but outside the $E = \sigma/\varepsilon_0$. However, the electric field is zero if the point lies inside the conductor.

At any point P just outside the surface of the conductor, the electric field E is in fact the sum of two fields.

1. E_1 due to the charge on a very small area just in the vicinity of point P and
2. E_2 due to the charge on the rest of the surface. Thus,

$$E = E_1 + E_2, \qquad ...(4.35)$$

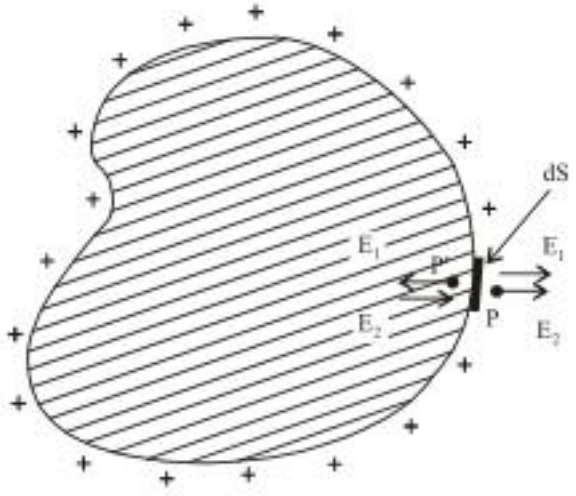


Fig. 4.33

Again, let the point P' be just inside the conductor as shown in Fig. (4.33). The electric field due to the charge on a very small area, *i.e.*, dS in the neighbourhood of the point P' is reversed *i.e.*, $-E_1$, while the electric field due to the rest charge remains unaltered in direction *i.e.* E_2. Therefore the resultant field at P' is $E_2 - E_1$ which must be zero since P' is inside the conductor, *i.e.*,

$$E_2 - E_1 = 0$$

or $$E_2 = E_1$$

Putting this in Eq. (4.35), we get,

$$E_1 = E_2 = \frac{E}{2} = \frac{\sigma}{2\varepsilon_0} \text{ N/Coul},$$

Since $E = \sigma/\varepsilon_0$, (See Eq. (4.34),

where σ is surface-density of charge. Now the charge on the surface immediately close to P is cr *dS*. Therefore the total force experienced by this charge due to the charge on the rest surface is given by. $F = \sigma\, dS.E_2$

because E_2 is the field due to rest charge. Thus,

$$F = \sigma\, dS.\, E_2 = \frac{\sigma^2 dS}{2\varepsilon_0} \qquad \text{(On putting } E_2\text{, given above)}$$

or $$F = \frac{\sigma^2 dS}{2\varepsilon_0} \text{ Newton}$$

And the mechanical force per unit area is pressure,

$$P = \frac{\sigma^2}{2\varepsilon_0} \cdot \frac{dS}{dS} = \frac{\sigma^2}{2\varepsilon_0} \text{ Newton/meter}^2 \qquad ...(4.36)$$

But $$E = \frac{\sigma^2}{\varepsilon_0}$$

therefore $\sigma = \varepsilon_0 E$

Eqn. (4.36) reduces, after substitution,

$$P = \frac{\varepsilon_0^2 E^2}{2\varepsilon_0} = \varepsilon_0 \frac{E^2}{2} \text{ Newton/meter}^2 \qquad ...(4.37)$$

This force is directed along the outward drawn normal to the surface. *Thus, the surface of a charged conductor is always under stress or electrostatic pressure acts outwards.*

Demonstration of Mechanical Force

The outward mechanical force can be demonstrated by gradually charging a soap bubble, when it is found to expand. The excess of pressure inside a soap bubble of radius r due to surface tension T is given by,

$$P = \frac{4T}{r}$$

When a bubble is given a charge q, it's surface-density of charge σ is given by,

$$\sigma = \frac{q}{4\pi r^2}$$

And the additional outward pressure due to charge is calculated from Eq. (4.36), *i.e.*,

$$\frac{\sigma^2}{2\varepsilon_0}$$

Due to this pressure, bubble expands and the excess of pressure inside decreases to P′. So the excess of pressure, after charging the bubble, becomes $P' + \frac{\sigma^2}{2\varepsilon_0}$ For equilibrium,

$$P' + \frac{\sigma^2}{2\varepsilon_0} = \frac{4T}{r'}$$

or $$P' = \frac{4T}{r'} - \frac{q^2}{32\pi^2\varepsilon_0 r'^4}$$

or $$P' = \frac{4}{r'}\left(T - \frac{q^2}{128\pi^2\varepsilon_0 r'^3}\right) \quad ...(4.38)$$

where, r' is the new radius of expanded bubble. Thus, *the charging of the bubble reduces the value of surface tension, therefore the bubble increases in size.*

The increase in radius due to expansion can be calculated as follows. Since the mass of air enclosed in a soap bubble is fixed, we can use Boyl's law, *i.e.,*

$$P \propto \frac{1}{V} \quad \text{or} \quad P \propto \frac{1}{r^3}$$

here V is volume and r is radius of the bubble. Hence we get,

$$P = \frac{k}{r^3}$$

$$\frac{dP}{dr} = -\frac{3k}{r^4} = -\frac{3P}{r}$$

or $$dr = -\left(\frac{r}{3P}\right)dP$$

$(-dP)$ is additional outward pressure and is given by Eq. (4.38).

$$-dP = \frac{q^2}{32\pi^2\varepsilon_0 r^4}$$

Hence, $$dr = \frac{q^2}{96\varepsilon_0\pi^2 P r^3} \quad ...(4.39)$$

Special Case

When inside pressure = The outside pressure, $P' = 0$

Hence (4.38) reduces to,

$$\frac{4T}{r'} - \frac{q^2}{32\pi^2\varepsilon_0 r'^4} = 0$$

or $$q = 8\pi r'\sqrt{2\varepsilon_0 T r'} \text{ Coul.} \quad ...(4.40)$$

4 26. Electrostatic Energy in the Medium Surrounding the Charged Conductor

The electrostatics energy due to a charge on a conductor, according to Faraday, remains in the medium surrounding the conductor. The charge on the conductor simply defines the surface condition existing in the electric field. So the energy is due to electric field which surrounds the medium.

We have calculated in the previous article the outward force per unit area experienced by the charged surface *i.e.,*

$$P = \varepsilon_0 \frac{E^2}{2} \text{ newton/meter}^2$$

Therefore the force on the area, dS

$$F = \varepsilon_0 \frac{E^2}{2} dS \text{ Newtons}$$

If the surface is displaced along the direction of force, *i.e.* normal to the dS through a distance dl, as shown in Fig. (4.34) then –

$$\text{The work done} = \text{Force} \times \text{distance}$$

$$= \varepsilon_0 \frac{E^2}{2} dS\, dl$$

$$= \varepsilon_0 \frac{E^2}{2} dS\, dl \text{ joules}$$

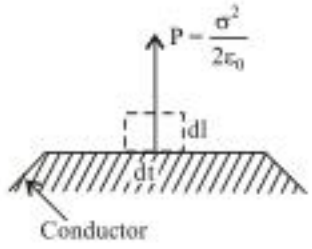


Fig. 4.34

This work done is stored up as energy in the medium.

The volume swept out = $dS\, dl$

Hence work done (W) in producing unit volume of the field, which is the electrostatics energy stored up in unit volume of the medium –

$$\text{Energy per unit volume} = \varepsilon_0 \frac{E^2}{2} \text{ joule/meter}^3 \qquad ...(4.41)$$

Ex. **4.13.** *Four grams of gold are beaten into a thin leaf of area 1 sq.m. A small piece is cut off from this and placed upon a conductor. Calculate the charge density required by the conductor so that the piece of gold is just lifted up.*

Sol. If σ is the charge density on the surface of the conductor the upward force per sq.m due to charge,

$$= \frac{\sigma^2}{2\varepsilon_0} \text{ Newton/m}^2$$

The downward force on the gold due to its weight per square meter

$$= 4 \times 10^{-3} \times 9.8 \text{ Newton/m}^2$$

For just lifting, we have,

$$\frac{\sigma^2}{2\varepsilon_0} = 4 \times 10^{-3} \times 9.8$$

or $$\sigma^2 = 2 \times \varepsilon_0 \times 4 \times 10^{-3} \times 9.8$$

or $$\sigma^2 = 2 \times 8.5 \times 10^{-12} \times 4 \times 10^{-3} \times 9.8$$

or $$\sigma = 8.2 \times 10^{-10} \text{ Coul/meter}^2$$

Ex. 4.14. *A soap bubble has a radius of 0.02 m. To what potential must the bubble be raised so that the pressure inside the bubble equals to the pressure outside, ($T = 24 \times 10^{-3}$ Nt/m).*

Sol. Using Eq. (4.40), for special case when P = 0,

$$q = 8\pi r \sqrt{2\varepsilon_0 T r}$$

and $$V = \frac{q}{4\pi\varepsilon_0 r}$$ (See Art (5.5) for potential)

$$V = 2\sqrt{\frac{2Tr}{\varepsilon_0}} = 2 \times \sqrt{\frac{2 \times 24 \times 10^{-3} \times 0.02}{8.85 \times 10^{-22}}}$$

$$= 20{,}840 \text{ volts.}$$

Ex. 4.15. *Find the charge required to double the radius of a soap bubble.*

Sol. Let, r be the initial radius of the soap bubble. Further, suppose A is the atmospheric pressure and T is the surface tension of the soap bubble. Then the initial pressure inside the soap bubble,

$$P = A + \frac{4T}{r} \qquad ...(1)$$

where $4T/r$ is the excess pressure due to surface tension. Let q be the charge required to double the radius ($2r$). The outward force per unit area due to the charge q is,

$$= \frac{\sigma^2}{2\varepsilon_0}, \text{ where } \sigma = \frac{q}{4\pi(2r)^2}$$

$$= \frac{q}{2\varepsilon_0 \times 16 \times \pi^2 \times 16r^4}$$

$$= \frac{q^2}{512\varepsilon_0\pi^2 r^4}$$

Now the pressure inside the bubble

$$= A + \frac{4T}{2r} - \frac{q^2}{32\varepsilon_0\pi^2(2r)^4}$$

when the bubble expands to twice its radius, the volume becomes 8 times the initial volume and the pressure inside the bubble is P/8.

Therefore, $$\frac{P}{8} = A + \frac{4T}{2r} - \frac{q^2}{512\varepsilon_0\pi^2 r^4}$$

Substituting the value of P from (1) this expression becomes,

$$A + \frac{4T}{2r} = 8A + \frac{16T}{r} - \frac{q^2}{64\varepsilon_0\pi^2 r^4}$$

or $$7A + \frac{12T}{r} = \frac{q^2}{64\varepsilon_0\pi^2 r^4}$$

or $$q^2 = 64\varepsilon_0\pi^2 r^4\left(7A + \frac{12T}{r}\right)$$

or $$q^2 = 64\varepsilon_0\pi^2 r^3(7Ar + 12T)$$

or $$q = 8[\varepsilon_0\pi^2 r^2(7Ar + 12T)]^{1/2}$$

which is the required charge to doable the radius.

Ex. 4 16. *A soap bubble of radius 10 cm is given a charge of 20 e.s u. Calculate the change in the radius of the bubble. (Atmospheric pressure is 10^6 dynes/cm^2).*

Sol. Radius $r = 10$ cm $= 0.1$ m

Charge $q = 20$ *e.s.u.* $= 6.6 \times 10^{-9}$ Coul.

Since 1 coul $= 3 \times 10^9$ *e.s.u.*

Atmospheric pressure 10^6 dynes/cm^2 $= 10^5$ Nt/m^2

Since 1 Newton $= 10^5$ dynes

The change in radius (or increase) have been calculated and given in Eq. 4.39.

i.e.,
$$dr = \frac{q^2}{96\varepsilon_0 \pi^2 P r^3}$$

or
$$dr = \frac{(6.6\times10^{-9})^2 \text{ Coul}^2}{96\times8.85\times10^{-12} \text{ Coul}^2\text{Nt.m}^2 \times (3.14)^2 \times 10^5 \text{ Nt/m}^2 \times (0.1)^3 \text{m}^3}$$

or
$$dr = 52 \times 10^{-12} \text{ meter}$$

or
$$dr = 52 \times 10^{-10} \text{ cm}$$

Ex. 4.17. *Two large metal plates of area 1.0 m² face each other are separated by 5 cm and carry equal and opposite charges on their inner surfaces. If E between the plates is 55 Nt/Coul, find the charge on the plates.*

Sol. Area of plate =1 m²

From Eqn. (4.33),
$$E = E_1 + E_2 = \frac{\sigma}{\varepsilon_0}$$

which is independent of separation, hence constant for the distance 5 cm.

Let q be charge on the plate
$$\sigma = \frac{q}{A}$$

Hence,
$$E = \frac{q}{A\varepsilon_0}$$

Putting the values,
$$55 = \frac{q}{1\times8.85\times10^{-12}}$$

or
$$q = 55 \times 1 \times 8.85 \times 10^{-12} \text{ Coul}$$
$$= 4.88 \times 10^{-10} \text{ Coul.}$$

Ex. 4 18. *Find the force per unit area with which plates of parallel plate capacitor attract each other if they are separated by 1 mm and maintained at 100 volts potential difference and the electric cons-tant of the medium is unity.*

Sol. Force of attraction per unit area is given by Eq, (4.37).
$$F = \frac{\varepsilon_0 E^2}{2} \text{ Newton/m}^2 \text{ and } E = V/d$$

$$F = \frac{\varepsilon_0 V^2}{2d^2} = \frac{8.85\times10^{-12}\times100\times100}{2\times(0.001)^2} \text{ Newton/m}^2$$
$$= 4.43 \times 10^{-2} \text{ Newton/m}^2$$
$$= 0.044 \text{ Newton/m}^2$$

4-27. Millikan Experiment; for Demonstration of Charge Quantization

In 1909 Millikan performed a series of experiments to demonstrate the existence of elementary charge. He used tiny oil drops while latter on plastic spheres of known mass were used instead of oil drops.

Apparatus consists of two parallel metal plates which are connected to positive and negative terminals of a battery through potential divider as shown in Fig. (4.35). The observation is taken with the microscope, hence the spheres (plastic balls) are illuminated by intense light.

The plastic balls which have been, charged by friction are thrown between two plates through a tube by a blower. At first, when the key is open, these spheres start falling due to force of gravity. Since their masses are equal and an equal viscous force due to air acts on them, all spheres have the same constant velocity called terminal velocity. Now, the key is closed, the plates are charged and there is an electric field between the plates. The charged spheres experience two force, namely force of gravity and electric force. If lower plate is connected to the positive terminal of battery, the direction of electric force is upward. As the charge on the spheres is different they shall experience different electric force.

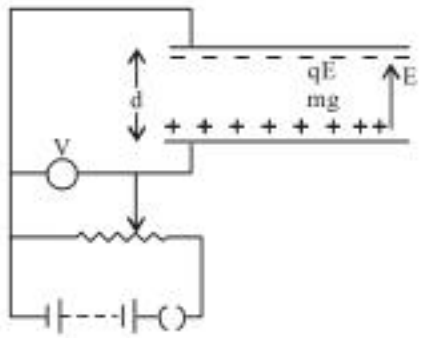


Fig. 4.35

By adjusting the potential by potential divider an equilibrium is established *i.e.* the two forces are equal in magnitude and opposite in direction. Then, for a sphere of mass m and charge q, we have,

$$qE = mg$$

If d is the separation of two plates and V is potential difference, we get,

$$E = \frac{V}{d}$$

Then,
$$q\frac{V}{d} = mg$$

or
$$q = \frac{mg\,d}{V}$$

Since mass of all spheres is same, hence mgd is constant. Therefore,

$$q \propto \frac{1}{V} \qquad ...(4.42)$$

In experiment, various potentials are applied to balance the spheres of different charge. If spheres have charges q_1, q_2, q_3...then corresponding potentials are V_1, V_2, V_3, hence according to Eq. (4.42),

$$q_1V_1 = q_2V_2 = q_3V_3 \ldots = \text{constant}$$

The potentials V_1 V_1, V_3, etc. are not arbitrary but the ratio $\frac{1}{V_1}, \frac{1}{V_2}, \frac{1}{V_2}$ etc. are in integral. multiple ratio, because $\frac{1}{V}$ is proportional to charge q of the ball. Hence charges q_1, q_2, q_3 on balls are also in the ratio of integral multiple. Clearly charge on each ball is integral multiple of minimum value. That minimum value of charge is electronic charge e. Thus,

$$q = ne \qquad ...(4.43)$$

428. Cavendish Proof of Inverse Square Law

The proof of inverse square law given by Coulomb himself using torsion balance was not satisfactory. However, the truth of the law can be verified by an indirect method, given by Cavendish.

Let 0 be the centre of the uniformly and positively charged spherical conductor and P a point inside the sphere, as shown in Fig. (4.36).

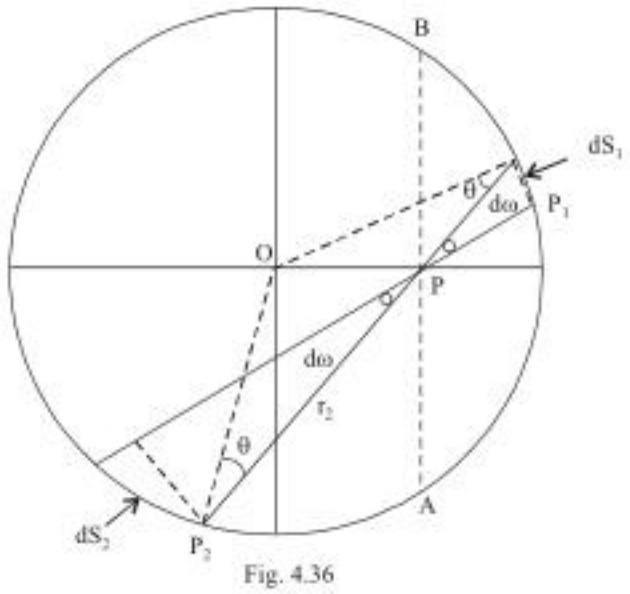


Fig. 4.36

Draw through point P, on each side, small cones intercepting areas dS_1 and dS_2 on the surface of the sphere and forming solid angle $d\omega$ *at* P.

as $PP_1 = r_1 \qquad PP_2 = r_2$

Then, we have,

$$d\omega = \frac{dS_1 \cos\theta}{r_1^2} = \frac{dS_2 \cos\theta}{r_2^2} \qquad ...(1)$$

(according to definition of Solid angle)

From (1), we get,

$$dS_1 = \frac{d\omega r_1^2}{\cos\theta} \text{ and } dS_2 = \frac{d\omega r_2^2}{\cos\theta}$$

If σ is the surface-density of charge,

then charge on $dS_1 = q_1 = \sigma dS_1 = \dfrac{\sigma d\omega r_1^2}{\cos\theta}$

and charge on $dS_2 = q_2 = \sigma dS_2 = \dfrac{\sigma d\omega r_2^2}{\cos\theta}$

Now if the electrostatic force varies as the, n-th power of the distance,
then electric intensity at P due to charge on dS_1

$$= \frac{q_1}{r_1^n} = \frac{\sigma d\omega r_1^2}{r_1^n \cos\theta} = \frac{\sigma d\omega}{r_1^{n-2}\cos\theta}$$

and electric intensity at P due to charge on dS_2

$$= \frac{q_2}{r_2^n} = \frac{\sigma d\omega r_2^2}{r_2^n \cos\theta} = \frac{\sigma d\omega}{r_2^{n-2}\cos\theta}$$

These intensities will be in the same line but will be oppositely directed. Therefore the resultant intensity at P

$$= \frac{\sigma d\omega}{\cos\theta}\left[\frac{1}{r_1^{n-2}} - \frac{1}{r_2^{n-2}}\right]$$

The following cases arise.

Case 1. If $n > 2$ and $r_2 > r_1$, as shown in Fig. the resultant intensity is positive because $\frac{1}{r_1^{n-2}} > \frac{1}{r_2^{n-2}}$.

In other words the intensity at P due to charge on dS_1, is greater than due to charge on dSt_2, hence there will be resultant field at P acting towards left, *i.e.,* directed toward the centre 0.

Case 2. If $n = 2$, the resultant intensity will be zero.

Case 3. If $n < 2$ and $r_2 > r_1$ the resultant intensity is negative. It can be similarly shown that there will be a resultant field at P acting toward right *i.e.,* directed away from the centre O.

The whole spherical surface can be regarded as a pair of solid cones of equal solid angle passing through the point P. The total intensity due to whole charged surface will have the same value as given above in the 3rd case.

The fact that inverse square law *i.e.*, $n = 2$ is true, can be verified if we show that the electric field at an external point due to charged spherical shell is zero, as discussed above in case (2).

Cavendish performed an experiment using the arrangement shown in Fig. (4.37). A conducting sphere B was supported and insulated inside a sphere A. A and B can be connected with a wire H.

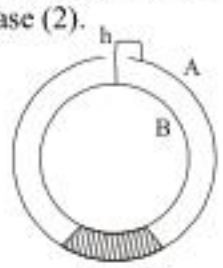


Fig. 4.37

The outer sphere A is given a positive charge then joined to B by H for a small moment. Then the two water disconnected. Then inner sphere B was tested by electroscope for any residual change on it. No appreciable charge could be detected; hence cavendish concluded that $n = 2$ *i.e.,* inverse square law holds good.

Later Faraday performed the experiment using a gold leaf electroscope and Maxwell using a quadrant electrometer. Both found that there is no charge on the inner sphere B. Maxwell concluded, taking into account the sensitivity of the electrometer, that the value of n cannot differ by more than 1/21, 600 of the value of 2. For all practical purposes the inverse square law is proved.

4.29. Millikan's Oil Drop Method for Determination of Electronic Charge

Millikan's apparatus is shown in Fig. 4.38. It consists of two horizontal parallel metallic plates A and B insulated from each other and separated by ebonite or glass plate. A potential difference can be applied between these plates by connecting a power supply through a reversing key with the help of which the direction of current can be changed. The upper plate A has some fine holes, through which drops (or droplets) of a non-volatile oil can be introduced between the plates from an atomiser. These drops get charged due to frictional effect at the atomiser nozzle. Hence the motion of these drops is affected by the electric field' existing between the plates A and B.

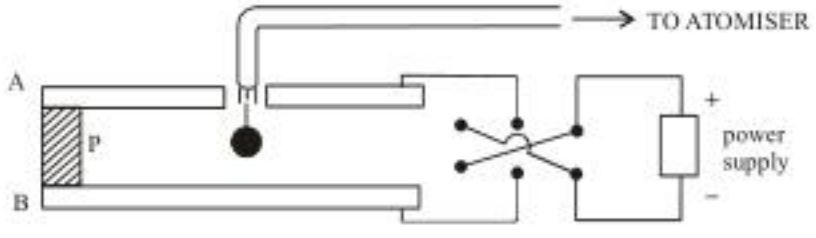


Fig. 4.38

When these drops are illuminated by a light beam they appear as tiny bright stars, when observed from the other side by means of a microscope.

Let the charge on a particular drop be *ne*. When electric field is switched on, then the electron is moving under the influence of two forces simultaneously (*i*) an electric force due to E *(it)* and gravitational force due to gravity. An electric field E created will exert an upward force (Ene) on the positively charged drop. However the weight *mg* acts vertically downwards. *Let the electrical force on the drop be large as compared to the gravitational force.* Thus the resultant upward force acting on the drop is given by

$$F = enE - mg \qquad ...(i)$$

When the electric field is switched off, the drop falls under gravity and the downward force is,

$$F = mg \qquad ...(ii)$$

According to Stoke's law the force due to viscous drag exerted on a sphere of radius *a* moving with velocity *v* through **a** medium of coefficient of viscosity η is given by,

$$F = 6\pi\eta a v \qquad ...(iii)$$

If *v* is velocity of drop known as *terminal velocity,* when both electrical and gravitational forces are present, we have by eq. (*i*).

$$neE - mg = 6\pi\eta a v \qquad ...(iv)$$

If v_0 is the terminal velocity when the electric field is switched off we have by eq. (2),

$$mg = 6\pi\eta a v_0 \qquad ...(v)$$

Adding eq. (*iv*) and (*v*) we get,

$$neE = 6\,\pi\eta a(v + v_0) \qquad ...(vi)$$

Dividing eq. (*vi*) by (*v*), we get.

$$\frac{neE}{mg} = \frac{v + v_0}{v}$$

$$ne = \frac{mg}{E}\left(\frac{v + v_0}{v_0}\right) \qquad ...(vii)$$

If ρ_d and ρ_a be the densities of drop and air respectively, then, the weight of the drop is

$$mg = \frac{4}{3}\pi a^3 (\rho_d - \rho_a)g$$

Substituting *a* from the eq. (5), we get,

$$mg = \frac{4}{3}\pi\left(\frac{mg}{6\pi\eta v_0}\right)^3 (\rho_d - \rho_d)g$$

or

$$mg = \frac{(6\pi r_1 v_0)^{3/2}}{\left[\frac{4}{3}\pi(\rho_d - \rho_a)g\right]^{1/2}}$$

Substituting this in eq. (*vii*), we get,

$$ne = \frac{(6\pi r_1 v_0)^{3/2}}{\left[\frac{4}{3}\pi(\rho_d - \rho_a)g\right]^{1/2} E}\,\frac{v + v_0}{v_0}$$

Hence by knowing *v* and v_0, the charge *ne* pn the oil drop is deter-mined. The electric field E = V/*d*, where V is the potential difference and *d* is the separation of the plates.

The velocity is determined by measuring with stopwatch the time taken by a drop in traversing from one point to other point on a micrometer scale attached to the eye-piece of the microscope.

Significance of the oil drop experiment results is that, when the experiment was repeated for number of times by taking different values of charge on drops, it was found that the charge determined in each case was an integer (whole number) multiple of elementary charge (electronic charge) *e*. It reveals very important conclusion, that is, the *"discrete nature of charge" (charge quantisation).* Thus available charge is always whole number multiple of electronic charge e.

REVIEW QUESTIONS

4.1. State the principle of conservation of charge. Give some *of* its examples.

4.2. What is the principle of quantisation of charge ? Give Millikan's oil drop method to verify the quantum nature of charge.

4.3. State and explain Coulomb's law in electrostatics. Hence define unit charge (Coul.).

4.4. What do you mean by electric field-intensity ? What is the principle of Superposition ? Apply it to calculate electric field due to :
   (*a*) Point charge distribution
   (*b*) Lire charge distribution
   (*c*) Surface charge distribution
   (*d*) Volume charge distribution.

4.5. What do you mean by the flux of an electric field ? Calculate the flux passing over a closed surface. Does it represent the flow of anything ?

4.6. Use Gauss's theorem to calculate –
   (*a*) Field due to a line charge
   (*b*) Field due to a conductor which is charged.

4.7. What are field lines ? How they differ from lines of forces ? State and prove the Gauss's law of electrostatics.

4.8. Apply Gauss's law to calculate the field due to two concentric spherical charged conductors.

4.9. Derive an expression for the force per unit area on the surfaces of a charged conductor.

4.10. Deduce Gauss's law in a differential form *i.e.*, $\Delta \vec{E} = \frac{\rho}{\varepsilon_0}$

4-11. Deduce Coulomb's law from Gauss's law.

4-12. Describe Millikan's oil drop experiment for determining the electronic charge. Explain the importance of this experiment.

PROBLEMS

4.1. You are given three point charges q_1, q_2, and q_3; when q_1 and q_2 are placed together to form a single point charge, the force on q_3 at distance L from this combination is a repulsion of 2 units magnitude. When q_3 and q_1, are so combined, the force on q_2 at the same distance L is an attraction of 18 units and when q_2 and q_3 are combined, the force on q_1 at distance L is an attraction of 4 units. Deduce the algebraic ratio of the charges. **(Ans.** $q_1 : q_3 : q_3 = 4 : -3 : 2$)

4.2. (a) Two positive point charges, each of magnitude $+ q$, are located on the y-axis at $y = +a$ and $y = -a$. Calculate –
   (1) The field at $(x, 0)$ on x-axis.
   (2) The force on third charge q placed at this point $(x, 0)$
   (3) the force on the third charge when it is placed on the origin, and
   (4) For what value of x is the force maximum ?

   (b) If one of the positive charges on the y-axis is replaced by an equal negative charge, show that the force experienced by the third charge on the x-axis is maximum when it is located at the origin.

4-3. The surface-density of a charge on a conductor, in dry air is 2.67×10^{-5} Coul'cm^2; what is the force per unit area due to this charge ? **[Ans.** $(4.4) \times 10^{-9}$ dynes/cm^2].

4.4. Two charges of the value 15×10^{-9} Coul and 30×10^{-9} Coul are placed at a distance 6 cm. Find the intensity of electric field at a point which is at a distance 2 cm from the first and 4 cm. from the second charge respectively. **(Ans.** 1.69×10^{-5} N/Coul towards the bigger charge)

4.5. What is the magnitude of a point-charge so that the electric field 50 cm away has a magnitude 2.0 N/Coul ?

(Ans. 5.5×10^{-11} Coul)

4.6. Two point charges of magnitude 2.0×10^{-7} coul. and 8.5×10^{-8} coul are 12 cm. apart. What is the field intensity of each charge at the site of the other ? **(Ans.** 12.5×10^{4} Newton/Coul and 5.3×10^{4} N/Coul.)

4.7. A positive point test-charge q_0 is placed at the mid point p between two equal positive charges q. What force acts on it at or near this point P ? **(Ans.** zero).

4.8. Two charges $q_1 = 1 \times 10^{-6}$ Coul and $q_2 = 2 \times 10^{-6}$ Coul are separated by 10 cm. At what point line joining the two charges is the electric field strength is zero. **(Ans.** 4.1. cm.)

4.9. Three identical small balls, each of mass 10 gm, are suspended separately from a common point by silk threads, each one meter long. The' balls are identically charged and hang at the corners of an equilateral triangle of side 10 cm. What is the charge on each ball ? **(Ans.** $q = 6.2 \times 10^{-8}$ Coul.)

4.10. An infinitely long wire is stretched horizontally 4 meters above the surface of the earth. It has a charge of 1 micro coulomb per cm of its length, Calculate the electric field at a point on earth vertically below the wire. **(Ans.** 4.5×10^{5} Nt/Coul.)

SELF TEST OBJECTIVE QUESTIONS

4.1. By induction we produce
- (*a*) equal charges
- (*b*) equal and like charges
- (*c*) equal and opposite charges
- (*d*) unequal and similar charges.

4.2. The creation of positively charged particle without the simultaneous creation of negatively charged particle has never been observed.
- (*a*) True
- (*b*) False

4.3. The charge of the ordinary electron has nothing intrinsically negative about it. What we call negative charge could just as well have been called positive and vice versa. The choice was a historical accident.
- (*a*) True
- (*b*) false

4.4. The force of attraction or repulsion between charges follows
- (*a*) square law
- (*b*) inverse square law
- (*c*) both (*a*) and (*b*)
- (*d*) none of (*a*) and (*b*)

4.5. When the distance between two equal charges is decreased to half and their magnitude of charge also decreased to half, the force between them—
- (*a*) remains unchanged
- (*b*) reduces to half
- (*c*) becomes double
- (*d*) becomes four times.

4.6. Which of the following statement is correct ?
- (a) Dielectric constant and permeability are the same
- (*b*) Dielectric constant and relative permittivity are the same
- (*c*) Permittivity and permeability are one and the same thing
- (*d*) None of the above.

4.7. Two charges q_1 and g_2 are placed close to each other; if a third charge q_3 is brought near it, the force now exerted by q_1 on q_2
(*a*) increases
(*b*) decreases
(*c*) remains the same
(*d*) reduces to practically zero.

4.8. Electric field intensity due to a point charge—
(a) Falls inversely proportional to the distance
(*b*) Falls inversely proportional to the square of the distance
(*c*) Falls inversely proportional to the square root of the distance
(*d*) It does not change with distance.

4.9. The electric field intensity of the an infinite long charge line varies—
(*a*) Inversely proportional to the square of the distance from it
(*b*) Inversely proportional to the distance from it
(*c*) Inversely proportional to the square root of the distance from it.
(*d*) It remains constant.

4.10. The electric charge in a conductor—
(*a*) resides on the inner side of isolated conductor
(*b*) resides on the outer side of isolated conductor
(*c*) is equally distributed throughout
(d) is concentrated at the centre of the conductor.

4.11.The electric field inside a perfectly conducting media is—
(*a*) ∞
(b) 0
(*c*) $120\ \pi$
(*d*) Depends on the value of charge

4.12. Gauss's' law can be used to calculate if :
(*a*) the symmetry of the charge distribution is high
(*b*) the symmetry of the charge distribution is low,

4.13. In a charged bubble the mechanical force due to charge is counter-balanced by
(a) force of gravitation
(*b*) viscosity
(*c*) surface tension
(*d*) none of these.

4.14. The flux-density depends only on the charge and is independent of the permittivity of the medium.
(*a*) True
(*b*) False

4.15. Gauss's law is one of the fundamental equations of electromagnetic theory.
(*a*) False
(*b*) True

Ans. 1. (*c*), **2.** (*a*), **3.** (*a*), **4.** (*b*), **5.** (*a*), **6.** (*b*), **7.** (*c*), **8.** (*b*), **9.** (*b*), **10.** (*b*), **11.** (*b*), **12.** (*a*), **13.** (*c*), **14.** (*a*), **15.** (*b*).

CHAPTER 5

ELECTROSTATICS – ELECTRIC POTENTIAL

5.1 A QUANTITATIVE DISCUSSION

In the art (4.17) we have discussed the flux (or divergence) law, *i.e.* the flow of a vector field $\vec{E}$ which is the first important property of a vector field. Let us now consider the circulation (curl) law *i.e.* the second property of a vector field.

If we compare various electrostatic lines of force (or field lines) pictures, we shall find that all these pictures have one common remarkable property, that is, there are *no closed lines* of force in any of these pictures. All lines of force (or field lines) begin and end on charged bodies. The *absence of closed lines is the property of vector field whose curl is zero*. Therefore, an electric field is a *non-curl* or *curl-free field* i.e. irrotational. Thus in vector notation,

$$\vec{\nabla} \times \vec{E} = 0 \qquad \ldots(5.1)$$

We shall prove it later on. According to theorem of vector analysis [(See Eq. (3.24)] that is the *curl of the gradient of a scalar is zero.*

In other words, $\vec{E}$ *whose curl is zero can be expressed as a gradient of a scalar function.* Therefore, the electric field is described not only by a (vector) intensity of electric field $\vec{E}$ but also by a scalar quantity, the electric potential (V). $\vec{E}$ *and V are related to each other*, it is only a *matter of convenience* which one is used in a given problem.

5.2 POTENTIAL DIFFERENCE

Suppose we have two points A and B in an electric field; move a test-charge q_o from A to B and if the work done by the agent moving the charge is W_{ab}, then electric potential difference is defined as,

$$V_b - V_a = \frac{W_{ab}}{q_o} \qquad \ldots(5.2)$$

The work done W_{ab} may be (1) positive, (2) negative or (3) zero, so V_b the potential at B will be (1) higher, (2) lower or (3) the same as V_a respectively.

Generally point A is considered to be at infinite distance and potential there V_a is taken as zero, *i.e.* $V_a = 0$, then potential V at B is given by,

$$V = \frac{W_{ab}}{q_0} \qquad \ldots(5.3)$$

Thus we define the electric potential at a point as the work done in moving the test, charge q_o from infinity to that point.

Unit: In rationalised MKS the unit of potential difference is volt. *One volt is defined as the difference in potential between the points, so that one joule of work is done in carrying one coulomb of positive charge from one point to other.*

$$1.\text{Volt} = \frac{1\ \text{Joul}}{1\ \text{Coul}}$$

$$1.\text{Volt} = \frac{1}{300}\ \text{e.s.u. of potential}$$

$$= 10^8\ \text{e.m.u. potential}$$

$$\text{Since V} = \frac{\text{W}_{ab}}{q_o}$$

Therefore, $\frac{\text{Newton} \times \text{metre}}{\text{Coul.}} = \frac{\text{Joules}}{\text{Coul}} = \text{Volts}$

Dividing by metre this above relation we obtain,

$$\frac{\text{Newton}}{\text{Coulomb}} = \frac{\text{Volt}}{\text{Metre}}$$

But Newton/Coulomb is the unit of electric field-intensity. *Therefore, the electric field-intensity* $\vec{\text{E}}$ *can be expressed either Newton/Coulomb or volt/metre.*

5.3 RELATION BETWEEN $\vec{\text{E}}$ AND V

(1) When Electric Field is Uniform

Suppose the points A and B are situated in a uniform electric field $\vec{\text{E}}$, as shown in Fig. (5.1). The separation between them is d. The electric force on the positive test charge q_o is $q_o E$ and points in right direction. Let us now move the test charge q_o from A to B.

To do this we have to apply an external force $\vec{\text{F}}$ of the same magnitude and in left direction, as shown in Fig. (5.1). The work done by the agent producing this force is,

$$\text{W}_{ab} = \text{F}d$$

$$= q_o\ \text{E}\,d$$

Putting the value of W_{ab} in Eq. (5.2), we get

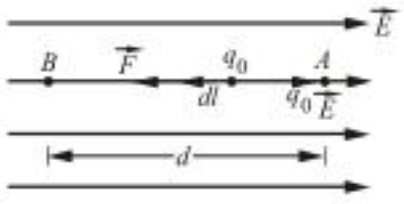


Fig. 5.1

$$\text{V}_b - \text{V}_a = \frac{\text{W}_{ab}}{q_o} = \frac{q_o\ \text{E}\,d}{q_o} = \text{E}\,d \qquad \ldots(5.4)$$

We again see that second unit of E is volt per metre. The equation (5.4) represents the relation between E and V in a simple case.

(2) When field $\vec{\text{E}}$ is not Uniform

Consider the case of a non-uniform field which exists in the vicinity of the positive point charge $+q$; the electric field $\vec{\text{E}}$ is radial and is inversely proportional to the square of the distance r from the charge q. This field is represented in Fig. (5.2) by dense lines near the charge and spaced lines away from the charge.

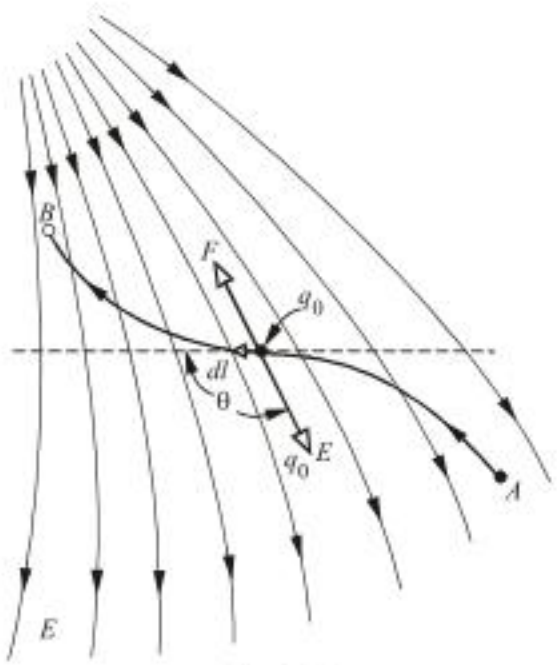


Fig. 5.2

The electric field exerts a force q_oE on the charge q_o. In order to move q_o towards B an external agent must apply a force $\vec{F} = -q_o \vec{E}$ .The work done for a small displacement $\vec{dl}$ along $AB = \vec{F}.d\vec{l}$. Therefore, the total work done to move test-charge from A to B is given by,

$$W_{AB} = \int_A^B \vec{F}.d\vec{l} = -q_o \int_A^B \vec{E}.d\vec{l} \qquad \ldots(5.5)$$

The negative sign is due to the fact that the motion A to B is opposite to the direction of the field. And as we discussed in art. (3.9), this is a line integral of $\vec{E}$. Hence,

$$V_B - V_A = \frac{W_{AB}}{q_o} = -\int_A^B \vec{E}.d\vec{l} \qquad \ldots(5.6)$$

If point A is situated at infinity and potential V_A there is zero, then the potential V at B,

$$V = -\int_\infty^B \vec{E}.d\vec{l} \qquad \ldots(5.7)$$

Thus, we can calculate V by this equation at any point. These equations (5.6) and (5.7) are applicable in general. Sometimes we write,

$$V_B - V_A = V_{AB}$$

Example. 5.1 *In Fig. 5.3, $\vec{E}$ is everywhere in the positive x-direction and equals to 10 volts/metre (a uniform field). If x_f = 1 metre, calculate (a) V_{AB} (b) V_{BA} (c) V_{-AB}.*

Solution. From Eqn. (5.6)

$$V_{AB} = -\int_A^B \vec{E}.d\vec{l}$$

$$= -\int_A^B E\cos\theta.dl$$

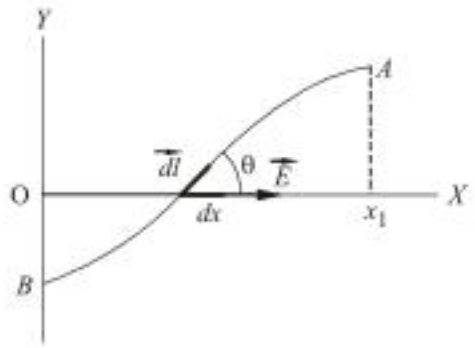


Fig. 5.3

Under given condition,

$$V_{AB} = -\int_{x_1}^{0} E\, dx, \qquad \left(\text{Since } \cos\theta = \frac{dx}{dl}\right)$$

$$= -E\int_{x_1}^{0} dx = -E\,|x|_{x_1}^{0} = E\,x_1 = \frac{10 \text{ volts}}{\text{metre}} \times 1 \text{ metre}$$

$$= 10 \text{ volts}$$

(b) As a variation of the above example, suppose that path is from B to A, then $V_{BA} = -10$ volts.

(c) As a third variation, let the direction of $\vec{E}$ be reversed (negative x-direction) but the path be from A to B, then

$$V_{AB} = -\int_{A}^{B} (-E)\cos\theta\, dl = \int_{x_1}^{0} E\, dx = -10 \text{ volts}$$

Example 5.2 *Prove that potential difference is path independent.*

Solution. In Fig. 5.4 let a test charge be moved without acceleration from A to B over the path ACB. For the path AC we have, $\theta = 135°$ and from equation (5.6),

$$V_0 - V_A = V_{CA} = -\int_{A}^{C} E\cos 135\, dl$$

$$= \frac{E}{\sqrt{2}}\int_{A}^{C} dl$$

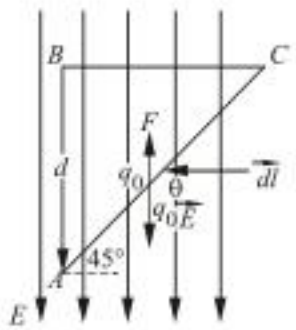


Fig. 5.4

$\int_{A}^{C} dl$ is the length AC $= \sqrt{2}d$, therefore,

$$V_{CA} = \frac{E}{\sqrt{2}}\sqrt{2}\,d = E\,d$$

For the path BC, there is no work done because E and dl are at right angles for all points, hence,

$$\vec{E}.d\vec{l} = 0,$$

Therefore, the points B and C are at the same potential. Thus,

$$V_B - V_A = V_C - V_A = E\,d$$

Let us calculate the work done for the direct path A to B.

$$V_B - V_A = -\int_{A}^{B} \vec{E}.d\vec{l} = -\int_{A}^{B} E\cos 180^\circ\, d\vec{l}$$

Because $\vec{E}$ points downward and $d\vec{l}$ points upward, hence angle between them is 180°. Simplification of the above equation leads,

$$V_B - V_A = \int_{A}^{B} E\,dl = E\int_{A}^{B} dl = E\,d$$

Hence the potential difference is same for direct path AB and an indirect path ACB. Therefore the potential difference is path independent or scalar function.

Example 5.3 *Prove that potential is a scalar quantity.*

Solution. Let there be a point charge $(+q)$, the field is radial. Suppose we choose a path consisting of components along the radial direction and perpendicular to it as shown in Fig. 5.5. The work done for the perpendicular components is zero because $\vec{E}.d\vec{l} = 0$, hence the work done is only for radial components. And sum of these components is equal to AB. Thus, we conclude that potential difference is same for both paths or potential is a scalar quantity.

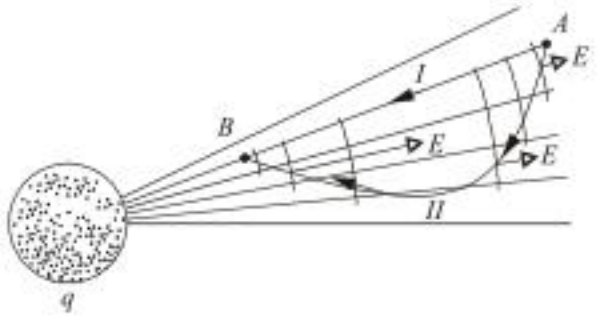


Fig. 5.5

5.4 CONSERVATIVE NATURE OF ELECTROSTATICS FIELD

The work done to move a test charge around any closed path in a static field is zero, since the path starts and ends at the same point.

Let us evaluate the work done *i.e.* the value of line integral $\int \vec{E}.d\vec{l}$ around a closed path, as shown in Fig. 5.6 (a).

The work done in moving the test charge q_o from point A to B is given by

$$W_{AB} = -q_o \int_{AP_1B} \vec{E}.d\vec{l}, \qquad \text{(see Eqn. 5.5)}$$

and for the path B to A,

$$W_{BA} = -q_o \int_{BP_2A} \vec{E}.d\vec{l}$$

Therefore, the total work done is equal to,

$$= -q_0 \int_{AP_1B} \vec{E}.d\vec{l} - q_o \int_{BP_2A} \vec{E}.d\vec{l}$$

$$= -q_0 \left[\int_{AP_1B} \vec{E}.d\vec{l} + \int_{BP_2A} \vec{E}.d\vec{l} \right]$$

$$= -q_0 \int_{AP_1BA} \vec{E}.d\vec{l}$$

$$= -q_0 \oint E.d\vec{l} \qquad \ldots(5.8)$$

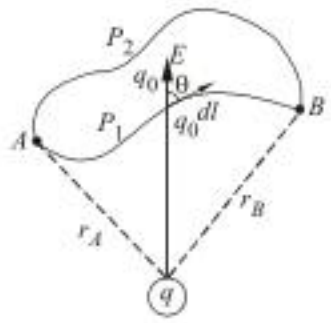


Fig. 5.6 (a)

where $\oint = \int_{AP_1BA}$ *i.e.* the integral for closed path.

Since the path starts and ends at the same point *dl* is zero, hence the work done for the closed path is zero. That is, *the work done by* $\vec{E}$ *on a particle that moves between the points depends only on the location of these points and not on the path followed by the particle. Such field is called conservative field.*

It follows, that the potential difference for a closed path is also zero. It means the potential for the two paths is same, or in other words the potential difference between any two points is independent of the path. Thus the potential difference between two points depends only on the location of these points and not on the path followed by the particle.

5.5 POTENTIAL DUE TO A POINT CHARGE

Let + q be a point charge, the field E is outward along a radial line. Let us calculate the potential difference between points A and B. Further, let that a test charge q_o be moved along a radial line from A to B as shown in Fig. 5.6 (b). As $\vec{E}$ points in the right direction and test charge is moved towards B *i.e.*, left. Hence $\vec{E}$ and $d\vec{l}$ are 180° apart, thereore,

$$\vec{E}.d\vec{l} = E \cos 180° \, dl = -E \, dl$$

Since the distance r is measured from charge *i.e.*, right and $d\vec{l}$ is measured towards B *i.e.*, left, hence,

$$dl = -dr$$

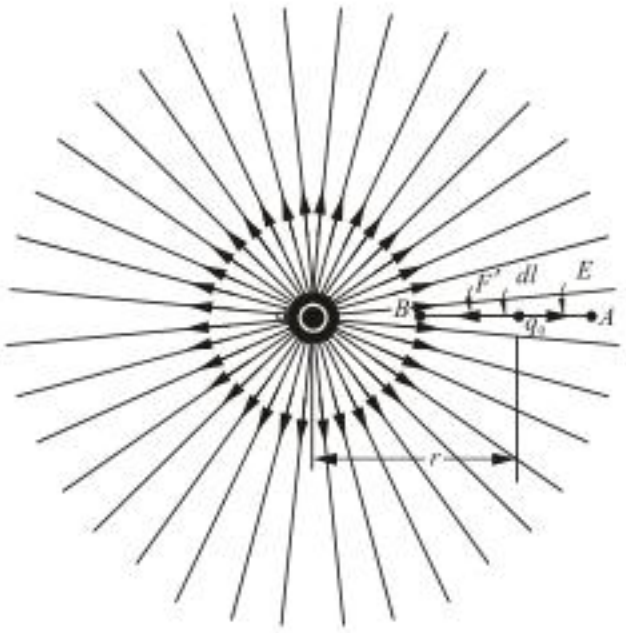


Fig. 5.6 (b)

According to Eq. (5.6), the potential difference is given by,

$$V_B - V_A = -\int_A^B \vec{E}.dl = -\int_A^B -E\,dl$$

$$= -\int_{r_A}^{r_B} E\,dr$$

where r_A and r_B are the distances of the points A and B from the point charge. But,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

Therefore,

$$V_B - V_A = -\frac{q}{4\pi\varepsilon_0}\int_{r_A}^{r_B}\frac{dr}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] \qquad \ldots(5.9)$$

If A is situated at infinity and potential, there be zero, i.e., $r_A \to \infty$ and $V_A = 0$, then potential at B if denoted by V,

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad \ldots(5.10)$$

Example 5.4 *Imagine a positive charge of value 2.23 × 10^{-10} coul in air. Calculate the potential at the distance 10 cm and 40 cm from the point charge and the potential difference between the points A and B, as shown in Fig. 5.7.*

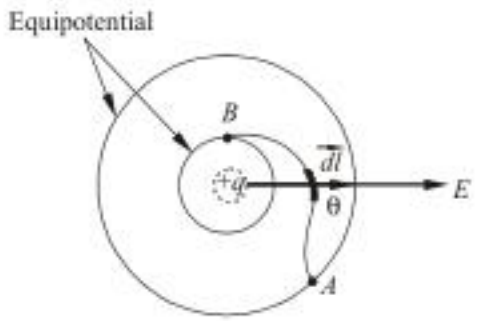


Fig. 5.7

Solution. $q = 2.23\times10^{-10}$ coul.

Using the equation (5.10)

$$V_A = \frac{q}{4\pi\varepsilon_0}\frac{1}{(0.4)}$$

$$= \frac{2.23\times10^{-10}\text{ coul}}{4\times3.14\times8.85\times10^{-12}\text{ coul}^2/\text{N.m}^2\times0.4\text{ m}}$$

$$= 5\text{ N.m/coul}$$

$$= 5\text{ joul/coul} = 5\text{ volts}$$

Similarly,

$$V_B = \frac{2.23\times10^{-10}}{4\times3.14\times8.85\times10^{-12}\times0.1}\text{ volts}$$

$$= 20\text{ volts}$$

$$V_{AB} = 20-5 = 15\text{ volts}$$

5.6 ELECTRIC POTENTIAL OF CHARGE DISTRIBUTIONS AND THE PRINCIPLE OF SUPERPOSITION OF POTENTIAL

Since the electric scalar potential due to a single point charge is a linear function of the value of its charge, therefore the *potentials of more than one point charge are linearly added by scalar addition.* This is known as the principle of superposition applied to electric potential.

The principle states that the *total electric potential at a point is the algebraic sum of the individual component potentials at the point.*

1. The Potential due to a Group of point Charges

The potential due to a group of point charges is the algebraic sum of the potentials due to individual charge.

Electric potential at P is the algebraic sum of the potentials due to the (1) point, (2) line, (3) surface and (4) volume distribution of charge as shown in Fig. 5.8.

Potential due to n charges Q_1 , Q_2, Q_3,..., Q_n at distances r_1, r_2, ..., r_n from P is given by,

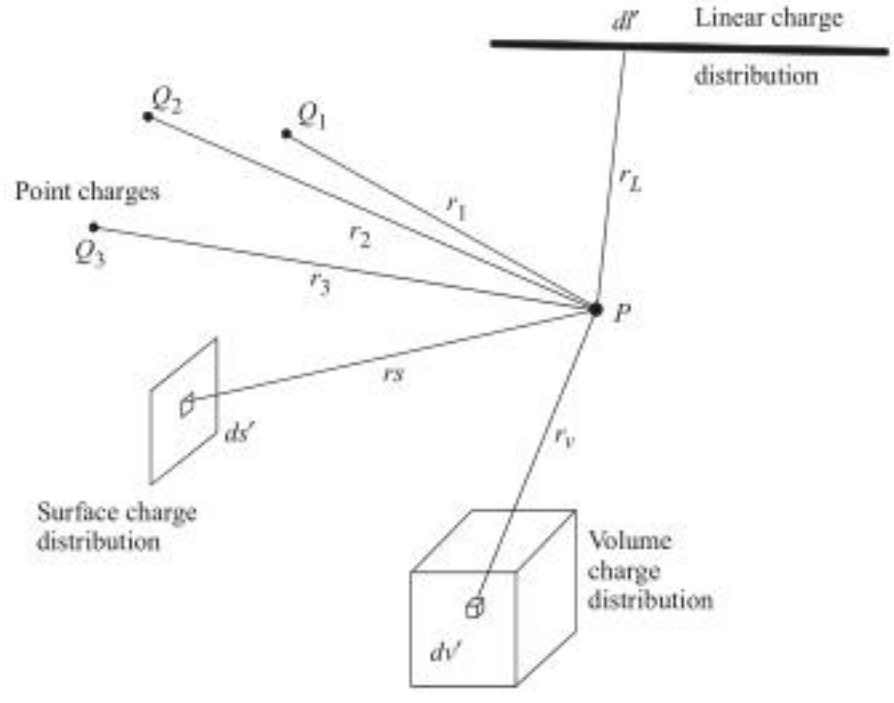


Fig.5.8

$$V_P = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{i=n} \frac{Qi}{r_i} \qquad \ldots(5.11)$$

2. The Potential due to Linear Charge Distribution

If the charge is not concentrated at a point but is distributed along a line as shown in Fig. 5.8. Then the potential due to this linear charge distribution is,

$$V_L = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_L}{r_L} dl' \qquad \ldots(5.12)$$

where ρ_l = Linear charge density (coul/metre)

dl' = Element of length of the line (metre)

the integration is carried out over the entire line of charge.

3. The Potential due to a Surface Charge Distribution

When the charge is distributed over a surface as shown in Fig. 5.8, the potential at P is,

$$V_s = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\sigma}{r_s} dS' \qquad \ldots(5.13)$$

where σ = surface charge density (Coul/metre^2)

$\boxed{dS'}$ = the element of the surface (metre^2)

And the integration limits are over the entire surface of the charge.

4. The Potential due to a Volume Charge Distribution

When the charge is distributed over a volume, as shown in Fig. 5.8, the potential at P is,

$$V_v = \frac{1}{4\pi\varepsilon_0}\iiint_v \frac{\rho dv'}{r_v} \qquad \ldots(5.14)$$

where ρ = volume charge density (couls/metre^3)

dv' = element of volume (metres^3)

The integration is taken throughout the volume containing charge. If all distributions are present simultaneously, then total electric potential at P by the superposition principle is the algebraic sum of the individual component potential. Thus we have,

$$V = V_P + V_L + V_s + V_v \qquad \ldots(5.15)$$

or after substituting from the above equations,

$$V = \frac{1}{4\pi\varepsilon_0}\left(\sum_{i=1}^{i=n}\frac{Q_i}{r_i} + \int_L \frac{\rho_L}{r_L}dl' + \iint_S \frac{\sigma dS'}{r_s} + \iiint_v \frac{\rho dv'}{r_v}\right) \qquad \ldots(5.16)$$

However, if all of the charge distribution are such that they occupy finite volumes instead of being distributed to points, lines, and surface, then Eq. (5.16) reduces to (5.14).

Example 5.5. *A square of side 1 metre in air has a point charge $q_1 = 10^{-12}$ coul at the upper left corner, a point charge $q_2 = -10^{-11}$ coul at the lower left, and a line distribution of charge of uniform density $\rho_L = 10^{-11}$ coul/metre along the right edge. Calculate the potential at the point P at the centre of the square, as shown in Fig. 5.9.*

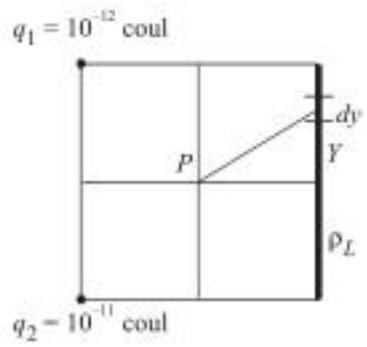


Fig. 5.9

Solution. The potential at P due to the point charges is by Eq. (5.11), *i.e.*,

$$V_P = \frac{1}{4\pi\varepsilon_0}\left(\frac{10^{-12}}{\sqrt{(0.5)^2+(0.5)^2}} - \frac{10^{-11}}{\sqrt{(0.5)^2+(0.5)^2}}\right)$$

$$= \frac{1}{4\times 3.14\times 8.85\times 10^{-12}}\left(\frac{10^{-12}}{0.707} - \frac{10^{-11}}{0.707}\right)$$

$$= -0.115 \text{ volt}.$$

The potential at P due to line distribution of charge is calculated by Eq. 5.12, *i.e.*, after substituting the given values, we have

$$V_L = \frac{1}{4\pi\varepsilon_0}\int_{Y=-0.5}^{Y=0.5} \frac{10^{-11}}{\sqrt{(0.5)^2+y^2}}dy$$

After evaluating the integral,

$$V_L = 0.158 \text{ volt}$$

The total potential at P is then,

$$V = V_P + V_L = 0.043 \text{ volt}$$

5.7 CALCULATION OF $\bar{E}$ FROM V

Potential Gradient

The potential rise between two points along an electric field is called the gradient of the potential. As we have discussed in Art (3.4),

$$\text{(gradient of } V) = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L}$$

where ΔV is change in potential for ΔL change in length.

By definition this is also the ratio of the infinite small potential rise dV to the infinitesimal length dl thus,

$$\text{(gradient of V)} = \frac{dV}{dl} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L}$$

If the element of the length dl is at an angle θ with respect to the electric field as Fig. 5.10, then the work done for the displacement dl to move a unit positive charge is the potential change dV

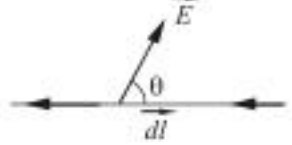


Fig. 5.10

$$dV = -\bar{E}.d\vec{l}$$

$$= -E\, dl \cos\theta.$$

The – ve sign indicates that $E\cos\theta$ is opposite to the direction of motion, hence,

$$\text{grad } V = -\frac{dV}{dl} = -E\cos\theta$$

If $\theta = 0$, $d\vec{l}$ is along the field line, the ratio $\frac{dV}{dl}$ is maximum, hence,

$$\text{gradient of } V = \frac{dV}{dl} = -E$$

Therefore the gradient of the potential (maximum rise in potential with distance) is equal in magnitude to the electric field intensity. *This maximum rise in potential with distance is gradient.* It has direction opposite to the field *i.e.*,

$$\bar{\nabla} V = -\bar{E} \text{ or } \bar{E} = -\bar{\nabla} V \qquad \ldots(5.18)$$

since grad $V = \bar{\nabla} V$ in vector notations

The significance of the negative sign is that to experience a rise in potential, a positive test-charge must be moved opposite to the electric field direction.

The equation (5.18) is used to calculate electric field from the potential.

If E_x, E_y and E_z are the components of E along x, y, z axes respectively, then,

$$E_x \hat{i} + E_y \hat{j} + E_z \hat{k} = -\left(\frac{\delta V}{\delta x}\hat{i} + \frac{\delta V}{\delta y}\hat{j} + \frac{\delta V}{\delta z}\hat{k}\right)$$

Therefore,

$$E_x = -\frac{\delta V}{\delta x}, E_y = -\frac{\delta V}{\delta y}, E_z = -\frac{\delta V}{\delta z}$$

As we have discussed in Eq. (3.24) that the curl of the gradient of scalar is zero, taking curl on both sides of equation (5.18), we get

$$\bar{\nabla} \times \bar{E} = 0 \qquad \ldots(5.19)$$

Thus, the *electric field is a non-curl or curl free, or ir-rotational field. Hence all the lines of force begin and end on the charged bodies and lines are not closed. In other words, the physical significance is that* $\bar{E}$ *originates from the point sources (charges).*

Example 5.6 *The potential function is given by ,*

$$V = \frac{10}{x^2 + y^2}$$

where V is in volts and x, y are in cm. Find (a) the expression for the gradient of the potential, (b) the value of gradient at the point (2, 1) cm and (c) the electric field intensity at this ponit.

Solution. *(a) Since potential function is independent of z.*

$$\frac{\delta V}{\delta z} = 0,$$

hence, $\text{grad } V = \bar{\nabla} V = \hat{i}\frac{\delta V}{\delta x} + \hat{j}\frac{\delta V}{\delta y}$ and

Similarly, $\frac{\delta V}{\delta x} = -\frac{10(2x)}{(x^2 + y^2)^2} = \frac{20x}{(x^2 + y^2)^2}$

$$\frac{\delta V}{\delta y} = -\frac{20y}{(x^2 + y^2)^2}$$

Therefore,

$$\text{grad } V = \bar{\nabla} V = -\frac{20}{(x^2 + y^2)^2}(\hat{i}x + \hat{j}y)$$

(b) At the point (2, 1) *i.e.*, $x = 2, y = 1$

$$\bar{\nabla} V = -\frac{20}{25}(\hat{i}2 + \hat{j}1)$$

The magnitude of gradient = – 1.79 volt/cm.

(c) The electric field $\vec{E} = -\vec{\nabla} V$

The magnitude of E = – 1.79 volv/cm.

In (b) and (c) the direction $\cos\theta$ is calculated with *x*-axis.

$$\cos\theta = \frac{2}{1.79}, \theta = 206°$$

Then, $V = 1.79 \angle 206°$ volts/cm

$\vec{E} = -1.79 \angle 206°$ *i.e.*, oppositely directed

$\vec{E} = -1.79 \angle 26°$.

5.8 EQUIPOTENTIAL SURFACES (OR CONTOURS)

An equipotential surface is defined as suface drawn through all points at the same potential in an electric field. In other words, the locus of points, all of which have the same electric potential, is called an equipotential surface. Equipotential surfaces around a positive charge are concentric spheres. To be more specific, any surface or volume over which the potential is constant is called an "equipotential".

In Uniform Field

In uniform field the field lines are parallel. A single field line gives no information as to the intensity of the field, it simply indicates the direction of $\vec{E}$. But, by measuring the work per coulomb to move a positive test charge along a field line the potential differences along the line can be determined. Larger is the value of potential difference between the two points, stronger is the field. In uniform electric field, the potential gradient is constant so that the equipotential lines are equally spaced as shown in Fig. 5.11(a).

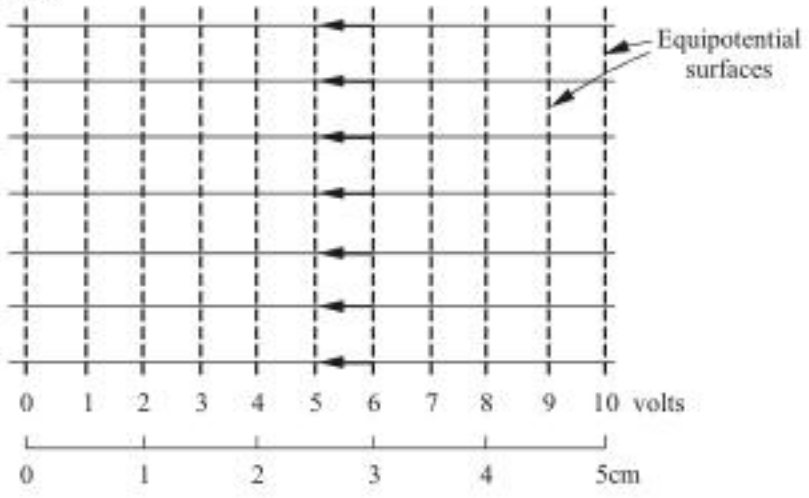


Fig. 5.11 (a)

Equipotential Surfaces in Non-Uniform Field

However, in the case of non-uniform field such as the field due to positive point charge. In such case *we can represent geometrically the two fundamental laws of electrostatics-one that the flux is proportional to the charge i.e. Gauss's laws,* and the other *that the electric field is the gradient of potential i.e.,* $\vec{E} = -\vec{\nabla} V$.

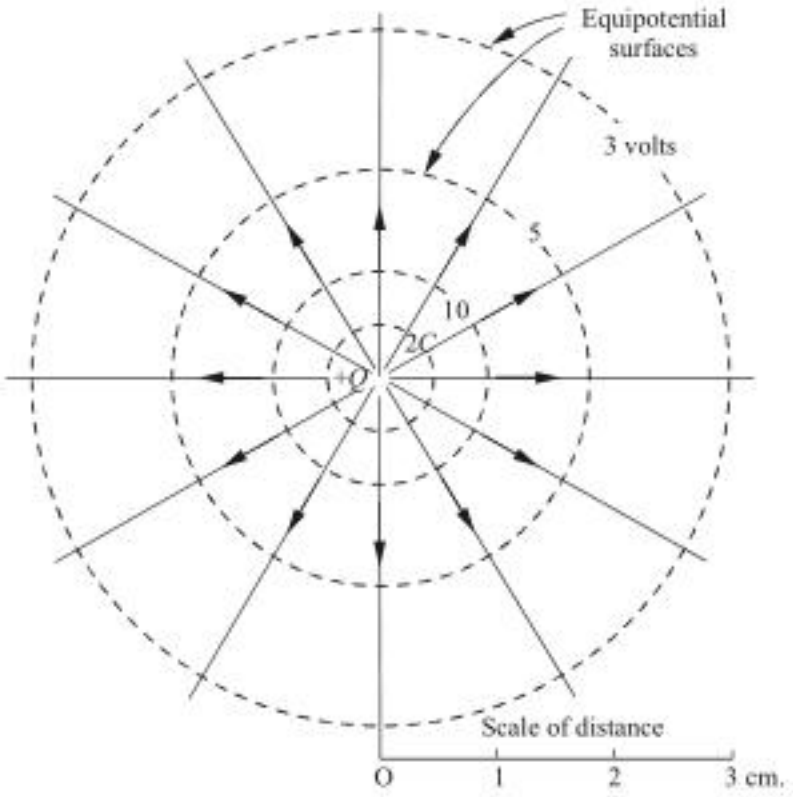


Fig. 5.11 (b)

For a point-charge the field lines are shown in Fig. 5.11(b). The density of field lines represents the magnitude of $\vec{E}$. For a point charge *the density of lines must decrease as $1/r^2$ becasue intensity varies inversely as the square of the distance*. However, the flux $\oint \vec{E}.d\vec{S}$ is constant for all distances because dS increases as r^2 at any point. So that the *field lines become more widely separated as the distance from the charge increases.* We draw the surface for which V is constant, call them equipotential surfaces. The relation $\vec{E} = -\vec{\nabla} V$, means the gradient is in the direction of most rapid change of potential and is perpendicular to the equipotential surface. If $\vec{E}$ is not perpendicular to surface, then surface is not equipotential because $\vec{E}$ has component in the surface. Therefore, equipotential surfaces are everywhere at right angles to the electric field lines.

5.9 DERIVATION OF POISSON'S LAPLACE'S EQUATIONS AND UNIQUENESS THEOREM

The Poisson's and Laplace's equations are the result of combining Gauss's law with the gradient of electric field. The Gauss's law (or theorem) in differential form is—

$$\vec{\nabla}.\vec{E} = \rho/\varepsilon_0 \qquad \text{(see Eq. 4.24)}$$

where ρ is the volume density of charge. But $\vec{E}$ is related to a scalar potential function V by ,

$$\vec{E} = -\vec{\nabla} V$$

Substituting in the above equation, we get

$$\vec{\nabla}.\vec{\nabla} V = -\frac{\rho}{\varepsilon_0}$$

or $$\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}, (\text{since } \vec{\nabla}.\vec{\nabla} = \nabla^2) \qquad \ldots(5.20)$$

This is known as Poisson's equation.

The solution of Eq. (5.20) gives the value of potential V. If there is no charge or, in other words, there is no kind of charge distribution, *i.e.*, the region of field is free from charge ($\rho = 0$), then equation (5.20) reduces to,

$$\nabla^2 V = 0 \qquad \ldots(5.21)$$

This equation is called Laplace equation.

Importance of These Equations

Mathematically entire subject of electrostatics is merely the study of the solution of Poisson's equation. Once the Poisson's equation is solved, we get the value of scalar potential function V. Hence $\vec{E}$ electric field is calculated from relation $\vec{E} = -\vec{\nabla} V$. Now when $\vec{E}$ and V are known, all rest electrical properties can be understood.

In charge- free space, the solution of Laplace equation gives the potential there. However, the Laplace equation is solved under certain given boundary conditions. If the solution exists under these boundary conditions or, in other words, the solution satisfies the given boundary conditions, it gives the value of potential at every point in the region.

As for example the boundary at the surface of a charged conductor are that

1. There is no component of electric field along the surface, *i.e.* there is only normal component which is equal to $(-\vec{\nabla} V)$.
2. The potential at the surface is equal to that of conductor (*i.e.* its exterior).

Uniqueness Theorem Regarding Electric Potential

As we have discussed that if there are no charge at a point *i.e.* in charge free region, then the potential V at that point must satisfy,

$$\nabla^2 V = 0$$

or in cartesian coordinates

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^3 V}{\delta y^2} + \frac{\delta^2 V}{\delta_z{}^2} = 0 \qquad \ldots(1)$$

which is *Laplacian equation* in cartesian coordinates. This equation is valid for every charge- free region. Therefore, outside the charged conductor the electric potential must satisfy the Laplace's equation. Once V (x, y, z) is found by solving Eq. (1) under given conditions, we can easily find the electric field by using relation $\vec{E} = -\text{grad } V$. Thus we have to solve the Eq. (1) under given specified conditions. These conditions might have been set in various ways.

Let us first condsider an electrostatics problem of large number of conductors. Further consider that the potential of each conductor $V_1, V_2 \ldots, V_k$ is fixed or known by permanently connecting the condurctors to batteries or other constant potential power supplies. Then our solution $V_{(x,y,z)}$ has to assume the correct value at all points on each of the conducting surfaces.

On the other hand, if we have specified the total charge on each conductor, say, it has value $q_1, q_2, \ldots q_k$. By Gauss's theorem, we have,

$$q = \varepsilon_0 \int_S \vec{E}.d\vec{S} \text{ and } \vec{E} = -\text{grad } V,$$

Hence $q = -\varepsilon_0 \int_S \text{grad } V.d\vec{S}$.

Thus by fixing charges we have effectively fixed the value of surface integral of grad V over the surface of each conductor. Therefore, now the problem is mathematically slightly different under this condition. Even in a case we can have two kinds of boundary condition (fixed potentials and fixed charges) to be satisfied simultaneously.

Now a general question of interest is with given boundary conditions, Does the problem have no solution, one solution or more than one solution? The answer to this question is given by *uniqueness theorem.*

Uniqueness theorem states that Laplace's equation satisfying given boundary conditions has one and only one solution i.e. unique solution.

To prove uniqueness theorem consider a system consisting of large number of conductors $C_1, C_2....C_k$ having fixed potential $V_1\ V_2...V_k$ and the potential V_k of each conductor approaches zero at infinite distance. Let us assume that V and ψ be two possible solutions of the problem i.e. Eq. (1) satisfying the given boundary conditions. As the Laplace equation is linear and V and ψ are the two possible solutions therefore any linear combinations of these such as $C_1 V + C_2 \psi$ where C_1 and C_2 are constants must also be its solution. Let us suppose that the difference between our two solutions $V - \psi$ represented by function W is a possible solution of Eq. 1. Thus,

$$W(x, y, z) = V(x, y, z) - \psi(x, y, z) \qquad \ldots(2)$$

Of course, W does not satisfy the boundary conditions (*i.e.* potential at conductor is finite and at a infinite zero). *Since V and ψ take the same value at the surface of every conductor, hence W is zero at the surface of every conductor which is contrary to boundary condition.*

Thus the solution W does not satisfy the given boundary conditions, hence it is not the solution of the given problem. In fact it represents a solution for an entirely different boundary value problem where all the conductors are held at zero potential and the potential at infinity is also zero. Thus the uniqueness theorem that corresponding to given boundary condition there exists one solution is proved.

Example *Show that in the charge-free region inside a hollow conductor of any shape whatever, the electric field is zero.*

Solution. In a charge- free region *i.e.* inside hollow conductor, we have Laplace equation.

$$\nabla^2 V = 0$$

The potential function $V(x, y, z)$ must satisfy the Laplace equation. As the potential $V(x, y, z)$ has a constant value over the entire boundary (or surface) of the conductor in other words we have an equipotential surface. So we have $V = V_0$, a constant everywhere on the boundary. Hence by uniqueness theorem there is one solution $V = V_0$ in charge free region i.e. inside hollow conductor. Mathematically, $V = V_0 =$ constant.

The field is determined by relation $\vec{E} = -\text{grad } V$ as potential V is constant clearly E in the charge-free space inside a hollow cylinder is zero.

5.10 CALCULATION OF POTENTIAL AND FIELD

(1) Potential and field due to an electric dipole. Two charges of equal magnitude and of opposite sign situated very close together are said to form a dipole. Fig. (5.12) shows a short dipole consists of two charges $+q$ and $-q$ separated by distance d along the x-axis. Let us calculate the potential at P, as shown in Fig. 5.12.

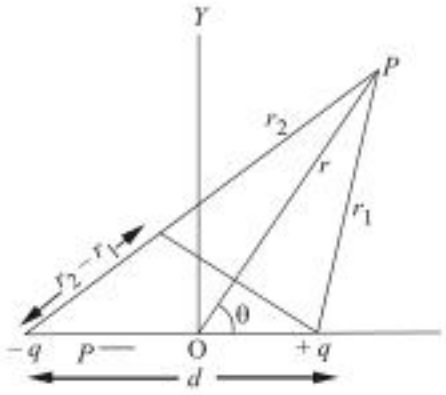


Fig. 5.12

The potential at P due to $+q$ charge,

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1}$$

and potential at P due to $-q$ charge

$$= -\frac{1}{4\pi\varepsilon_0}\frac{q}{r_2}$$

Potential at P due to the dipole,

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 \cdot r_2}\right]$$

If d is small in comparison to r, then from Fig. 5.12,

$$r_2 - r_1 \cong d\cos\theta \text{ and } r_1 r_2 \cong r^2$$

Substituting these values, the potential at P

$$= \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}$$

or $$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} \text{ volts} \qquad \ldots(5.22)$$

In which $p = qd$ is called the *electric dipole moment* which *is a vector i.e.* $\vec{p}$, *whose direction is from* $-q$ *to* $+q$. *Evidently,* V depends on angle θ and distance r from the centre of the dipole.

If $\theta = 90°$, *i.e., the equatorial plane* in other words along the *perpendicular bisector of the line joining the charges,* $V = 0$ from equation (5.22). Therefore, *potential vanishes everywhere in the equitorial plane.* It means that no work will be done, if a test charge is brought from infinity near to the dipole along the perpendicular bisector of the dipole.

However, for a given value of r,

1. If $\theta = 0$, V is maximum and positive.
2. If $\theta = 180°$ V is maximum and negative.

So that potential is maximum along the line joining the charges, on one side it is positive (corresponds to $\theta = 0$) and on other (opposite side) (corresponds to $\theta = 180°$) it is negative. Clearly, *V depends on p = qd i.e., product qd and not separately on q and d.*

The equation (5.22) can be written as,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{r}}{r^2} \qquad \ldots(5.23)$$

where $\hat{r}$ is a unit vector along r *i.e.* OP.

Since $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ (see Eq. 3.60)

We can express Eq. (5.23) as follows:

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\vec{p}.\vec{\nabla}\left(\frac{1}{r}\right) \qquad \ldots(5.24)$$

Field of a Dipole

The field at a point having polar co-ordinates (r, θ) is obtained by, $E = -\vec{\nabla} V$, therefore the component of the field along the radius vector, $\vec{r}$ is given by the radial component.

$$E_r = -\frac{\delta V}{\delta r}, \text{ since V is function of } r$$

Substituting V from (5.23), we get,

$$E_r^* = -\frac{1}{4\pi\varepsilon_0}\frac{\delta}{\delta r}\left(\frac{p\cos\theta}{r^2}\right)$$

$$E_r = \frac{p\cos\theta}{2\pi\varepsilon_0 r^2} \qquad \ldots(1)$$

The field component perpendicular to the radius vector $\vec{r}$, *i.e.* the transverse component.

$$E_\theta^* = -\frac{1}{r}\frac{\delta V}{\delta\theta}$$

$$= -\frac{1}{r}\frac{\delta}{\delta\theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right)$$

---

*Gradient in spherical coordiantes.

$$\nabla V = \hat{a}_r\frac{\delta V}{\delta r} + \hat{a}\theta\frac{1}{r}\frac{\delta V}{\delta\theta} + \hat{a}\phi\frac{1}{r\sin\theta}\frac{\delta V}{\delta\phi}$$

$$\vec{E} = -\hat{a}_r\frac{\delta V}{\delta r} \text{ and } \vec{E}_\theta = -\hat{a}\theta\frac{1}{r}\frac{\delta V}{\delta\theta}$$

$$= \frac{p \sin\theta}{4\pi\varepsilon_0 r^3} \qquad \ldots(2)$$

The magnitude of the resultant dipole field is given by,

$$E = \sqrt{E_r^2 + E_\theta^2}$$

$$= \sqrt{\left(\frac{p\cos\theta}{2\pi\varepsilon_0 r^2}\right)^2 + \left(\frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\right)^2}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}(4\cos^2\theta + \sin^2\theta)^{1/2}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta + 1)^{1/2} \text{ volts/m} \qquad \ldots(5.25)$$

In vector form we can write,

$$\vec{E} = \vec{E}_r + \vec{E}_\theta$$

$$\vec{E} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}\hat{r}_r + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\hat{r}_\theta$$

where, $\hat{r}_r$ = Unit vector in r direction (see Fig. 5.13)

$\hat{r}_\theta$ = Unit vector in θ direction

The angle ϕ which the resultant field makes with $\vec{r}$ is given by,

$$\phi = \tan^{-1}\frac{E_\theta}{E_r} = \frac{1}{2}\tan\theta \qquad \ldots(5.26)$$

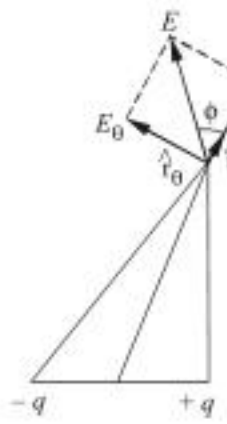


Fig. 5.13

In Cartesian Coordinates

It is evident from the relation,

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

that, $r = \sqrt{x^2 + y^2 + z^2}$

where r is the distance of the point P and origin of the coordinate system is at 0, the centre of the dipole, and $\cos\theta = \frac{x}{r}$

$$\vec{E} = -\nabla V$$

or in components,

$$E_x\hat{i} + E_y\hat{j} + E_z\hat{k} = -\left(\frac{\delta V}{\delta x}\hat{i} + \frac{\delta V}{\delta y}\hat{j} + \frac{\delta V}{\delta z}\hat{k}\right)$$

Hence, $$E_x = -\frac{\delta V}{\delta x} = -\frac{\delta}{\delta x}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right)$$

$$= -\frac{p}{4\pi\varepsilon_0}\frac{\delta}{\delta x}\left(\frac{x}{r^3}\right), \qquad \left(\text{since } \cos\theta = \frac{x}{r}\right)$$

$$= -\frac{p}{4\pi\varepsilon_0}\left[\frac{1}{r^3} + x\frac{\delta}{\delta x}\frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{p}{4\pi\varepsilon_0}\left[\frac{1}{r^3} - x\frac{3}{2}\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= -\frac{p}{4\pi\varepsilon_0}\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$$

$$= -\frac{p}{4\pi\varepsilon_0}\left[\frac{1}{r^3} - \frac{3\cos^2\theta}{r^5}\right]$$

$$= -\frac{p}{4\pi\varepsilon_0} - \left[\frac{3\cos^2\theta - 1}{r^3}\right] \qquad \ldots(3)$$

Similarly,

$$E_y = -\frac{\delta V}{\delta y}$$

$$= -\frac{p}{4\pi\varepsilon_0}\frac{3yx}{r^5} \qquad \ldots(4)$$

and, $$E_z = -\frac{\delta V}{\delta z}$$

$$= \frac{p}{4\pi\varepsilon_0}\frac{3zx}{r^5} \qquad \ldots(5)$$

The resultant of y and z components of field gives the component of field in y-z plane which is directed perpendicular to the x-axis that is the axis of the dipole. We call it transverse component of field and denote it by E_T. Then,

$$E_T = \sqrt{E_y^2 + E_z^2}.$$

On substitution from (4) and (5),

$$E_T = \frac{3px}{4\pi\varepsilon_0 r^5}(y^2+z^2)^{1/2}$$

or, $$E_T = \frac{3px}{4\pi\varepsilon_0 r^5}(r^2-x^2)^{1/2}, \qquad (\text{since } r^2 = x^2+y^2+z^2)$$

$$= \frac{3px}{4\pi\varepsilon_0 r^5}(r^2 - r^2\cos^2\theta)^{1/2}, \text{ since } \cos\theta = \frac{x}{r}$$

$$= \frac{3px\, r\sin\theta}{4\pi\varepsilon_0 r^5}$$

$$= \frac{p}{4\pi\varepsilon_0}\frac{3\cos\theta\sin\theta}{r^3} \qquad \text{(by putting } x = r\cos\theta) \qquad \ldots(6)$$

The total field,

$$E = \sqrt{E_x^2 + E_y^2 + E_z^2}$$

$$= \sqrt{E_r^2 + E_x^2}$$

Substituting E_r and E_x from the above equations (6) and (3),

$$E = \frac{p}{4\pi\varepsilon_0 r^3}\left[(3\cos^2\theta - 1)^2 + (3\cos\theta\sin\theta)^2\right]^{1/2}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left[9\cos^4\theta - 6\cos\theta + 1 + 9\cos^2\theta\sin^2\theta\right]^{1/2}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left[9\cos^4\theta - 6\cos^2\theta + 1 + 9\cos^2\theta(1 - \cos^2\theta)\right]^{1/2}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left(3\cos^2\theta + 1\right)^{1/2} \qquad \ldots(5.27)$$

The result is same as given in Eq. (5.25).

Variation of Field

Clearly, from Eq. (5.27) if $\theta = 0$, field is maximum for the point on the axis,

$$E = \frac{p}{2\pi\varepsilon_0 r^3} \qquad (5.28)$$

That is, on the axis of dipole field is given by Eq. (5.28). However, if $\theta = 180°$, *i.e.* the point on the axis but in opposite side than to the point corresponding to $\theta = 0$ field is *maximum but points in opposite direction*.

In equation (5.27), if $\theta = 90°$ *i.e.*, point P is perpendicular to the axis, the field is,

$$E = \frac{p}{4\pi\varepsilon_0 r^3} \qquad \ldots(5.29)$$

Comparison of (5.28) and (5.29) equations shows that the field on the dipole axis is twice strong as that perpendicular to the axis.

The field lines and equipotential surfaces for a dipole are shown in Fig. 5.14.

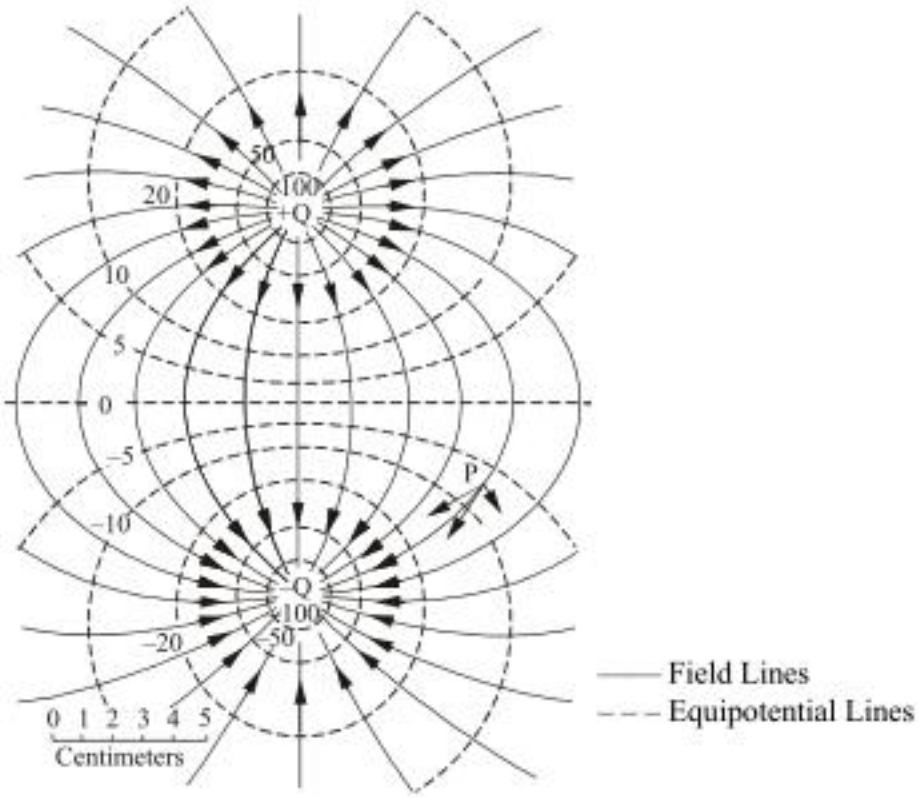


Fig. 5.14

Example 5.7 *Consider a dipole, as shown in Fig. 5.15, located at the origin of xy system and dipole pointing along the positive y-axis. Calculate E at point P (x, y).*

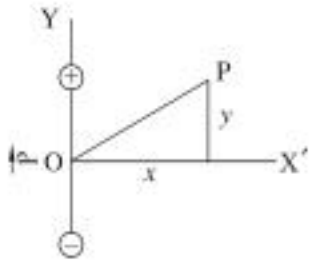


Fig. 5.15

Solution. The value of V from Eq. (5.22),

$$V(x, y, z) = \frac{p}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2}$$

In this case,

$$r = (x^2 + y^2)^{1/2}$$

and $$\cos\theta = \frac{y}{(x^2 + y^2)^{1/2}}$$

Hence, $$V = \frac{p}{4\pi\varepsilon_0} \frac{y}{(x^2 + y^2)^{3/2}}$$

The component of field along y-axis,

$$E_y = -\frac{\delta V}{\delta y} = -\frac{p}{4\pi\varepsilon_0}\left[-\frac{3}{2}\frac{(2y)\cdot y}{(x^2+y^2)^{5/2}} + \frac{1}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{p}{4\pi\varepsilon_0}\left[\frac{3y^2}{(x^2+y^2)^{5/2}} - \frac{1}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{p}{4\pi\varepsilon_0}\frac{1}{(x^2+y^2)^{3/2}}\left[\frac{3y^2}{(x^2+y^2)} - 1\right] \qquad \ldots(5.30)$$

Since, $r^2 = x^2 + y^2$ and $\cos\theta = \frac{y}{r}$; from Fig. 5.15.

$$E_y = \frac{p}{4\pi\varepsilon_0 r^3}\left[3\cos^2\theta - 1\right] \qquad \ldots(5.31)$$

Similarly, the component of field along x-axis,

$$E_x = -\frac{\delta V}{\delta x} = -\frac{\delta}{\delta x}\left[\frac{p}{4\pi\varepsilon_0}\frac{y}{(x^2+y^2)^{3/2}}\right]$$

$$= -\frac{p}{4\pi\varepsilon_0}\left[-\frac{3}{2}\frac{(x^2+y^2)^{1/2}\, y \,.\, 2x}{(x^2+y^2)^3}\right]$$

$$= \frac{3p}{4\pi\varepsilon_0}\frac{xy}{(x^2+y^2)^{5/2}} \qquad \ldots(5.32)$$

Since $\sin\theta = \frac{x}{r}$ and $r^2 = x^2 + y^2$, $\cos\theta\,\frac{y}{r}$

$$E_x = \frac{3\,p\,r\sin\theta\, r\cos\theta}{4\pi\varepsilon_0 r^5}$$

$$= \frac{3p}{4\pi\varepsilon_0}\frac{\sin\theta\cos\theta}{r^3} \qquad \ldots(5.33)$$

If we put, $\theta = 0$, *i.e.* point P is on the y-axis which is axis of dipole, then from equations (5.31) and (5.33), we get

$$E_x = 0, \text{ as } \sin\theta = 0$$

and $$E_y = \frac{2p}{4\pi\varepsilon_0 r^3}, \text{ as } \cos\theta = 1$$

E_y is positive *i.e.* pointing along + y-axis. If $\theta = 90°$ *i.e.* point P is on x-axis which is equitorial line, we get,

$$E_x = 0$$

$$E_y = -\frac{p}{4\pi\varepsilon_0 r^3}$$ *i.e.* the field is in the negative y-direction.

Example 5.8 *Calculate the potential and field due to a dipole of dipole moment 4.5×10^{-10} coul/metre at a distance 1 metre from it, (1) on its axis, (2) on its perpendicular bisector.*

Solution. The potential is given by Eq. (5.22)

$$V = \frac{4}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

(1) When point lies on the axis of dipole, $\theta = 0$, therefore,

$$V = \frac{4}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}\left(\text{since}\frac{1}{4\pi\varepsilon_0} = 9\times10^9\right) \text{ On putting } p \text{ and } r$$

$$= \frac{4.5\times10^{-10}\times9\times10^9}{1^2}$$

$$= 4.05 \text{ volts}.$$

And field is given by Eq. (5.25),

$$E = -\frac{p}{4\pi\varepsilon_0 r^3}(3\cos\theta^2 + 1)^{1/2}$$

since $\theta = 0$,

$$V = \frac{4}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

$$E = \frac{2p}{4\pi\varepsilon_0 r^3} \text{ volt/metre or newtons/coul}$$

$$= \frac{2\times4.5\times10^{-10}\times9\times10^9}{1^3} \text{ volt/metre}$$

$$= 8.1 \text{ volt/metre}$$

(2) When the point lies on the perpendicular bisector of the dipole, $\theta = 90°$, then, $V = 0$, since $\cos 90° = 0$

and $$E = \frac{p}{4\pi\varepsilon_0 r^3}$$

$$= 4.5\times10^{-10}\times9\times10^9 \text{ volt/metre}$$

$$= 4.05 \text{ volt/metre}$$

Comparison of the result for field shows that on axis, field is double.

Example 5.9 *Calculate the electric field in (x, y) plane due to a dipole of dipole moment 5×10^{-10} coul/metre, magnitude and pointing along + y-axis. Find the magnitude and direction of $\overline{E}$ at (1, 1, 0).*

Solution. The field is given by equations (5.30) and (5.32), because $\overline{p}$ point in the + y-direction, *i.e.*

$$\vec{p} = 5\times10^{-10}\,\hat{j}$$

$$\vec{E} = E_x\,\hat{i} + E_y\,\hat{j}\,.$$

Putting E_x and E_y from (5.32) and (5.30), we get

$$\vec{E} = \frac{3pxy}{4\pi\varepsilon_0(x^2+y^2)^{5/2}}\hat{i} + \frac{p}{4\pi\varepsilon_0}\left[\frac{3y^2}{(x^2+y^2)^{5/2}} - \frac{1}{(x^2+y^2)^{3/2}}\right]\hat{j}$$

At point (1, 1, 0), $x = 1, y = 1\ z = 0$

$$\therefore \qquad E\,(1,1,0) = \frac{1}{2^{5/2}}\left[\frac{3p}{4\pi\varepsilon_0}\hat{i} + \frac{p}{4\pi\varepsilon_0}(3-2)\,\hat{j}\right]$$

$$\text{or} \qquad \vec{E}\,(1,1,0) = \frac{1}{2^{5/2}}\frac{p}{4\pi\varepsilon_0}[3\hat{i} + 1\hat{j}]$$

On putting given value of p and $\dfrac{1}{4\pi\varepsilon_0}$, we get

$$= \frac{5\times10^{-10}\times9\times10^{9}}{2^{5/2}}(3\hat{i} + 1\hat{j})$$

$$= \frac{4.5}{2^{5/2}}(3\hat{i} + 1\hat{j})$$

Hence magnitude of field,

$$E\,(1,1,0) = \frac{4.5}{2^{5/2}}\sqrt{3^2+1^2}$$

$$= \frac{4.5}{2^{5/2}}\sqrt{10} \text{ volts/metre}$$

$$= \frac{4.5}{2^{5/2}}\sqrt{10}$$

$$= 2.5 \text{ volts/metre}$$

Direction of E is given by direction cosines which is left for your calculation.

5.11 ELECTRIC MULTIPOLES

Multipole is a collective term for certain point charge systems which , in the order of increasing complexity, are called the monopole (one-pole), the dipole (two-pole), the quadrupole (four-pole), the octupole (eight-pole), etc. These names are derived from the number of point charges, or poles comprising a given multipole. In general the number of poles in a multipole is always 2^n, where n, called the order of multipole, is a positive integer.

For n = 0, there is the monopole

For n = 1, there is the dipole

For n = 2, there is the quadrupole, etc.

ELECTRIC QUADRUPOLE

The quadrupole is an arrangement of two parallel dipoles of opposite polarity, but equal otherwise, separated by small distance. Opposite polarity means their dipole moments are in opposite direction. So we can regard quadrupole as an arrangement of two dipoles. Fig. 5.16 shows few quadrupole arrangements.

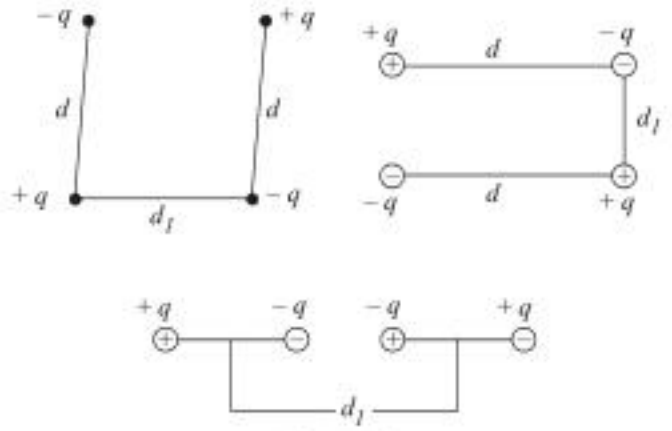


Fig. 5.16

When a multipole consists of point-charges lying on one single axis, the multipole is called an axial multipole or linear dipole.

5.12 POTENTIAL AND FIELD DUE TO A QUADRUPOLE

To calculate the potential and field, we shall consider the simple case of a linear quadrupole, as shown in Fig. 5.17, whose effects at external point do not cancel completely. In Fig. 5.17 the origin of coordinates is taken at the centre of the quadrupole.

The potential at P due to this quadrupole is,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{2q}{r} + \frac{q}{r^2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r}\right)$$

From the figure, we can write by cosine law, $r_1^2 = r^2 + d^2 - 2rd\cos\theta$. Hence,

$$\frac{1}{r_1} = \frac{1}{(r^2 + d^2 - 2rd\cos\theta)^{1/2}}$$

$$= \frac{1}{2}\left[1 + \left(\frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right)\right]^{-1/2}$$

Since $r \gg d$ we can expand by Binomial theorem, as

$$\left[(1-x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8}\right]$$

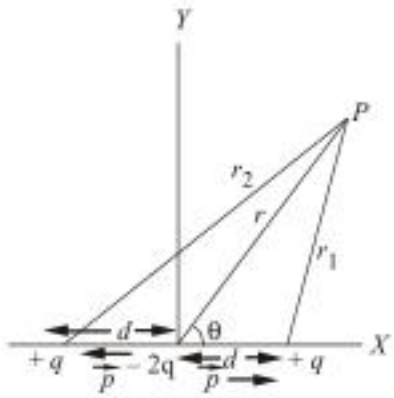


Fig. 5.17

Hence,

$$\frac{1}{r_1}=\frac{1}{r}\left[1-\frac{1}{2}\left(\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right)+\frac{3}{8}\,\frac{4d^2}{r^2}\right]\cos^2\theta+\ldots+\ldots\text{higher order terms}$$

Since $r \gg d$ higher terms are neglected,

$$\frac{1}{r_1}=\frac{1}{r}\left[1+\frac{d^2}{2r^2}\left(3\cos^2\theta-1\right)+\frac{d}{r}\cos\theta\right]$$

Similarly,

$$\frac{1}{r_2}=\frac{1}{r}\left[1+\frac{d^2}{2r^2}(3\cos^2\theta-1)-\frac{d}{r}\cos\theta\right]$$

After substitution in expression for V, we get,

$$\mathrm{V}=\frac{q}{4\pi\varepsilon_0}\,\frac{1}{r}\left[1+\frac{d^2}{2r^2}\left(3\cos^2\theta-1\right)+\frac{d}{r}\cos\theta+1+\frac{d^2}{2r^2}\left(3\cos^2\theta-1\right)-\frac{d}{r}\cos\theta-2\right]$$

$$\mathrm{V}=\frac{qd^2}{4\pi\varepsilon_0 r^3}\left(3\cos^2\theta-1\right) \qquad \ldots(5.34)$$

The product $2qd^2$ is called the *electric quadrupole moment* denoted by Q_d, than

$$\mathrm{V}=\frac{Q_d(3\cos^2\theta-1)}{8\pi\varepsilon_0 r^3} \qquad \ldots(5.35)$$

It is evident from this that the potential due to a linear quadrupole varies inversely as the cube of the distance.

Electric Field

By making the use of relation, $\vec{\mathrm{E}}=-\vec{\nabla}\,\mathrm{V}$, we can calculate the electric field. The x-component in cartesian coordinates is,

$$\mathrm{E}_x=-\frac{\delta\,\mathrm{V}}{\delta x}=-\frac{qd^2}{4\pi\varepsilon_0}\,\frac{\delta}{\delta x}\left(\frac{3\cos^2\theta-1}{r}\right)$$

$$=-\frac{qd^2}{4\pi\varepsilon_0}\,\frac{\delta}{\delta x}\left[\frac{3x^2}{r^5}-\frac{1}{r^3}\right],\ (\text{since } x=r\cos\theta)$$

$$=-\frac{qd^2}{4\pi\varepsilon_0}\left[\frac{6x}{r^5}-\frac{5}{2}\,\frac{2x}{r^7}.3x^2+\frac{3}{2}\,\frac{2x}{r^5}\right] \qquad (\text{since } r^2=x^2+y^2+z^2)$$

$$=-\frac{qd^2}{4\pi\varepsilon_0}\left(\frac{6x}{r^5}-\frac{15x^2}{r^7}+\frac{3x}{r^5}\right)$$

$$=-\frac{3qd^2}{4\pi\varepsilon_0 r^5}\left(\frac{5x^2}{r^2}-3x\right)$$

$$= -\frac{3qd^2}{4\pi\varepsilon_0 r^4}\left(5\cos^3\theta - 3\cos\theta\right)$$

And similarly y-component of field is obtained as

$$\mathrm{E}_y = -\frac{\delta \mathrm{V}}{\delta y} = -\frac{qd^2}{4\pi\varepsilon_0}\frac{\delta}{\delta y}\left(\frac{3\cos^2\theta - 1}{r^3}\right)$$

Proceeding in the same way we get,

$$\mathrm{E}_y = \frac{3qd^2}{4\pi\varepsilon_0 r^5}\left(5\cos^2\theta - 1\right) y$$

and z-component

$$\mathrm{E}_z = -\frac{\delta \mathrm{V}}{\delta z} = \frac{3qd^2}{4\pi\varepsilon_0 r^5}\left(5\cos^2\theta - 1\right) z$$

Therefore, the magnitude of the intensity,

$$\mathrm{E} = (\mathrm{E}_x{}^2 + \mathrm{E}_y{}^2 + \mathrm{E}_z{}^2)^{1/2}$$

$$= \frac{3qd^2}{4\pi\varepsilon_0}\left[\ \frac{1}{r^3}\left(5\cos^3\theta - 3\cos\theta\right)^2 + \frac{1}{r^{10}}\left(5\cos^2\theta - 1\right)^2 y^2 + \frac{1}{r^{10}}\left(5\cos^2\theta - 1\right)^2 z^2\ \right]^{1/2}$$

$$= \frac{3qd^2}{4\pi\varepsilon_0 r^4}\left[\ \left(5\cos^3\theta - 3\cos\theta\right)^2 + \frac{1}{r^2}\left(5\cos^2\theta - 1\right)^2\left(y^2 + z^2\right)\right]^{1/2}$$

$$= \frac{3qd^2}{4\pi\varepsilon_0 r^4}\left[\ \left(5\cos^3\theta - 3\cos\theta\right)^2 + \left(5\cos^2\theta - 1\right)\left(1 - \cos^2\theta\right)\right]^{1/2}$$

Since $y^2 + z^2 = r^2 - x^2$ and $x = r\cos\theta$.

Solving squares and product, we get,

$$\mathrm{E} = \frac{3qd^2}{4\pi\varepsilon_0 r^4}\left[5\cos^4\theta - 2\cos^2\theta + 1\right]^{1/2} \qquad \ldots(5.36)$$

Thus the field of a linear quadrupole varies inversely as the fourth power of the distance.

Field on the Axis of the Quadrupole

If $\theta = 0$, *i.e.* the point lying on the axis, the field is maximum,

$$\mathrm{E} = \frac{6qd^2}{4\pi\varepsilon_0 r^4} \qquad \ldots(5.37)$$

Taking $2qd^2 = \mathrm{Q}_d$, *the quadrupole moment,*

$$\mathrm{E} = \frac{3\,\mathrm{Q}_d}{4\pi\varepsilon_0 r^4}$$

Field along the Equitorial Line

If $\theta = 90°$, that is the point lying on the line (equitorial) perpendicular to the axis of the quadrupole and passing through the centre.

$$E = \frac{3qd^2}{4\pi\varepsilon_0 r^4} \qquad \ldots(5.38)$$

or $$E = \frac{3Q_d}{8\pi\varepsilon_0 r^4}$$ where Q_d is quadrupole monent.

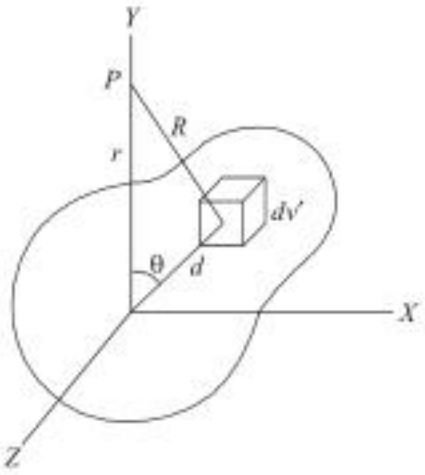


Fig. 5.18

If $\theta = 180°$, then the field is maximum but in a direction of negative x-axis.

5.13 POTENTIAL DUE TO MULTIPOLES

We can conclude from the results given above and establish a general rule. The potential due to a point charge (mono-pole) varies as $\frac{1}{r}$, where r is the distance of the point where potential is calculated. The dipole potential varies as $\frac{1}{r^2}$, the quadrupole potential varies as $\frac{1}{r^3}$. Similarly, it can be shown that the octupole potential varies as $\frac{1}{r^4}$. So we can write in general,

the potential varies as $\frac{1}{r^{n+1}}$ where n is called the order of multipole and

$n = 0$, for a monopole,

$n = 1$, for a dipole

$n = 2$, for a quadrupole and so on.

5.14 MULTIPLE APPROXIMATION TO AN ARBITRARY DISTRIBUTION OF CHARGE, (I.E. POTENTIAL OF AN ARBITRARY CHARGE DISTRIBUTION)

In atoms and molecules, charges are distributed over a small volume. Let us calculate the potential due to charge distribution. The potential due to a point charge q is given by,

$$V_P = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

But if the charge is distributed over a small volume, then according to principle of linear superposition as discussed in article (5.6), the potential at point P, as shown in Fig. 5.18, can be written as,

$$V_P = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho(x', y', z')}{R}\, dv' \qquad \ldots(5.39)$$

where dv' is a small volume around a point (x', y', z') within the charge distribution and R is the distance of the point P from dv'. Suppose the small element dv' is at a distance d from the origin and θ is the angle between d and y-axis, then from the law of cosine, we can write,

$$R = (r^2 + d^2 - 2rd\cos\theta)^{1/2}$$

Therefore,

$$\frac{1}{R} = \left(r^2 + d^2 - 2rd\cos\theta\right)^{-1/2}$$

$$= \frac{1}{r}\left[1 + \left(\frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right)\right]^{-1/2}$$

Since $r \gg d$, hence according to Binomial expansion, we get,

$$\frac{1}{R} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right) + \frac{3}{8}\frac{4d^2}{r^2}\cos^2\theta + \text{higher order terms}\right]$$

$$= \frac{1}{r}\left[1 + \frac{d}{r}\cos\theta + \frac{1}{2}\frac{d^2}{r^2}\left(3\cos^2\theta - 1\right)\right]$$

Putting this value of 1/R in equation (5.39) and taking r out of integration sign because it is constant for fixed position of P. Thus we get,

$$V_P = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_v \left[1 + \frac{d}{r}\cos\theta + \frac{d^2}{r^2}\left(\frac{3\cos^2\theta - 1}{2}\right) + \ldots\right]\rho\, dv'$$

or
$$V_P = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_v \rho\, dv' + \frac{1}{4\pi\varepsilon_0 r^2}\int_v \rho d\cos\theta\, dv' + \frac{1}{4\pi\varepsilon_0 r^3}\int_v \frac{\rho d^2\,(3\cos^2\theta - 1)}{2}\, dv'$$

+... higher order terms

Each integral on the right hand side depends only on the structure of the charge distribution. Let us denote them by $K_0, K_1, K_2, \ldots$, such that,

$$K_0 = \int_v \rho\, dv'$$

$$K_1 = \int_v d\cos\theta\,\rho\, dv'$$

$$K_2 = \int_v \frac{d^2}{2}\left(3\cos^2\theta - 1\right)\rho dv' \text{ and so on. Then,}$$

$$V_P = \frac{K_0}{4\pi\varepsilon_0 r} + \frac{K_1}{4\pi\varepsilon_0 r^2} + \frac{K_2}{4\pi\varepsilon_0 r^3} + \ldots \qquad \ldots(5.40)$$

Thus the potential at any point on the y-axis can be expressed as a power series in $1/r$ with constant coefficients, K_0, K_1, K_2 etc.

The equation (5.40) clearly shows that only the first non-zero term will contribute maximum to the potential for a point at a large distance from the charge distribution.

The first term in the equation (5.40) is merely the potential due to a point charge concentrated at the origin, then, $K_0 = \int_v \rho dv'$, is the net charge in the distribution and is known as monopole moment and the first term of eq. (5.40) is called monopole term.

Similarly, the second in the equation (5.40) is the potential due to a dipole and K_1 is the y-component of the dipole moment.

To make it clear, the product $d\cos\theta\rho dv'$ is the product of $d\cos\theta$ which is equal to y and measures the relative displacement along y-axis of the positive and negative charges in the distribution. The second term is called a dipole term and coefficient K_1 is called the dipole moment.

Similarly, the comparison of third term of Eq. (5.40) with equation (5.34) *i.e.* expression for potential of a quadrupole. We call this term as a quadrupole term and K_2 is called the quadrupole moment and so on for other higher terms.

Thus the potential due to a charge distribution is approximated as the sum of potential due to multipoles of different orders. Such as monopole, dipole, quadrapole and so on as a series.

For neutral charge distribution *i.e.* if we have equal amount of positive and negative charge, then $K_0 = 0$.

If $K_0 = 0$ and $K_1 \neq 0$, the potential along the y-axis varies as $1/r^2$ and value of K_1 is independent of the origin of coordinates. If $K_0 \neq 0$, the first term in series will dominate up to a sufficient large distance and beyond this the potential due to the charge distribution will become effectively equal to $\frac{K_0}{4\pi\varepsilon_0 r}$ *i.e.*, the potential due to a single point-charge.

ELECTRIC FIELD

The electrostatic field due to the charge distribution at a point is obtained by using the relation,

$$\vec{E} = -\vec{\nabla} V$$

5.15 (1) POTENTIAL DUE TO AN INFINITELY CHARGED LONG WIRE

Fig. 5.19(a) shows an infinitely long uniformly charged wire. Let us calculate potential differece between the two points A and B at a distance r_A and r_B from the wire respectively. We can calculate the potential difference by using equation 5.6 *i.e.*,

$$V_B - V_A = -\int_A^B \vec{E}.d\vec{l} = -\int_A^B E\, dl \cos\theta$$

where θ is the angle between $d\vec{l}$ and $\vec{E}$.

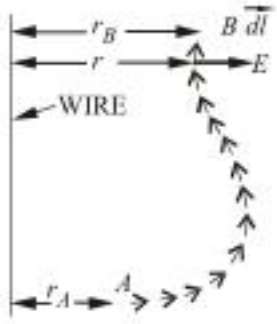


Fig. 5.19 (a)

We have calculated the electric field strength for such distribution by Gauss's law and given by Eq. (4.29) *i.e.*,

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

where λ is the charge per unit length of the wire. Therefore,

$$V_B - V_A = -\int_A^B \frac{\lambda}{2\pi\varepsilon_0 r} dl \cos\theta$$

since $dl \cos\theta = dr$, the component along field E i.e. r, hence,

$$V_B - V_A = -\frac{\lambda}{2\pi\varepsilon_0}\int_{r_A}^{r_B} \frac{dr}{r}$$

or
$$V_B - V_A = -\frac{\lambda}{2\pi\varepsilon_0}\left|\log_e r\right|_{r_A}^{r_B}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r_A}{r_B}\right) \qquad \ldots(5.41)$$

Potential at any point can be calculated by taking reference point A at infinity *i.e.*,

$$r_A = \infty$$

Then we get an infinite value for the potential at all points at finite distance from the wire which is correct since we have assumed an infinite charge on the wire.

However, in practice we always deal with potential difference between points with finite separation and absolute potential has no real significance.

Calculation of E and V

Suppose V is given, then we can calculate E by using $\overline{E} = -\overline{\nabla}\ V$ relation. However, we have to omit the limit of integration in the equation for $V_B - V_A$, because even then potential is a linear function of position. Let V be the potential difference between the points *i.e.* $V_B - V_A = V$

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\int \frac{dr}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\log_e r + \text{constant}$$

Now $\vec{E} = -\vec{\nabla} V$

$$= -\frac{\delta V}{\delta r}\hat{r}$$

where $\hat{r}$ is unit vector along r differentiating V w.r.t. r we get,

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

The magnitude of $\vec{E}$ is

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

This is same as the direct relation (4.29).

(2) Potential and Field due to a Uniformly Charged Disc

Fig. 5.19 (b) shows a flat disc of radius a carrying a positive charge Q spread uniformly over its surface. Let surface charge density *i.e.* charge per unit area be σ coul/met^2. The disc considered here is not of conducting material because if it is so the disc would become a surface of constant potential as the charge on the conducting disc would redistribute itself, crowding more toward the rim of the disc. Thus we have an insulating disc, like a sheet of plastic upon which charge has been sprayed so that energy per square centimetre of the disc has received, and holds fixed, the same amount of charge. Let us calculate the electric potential due to a uniformly charged disc in the following cases.

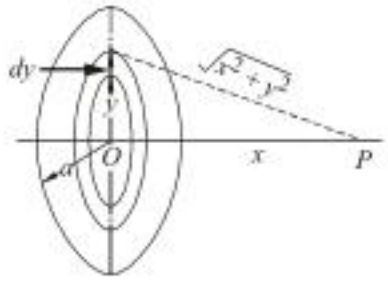


Fig. 5.19(b)

(a) Potential at an Axial Point

Suppose we have disc lying in YZ plane. Consider a point P lying on the x-axis at a distance x from the centre. The x-axis has been taken as the axis of the symmetry of the disc. (*i.e.* an axis passing through the centre of disc and perpendicular to its plane). The disc may be supposed to be formed by a large number of thin concentric ring-shaped elements. Consider one such ring of radius y and thickness dy.

The area of this ring element $= 2\pi\, ydy$.

The charge on this element $dq = 2\pi\, ydy\sigma$.

The distance of all poits lying on the ring from P is $\sqrt{y^2 + x^2}$.

Therefore, the potential at P due to the ring

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2 + y^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\pi\, ydy\,\sigma}{\sqrt{x^2 + y^2}} \qquad \ldots(1)$$

The potential at P due to the whole disc can be obtained by integrating dV from $y = 0$ to $y = a$. Thus,

$$V(x,0,0) = \frac{1}{4\pi\varepsilon_0}\int_0^a \frac{2\pi . y\, dy\, \sigma}{\sqrt{x^2 + y^2}}$$

$$= \frac{\sigma}{2\varepsilon_0}\left|(x^2 + y^2)^{1/2}\right|_0^a$$

$$V(x,0,0) = \frac{\sigma}{2\varepsilon_0}\left[(a^2 + x^2)^{1/2} - x\right] \text{ for } x > 0 \qquad \ldots(2)$$

This gives the potential at all points on the axis of symmetry on one side of the disc.

When the axial point P is along negative direction of x *i.e.* other side of the disc, we have

$$V(x,0,0) = \frac{\sigma}{2\varepsilon_0}\left[(a^2 + x^2)^{1/2} + x\right] \text{ for } x < 0 \qquad \ldots(3)$$

(b) Potential at Long Distance Point

When the point P is at a large distance from the centre on the axis of disc *i.e.* when $x \gg a$ Eq. (2) can be written as,

$$V(x,0,0) = \frac{\sigma}{2\varepsilon_0}\left[(a^2 + x^2)^{1/2} - x\right]$$

$$= \frac{\sigma x}{2\varepsilon_0}\left[\left(1 + \frac{a^2}{x^2}\right)^{1/2} - 1\right]$$

$$\simeq \frac{\sigma x}{2\varepsilon_0}\left[1 + \frac{a^2}{2x^2} - 1\right] \qquad \text{(by Binomial expansion and neglecting higher terms)}$$

$$V(x,0,0) \simeq \frac{\sigma a^2}{4\varepsilon_0 x} \text{ for } x \gg a$$

$$= \frac{\pi a^2 \sigma}{4\pi\varepsilon_0 x} = \frac{1}{4\pi\varepsilon_0}\frac{q}{x} \qquad \ldots(4)$$

where $\pi a^2 \sigma$ is total charge q on the disc and x is the distance of the disc from the point P. *Thus the potential is the same as due to a point-charge q. Therefore for external points far from the disc, it behaves as a point-charge.*

(c) Potential at the Centre of the Disc

At the centre of disc $x = 0$, so that electric poential V_0 at the centre of disc can be obtained by putting $x = 0$ in Eqs. (2) or (3) is given by

$$V_0 = \frac{\sigma a}{2\varepsilon_0} \qquad \ldots(5)$$

(d) Potential at a Point on the Rim of the Disc

To calculate the potential at a point P_1 lying on the edge of the disc, we consider the segment of the ring shown dotted centred at P_1 of radius r and thickness *dr*, as shown in Fig. 5.19 (c).

Length of the segment $= 2r\theta$

The area of the segment $= 2r\theta dr$

The charge on this segment $dq = 2r\theta dr\sigma$

Potential due to this segment at P_1

$$= \frac{1}{4\pi\varepsilon_0}\frac{2r\theta\, dr\sigma}{r}$$

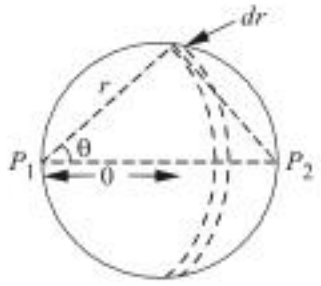


Fig. 5.19 (c)

The total electric potential V_{P1} at P_1 due to this whole charge on the disc

$$V_{P1} = \int_0^{2a} \frac{\sigma\theta}{2\pi\varepsilon_0} dr = \frac{\sigma}{2\pi\varepsilon_0}\int_0^{2a} \theta\, dr$$

But from the right-angled triangle shown in figure,

$$r = 2a\cos\theta$$

$$dr = -2a\sin\theta\, d\theta$$

When $r = 0$, $\cos\theta = 0$, hence $\theta = \frac{\pi}{2}$

and when $r = 2a$, $\cos\theta = 1$, hence $\theta = 0$. Therefore,

$$V_{P1} = \frac{\sigma}{2\pi\varepsilon_0}\int_{\pi/2}^{0} -2a\theta\sin\theta\, d\theta$$

$$= \frac{\sigma.a}{\pi\varepsilon_0}\int_0^{\pi/2} \theta\sin\theta\, d\theta$$

$$= \frac{\sigma.a}{\pi\varepsilon_0}\Big|\ \sin\theta - \theta\cos\theta\ \Big|_0^{\pi/2}$$

$$= \frac{\sigma.a}{\pi\varepsilon_0} \qquad \ldots(6)$$

The comparison of equation (6) with (5) shows that potential falls from the centre of the disc $\left(V_0 = \frac{\sigma a}{2\varepsilon_0}\right)$ to the edge $\left(V_{P1} = \frac{\sigma a}{\pi\varepsilon_0}\right)$.

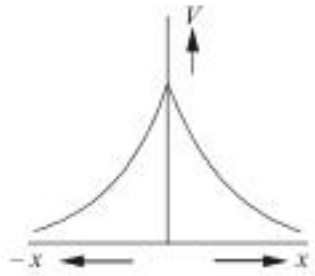


Fig. 5.19 (d)

Fig. 5.19(d) shows the variation of V with radial distance.

Electric Field

The electric field at axial point *P* i.e. the point on *x*-axis at a distance + *x* from the centre of the disc can be calculated directly from eq. (2), *i.e.*,

$$E_{+x} = -\frac{\delta V}{\delta x} = -\frac{\delta}{\delta x}\left[\frac{\sigma}{2\varepsilon_0}\left\{(a^2 + x^2)^{1/2} - x\right\}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{(a^2 + x^2)^{1/2}}\right] \text{ for } x > 0 \qquad \ldots(7)$$

Similarly for point on the $-x$ direction from Eq. (3),

$$E_{-x} = \frac{\sigma}{2\varepsilon_0}\left[1 + \frac{x}{(a^2 + x^2)^{1/2}}\right] \text{ for } x < 0 \qquad \ldots(8)$$

At centre $x = 0$, hence $E_0 = \dfrac{\sigma}{2\varepsilon_0}$

The electric field at a point lying at large distance from the centre on the x-axis of the discs *i.e.* for $x \gg a$ from Eq. (7),

$$E = \frac{\sigma}{2\varepsilon}\left[1 - \frac{x}{x\left(1 + \frac{a^2}{x^2}\right)^{1/2}}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \left(1 - \frac{a^2}{2x^2}\right)\right]$$

$$= \frac{\sigma a^2}{4\varepsilon_0 x^2} = \frac{\pi a^2 \sigma}{4\pi\varepsilon_0 x^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2} \qquad \ldots(10)$$

where $q = \pi a^2 \sigma$ the total charge. The variation in E with distance from the centre of the disc is shown in Fig. 5.19(e).

To calculate the electric field strength at other points normal to the disc, apply Gauss's theorem. If E_+ and E_- are the values of electric field on the two sides of the disc, construct a Gaussian surface as a thin flat box enclosing an areaw A. The sides of this box will not contribute to the flux. That can be achieved by taking sides of box as small as negligible. Hence by Gauss's theroem,

$$AE_+ - AE_- = \frac{\sigma A}{\varepsilon_0}$$

$$E_+ - E_- = \frac{\sigma}{\varepsilon_0}$$

or $$E_+ = E_- = \frac{\sigma}{2\varepsilon_0} \qquad \ldots(11)$$

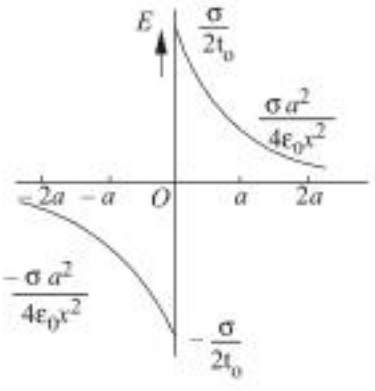


Fig. 5.19 (e)

This is same as infinite sheet of charge. Thus any sheet appears infinite if viewed from points very close to it.

5.15 (3) POTENTIAL DUE TO TWO CONCENTRIC SPHERICAL SHELLS OF CHARGE

Consider two spherical metallic shells having radii r_1 and r_2 ($r_2 > r_1$) with charge q_1 and q_2 uniformly distributed respectively as shown in Fig. 5.19 (f). We have calculated electric field by Gauss's law given in Eq. (4.28 b) *i.e.*,

$$E = \frac{q_1}{4\pi\varepsilon_0 r^2} \text{ between shells } r_1 \ll r \ll r_2 \qquad \ldots(A)$$

and $$E = \frac{1}{4\pi\varepsilon_0}\frac{q_1+q_2}{r^2} \text{ outside both shells } r \gg r_2 \qquad \ldots(B)$$

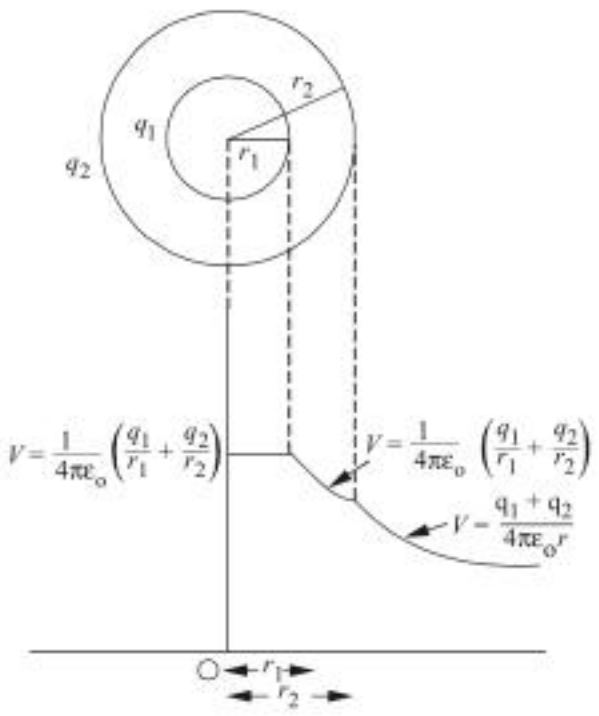


Fig.5.19 (e)

Let reference potential be zero at infinity. There are three cases.

(1) Potential at any point outside the outer shell i.e. $r \geq r_2$

The potential at any point distance r, from the centre to zero potential at infinity is given by,

$$V = -\int_{\infty}^{r} \vec{E}.d\vec{r}, \text{ substituting } E \text{ from above Eq. (B)}$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0}\frac{q_1+q_2}{r^2}dr$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1+q_2}{r} \text{ for } r \geq r_2 \qquad \ldots(1)$$

Therefore, the potential of the outer shell is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1 + q_2}{r}$$

(2) Potential at any point between the two shells

The potential difference between two points distance $r\ (r > r_1)$ and r_2 from the centre of concentric shells is given by,

$$V_r - V_{r2} = -\int_{r2}^{r} \vec{E}.d\vec{r}, \text{ substituting E from above Eq. } A$$

$$= -\int_{r2}^{r} \frac{q_1}{4\pi\varepsilon_0 r^2} dr$$

$$= \frac{q_1}{4\pi\varepsilon_0}\left|\frac{1}{r}\right|_{r2} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r} - \frac{q_1}{r_2}\right]$$

But from Eq. (1), (by putting $r = r_2$) we have

$$V_{r2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 + q_2}{r_2}$$

Therefore,

$$V_r = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r} - \frac{q_1}{r_2}\right] + \frac{1}{4\pi\varepsilon_0}\frac{q_1 + q_2}{r_2}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r} - \frac{q_1}{r_2} + \frac{q_1 + q_2}{r_2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r} + \frac{q_2}{r_2}\right] \text{ for } r_1 \le r \le r_2 \qquad \ldots(2)$$

(3) Potential at any point inside the inner shell

The potential at any point distant $r \le r_1$ from the centre of concentric metallic shells can be directly obtained by putting $r = r_1$ in Eq. (2). Thus,

$$V = \frac{1}{4\pi\varepsilon_0}\left|\frac{q_1}{r_1} + \frac{q_2}{r_2}\right|, r \le r_1 \qquad \ldots(3)$$

This is also the potential at all points inside the inner shell. Thus the potential inside the inner shell remains constant. Thus for case I the two concentric spherical metallic shells behave as a point where total charge $(q_1 + q_2)$ is concentrated at a point. The variation of V as a function of r is illustrated by Fig. 5.19 (f) shown here. It is to be noted that V is continuous since equation (2) equals eq. (3) for $r = r_1$ and eq. (2) equals eq. (1) for $r = r_2$.

If Charges are Equal and Opposite

If the shells carried equal and opposite charges $q_1 = -q_2$, then from Eq. (1) the potential outside the outer shell is zero. But inside the inner and in space between them, potential has finite value. Since inside the inner sphere potential remains constant, it means there will be no electric field $\left(\bar{E} = -\frac{dV}{dr} = 0\right)$. And under $q_1 = -q_2$, the *space between the two shells would have non-vanishing electric fied*.

5.15 (4) POTENTIAL DUE TO TWO CONCENTRIC METAL SHPHERES

Let us consider two concentric metallic spheres having radii R_1 and R_2, $R_2 > R_1$ carrying charge Q_1 and Q_2 respectively [as shown in Fig. 5.19(f) with $r = R$]. This is not any new problem. Since charge is *distributed uniformly, hence each sphere can be regarded as constituted by number of spherical shells*. And by the *principle of superposition* for potential the sum of the potentials due to these shells is the potential of sphere. Therefore, we can write directly from the result of the spherical shell:

1. The potential at any point outside the outer sphere: By eq. (1) of the previous example

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 + Q_2}{R} \text{ for } R \geq R_2 \qquad \ldots(4)$$

Therefore, the potential of the outer sphere is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 + Q_2}{R_2}$$

2. Potential at any point between the two Spheres: By eq. (2) of the previous example,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{R} + \frac{Q_2}{R_2}\right] R_1 \leq R \leq R_2 \qquad \ldots(5)$$

3. Potential at any point inside the inner Sphere: By eq. (3) of the previous example,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{R_1} + \frac{Q_2}{R_2}\right], R \leq R_1 \qquad \ldots(6)$$

This is also the potential at all points inside the inner spheres.

If Charges are Equal and Opposite

If the spheres carried equal and opposite charges $Q_1 = -Q_2$, then from eq. (4), the potential outside the outer sphere is zero. But inside the inner and in space between them potential has value. Since inside the inner sphere potential remains constant, hence field would be zero. But under the condition $Q_1 = -Q_2$, the *space between the two spheres would have non-vanishing electric field* because potential is function of r. Variation is same as shown in Fig. 5.19 (f).

5.16 ELECTROSTATIC POTENTIAL ENERGY OF A SYSTEM OF CHARGES

We define *the electric potential energy of a system of point-charges as the work required to assemble this system of charges by bringing them close together, as in the system from an infinite distance*. However, we assume that initial kinetic energy of the charges is zero at infinity, *i.e.* they are at rest at infinity. Thus, an external work is done against the forces between the charges and this external work done is stored in the system as the electrical potential energy of the configuration (or arrangement) of the charges. As the forces between the charges are of two types, *i.e.* attractive for opposite (or unlike)

charges and repulsive for similar (or like) charges therefore, the work done will be positive in the case of like (or similar) charges and this work done will be negative in the case of unlike (or dissimilar) charges so that, for *similar charges potential energy is positive and for dissimilar it is negative.*

Let us start with two charges q_1 and q_2 separated by distance r as shown in Fig. 5.20. Suppose q_2 is removed to infinity and is at rest. The electric potential at the point where q_2 is situated, due to q_1 is given by Eq. (5.10).

$$V = \frac{1}{4\pi\varepsilon_o}\frac{q_1}{r}$$

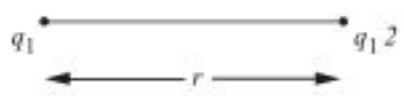


Fig. 5.20

According to definition this is the work done in bringing a unit positive charge from infinity to this point *i.e.* the point where q_2 is situated. Therefore the work required to bring q_2 from infinity to this point is given by,

$$W = V\, q_2$$

Substituting V we get,

$$W = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

Thus, *according to definition of the electric potential energy, this work W is precisely the electric potential energy U of the system* $q_1 + q_2$ *i.e,*

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \qquad \ldots(5.42)$$

We can generalise the equation (5.42) for a *number of charges*. The procedure is to compute the potential energy for every pair of charges separately and to add the results algebraically. Consider the four charges shown in Fig. (5.21).

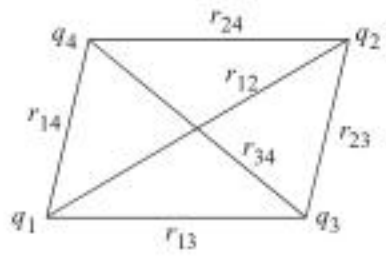


Fig. 5.21

Suppose, initially point charges q_1, q_2, q_3, q_4 were at very greate distance from each other. According to equation (5.42), the work done in bringing q_2 at a distance r_{12} from q_1 which is fixed.

$$U_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

If now another charge q_3 is broght at a distance r_{13} from q_1 and r_{23} from q_2, keeping q_1 and q_2 fixed, then the work done is given by,

$$U_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

Similarly, if the charge q_4 is brought near q_1, q_2 and q_3 at r_{14}, r_{24} and r_{34} respectively as shown in Fig. (5.21), then the work done is given by,

$$U_4 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right)$$

Hence, the total work done in forming the system of four charges as shown in Fig. (5.21) is,

$$U = U_2 + U_3 + U_4$$

or
$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{13}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 r_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right]$$

Now, each term on the right hand side can be divided into two equal terms. After rearrangement the equation reduces,

$$U = \frac{1}{8\pi\varepsilon_0}\left[\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}}\right) + \left(\frac{q_2 q_1}{r_{21}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}}\right) + \left(\frac{q_3 q_1}{r_{31}} + \frac{q_3 q_2}{r_{32}} + \frac{q_3 q_4}{r_{34}}\right) + \left(\frac{q_4 q_1}{r_{41}} + \frac{q_4 q_2}{r_{42}} + \frac{q_4 q_3}{r_{43}}\right)\right] \quad \ldots(5.43)$$

(here $r_{12} = r_{21}, r_{13} = r_{31}$ is written for convenience)

Let us consider the first set of the bracket on R.H.S.

$$\frac{1}{8\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}}\right] = \frac{1}{2} q_1 \left[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{12}} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_4}{r_{14}}\right]$$

$$= \frac{1}{2} q_1 \times \text{potential at the point where } q_1 \text{ is situated, due to } q_2, q_3, q_4.$$

Similarly, the remaining sets of the equation (5.43) for total work done can be shown to be equal to one half of the product of the charge and potential at the point occupied by them due to other charges. Thus, we *can generalise the equation (5.43)* the potential energy of a system of N charges is given by,

$$U = \frac{1}{2}\sum_{i=1}^{i=N} q_i V_i \quad \ldots(5.44)$$

or
$$U = \frac{1}{2}\sum_{i=1}^{i=N} \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_i q_j}{r_{ij}} \quad \ldots(5.45)$$

The meaning of the double summation is that when we start with $i = 1$ and sum over $j = 2, 3, 4, \ldots$N, then we take $i = 2$, and sum over $j = 1, 3, 4 \ldots$N. *The potential energy U of the system of charges is a unique property of the system and is independent of the order in which the charges are assembled.*

5.17 POTENTIAL ENERGY DUE TO A CHARGE DISTRIBUTION

(*Principle of super position for potential energy*)

If instead of point-charges, we have a continuous charge distribution. Suppose ρ be the volume-density of charge, then charge contained in a small volume dv is ρdv. Therefore, the total charge contained in volume, v is,

$$q = \int_v \rho dv$$

If V is the potential at the point occupied by the charge ρdv, then electric potential energy of such distribution is,

$$U = \frac{1}{2}\int_v \rho V \, dv \qquad \ldots(5.46)$$

Using Gauss's law, *i.e.*

$$\text{div } \vec{E} = \frac{\rho}{\varepsilon_0}$$

or $\rho = \varepsilon_0 \text{ div } \vec{E}$

Putting this value of ρ in equation (5.46), we get,

$$U = \frac{\varepsilon_0}{2}\int_v V \text{ div } \vec{E} \, dv$$

But $\text{div } (V\vec{E}) = V \text{ div } \vec{E} + \vec{E} \,.\, \text{div } V.$ (by Eq. (3.54)

or $V \text{ div } \vec{E} = \text{div } (V\vec{E}) - \vec{E} \text{ grad } V,$ (Since *div* = grad)

Hence expression for U becomes after substitution,

$$U = \frac{\varepsilon_0}{2}\int_v \text{div } (V\vec{E}) \, dv - \frac{\varepsilon_0}{2}\int_v \vec{E}.\text{grad } V \, dv$$

Applying divergence theorem to the first term of this expression, for changing volume integral to the surface integral *i.e.*

$$\int_v \text{div } V\vec{E} \, dv = \int_s V\vec{E}.\hat{n} \, dS$$

we get,

$$U = \frac{\varepsilon_0}{2}\int_s V\vec{E}.\hat{n} \, dS - \frac{\varepsilon_0}{2}\int \vec{E}.\text{grad } V \, dv$$

Since potential $V \propto \frac{1}{r}$, field $E \propto \frac{1}{r^2}$ and surface $dS \propto r^2$. The product $V\vec{E}.\hat{n}\,dS = \frac{1}{r}$. The surface integral $\int_s V\vec{E}.\hat{n} \, dS$ becomes vanishingly small, if r is very large, means if the surface S may be chosen at a very great distance from all the charges, then the first term can be neglected and equation reduces,

$$U = \frac{\varepsilon_0}{2}\int_v \vec{E}.\text{grad } V \, dv$$

or $U = \frac{\varepsilon_0}{2}\int_v \vec{E}.\vec{E} \, dv$, since $\vec{E} = -\text{grad } V$

or $U = \frac{\varepsilon_0}{2}\int_v E^2 \, dv$ joules, since $\vec{E} \,.\, \vec{E} = E^2$ $\ldots(5.47)$

The **energy density,** *i.e.* the energy stored per unit volume is,

$$U = \frac{\varepsilon_0}{2} E^2 \text{ joules/met}^2 \qquad \ldots(5.48)$$

This energy is stored in the field. As the electric field E is conservative, the energy stored can be recovered by allowing the charges to go apart.

5.18 ELECTROSTATIC ENERGY OF A UNIFORMLY CHARGED SPHERE

As an example, let us calculate the energy required to assemble a sphere of charge, that is, the work done in assembling charges together from infinity in the form of sphere.

Let the sphere be assembed by number of infinitesimal thin spherical shells. Suppose the radius of sphere at any instant be r, now an infinitesimal shell of charge of thickness dr be added to the sphere of radius r. This process is continued till the radius of the sphere of charge becomes, say, r_0.

The charge contained in the sphere of radius $r = \frac{4}{3}\pi r^2 \rho$ where ρ is volume density of charge.

The charge contained in the shell of thickness $dr = 4\pi r^2 dr\rho$ where $4\pi r^2 dr$ is the volume of the spherical shell.

The work done in adding the charge $4\pi r^2 dr\rho$ to the sphere containing the charge $\frac{4}{3}\pi r^2 \rho$

dW = Potential of the sphere of radius $r \times$ charge contained in the shell

Hence $$dW = \frac{\frac{4}{3}\pi r^2 \rho}{4\pi\varepsilon_0 r} . 4\pi r^2 dr\rho$$

$$= \frac{4\pi}{3\varepsilon_0}\rho^2 r^4 dr \qquad \ldots(5.49)$$

The total work done in assembling the sphere is obtained by integrating the above equation between the limit $r = 0$ and $r = r_0$, thus,

$$W = \int_0^{r_0} \frac{4\pi}{3\varepsilon_0}\rho^2 r^4 dr$$

$$= \frac{4\pi\rho^2 r_0^{\,3}}{15\varepsilon_0} \qquad \ldots(5.50)$$

If Q is the total charge on the sphere, then,

$$Q = \frac{4\pi r_0^{\,3}\rho}{3\varepsilon_0}$$

or $$\rho = \frac{3Q}{4\pi r_0^{\,3}}$$

Putting this value of ρ we get,

$$W = \frac{4\pi}{15\varepsilon_0} r_0^{\,5} \left(\frac{3Q}{4\pi r_0^{\,3}}\right)^2$$

This work done is stored as a potential energy of the system. Hence,

$$U = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 r_0} \qquad \ldots(5.51)$$

Clearly, the energy required to assemble a sphere of charge is directly proportional to the sphere of total charge and inversely proportional to the radius of the sphere.

Electron Volt

The electrical energy is sometimes expressed in electron volts (eV) or million electron volts (MeV). An electron volt is the energy associated with an electron when it moves between two points maintained at a difference of potential of 1 volt, *i.e.*

$$1\text{ eV} = 1.6\times10^{-19}\times 1\text{ joules}$$

or $$1\text{ eV} = 1.6\times10^{-19}\text{ joules}$$

and $$1\text{ MeV} = 10^6\text{ eV}$$

Example 5.10 *Three charges are arranged, as in Fig. (5.22). Calculate their mutual potential energy, where $q = 1.0 \times 10^{-7}$ coul and $a = 10$ cm.*

Solution. Three charges are fixed by external forces, as shown in Fig. 5.22. The total energy of the configuration is the sum of the energies of each pair of charges, *i.e.*

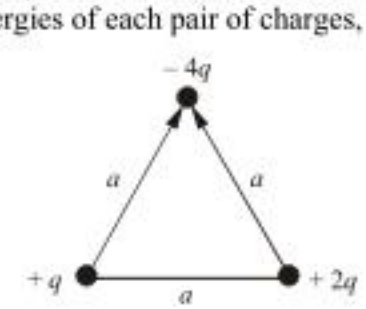


Fig. 5.22

$$U = U_{12} + U_{13} + U_{23}$$

$$U = \frac{1}{2}\sum_{i=1}^{i=N}\frac{1}{4\pi\varepsilon_0}\sum_{j\neq i}\frac{q_i q_j}{r_{ij}} \qquad \text{(by Eq. 5.45)}$$

$$= \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\left[\frac{q(2q)}{a}+\frac{q(-4q)}{a}+\frac{2q(q)}{a}+\frac{2q(-4q)}{a}+\frac{(-4q)q}{a}+\frac{(-4q)(2q)}{a}\right]$$

$$= -\frac{10}{4\pi\varepsilon_0}\frac{q^2}{a}$$

$$= -9\times10^{+9}\text{ Newton/metre}^2/\text{Coul}^2\times\frac{10\times(1.0\times10^{-7}\text{ Coul})^2}{0.10\text{ metre}}$$

$$= -9.0\times10^{-3}\text{ joule}.$$

The negative sign indicates that negative work is required to assemble the charges or $+ 9.0 \times 10^{-3}$ joule of work must be done to dismantle the structure shown in Fig. 5.22.

Example 5.11 *Show that each ion in an infinite line of equally spaced alternate positive and negative ions carrying charges of equal magnitude, has equal potential energy.*

Solution. Suppose q be the charge on each ion and x the spacing between any two adjacent ions.

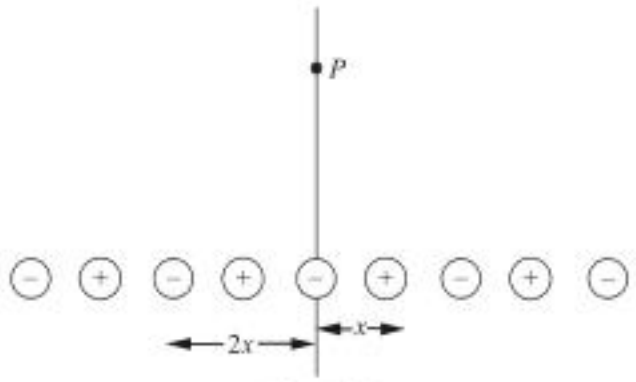


Fig. 5.23

If we consider an ion denoted by P, there are two positive ions at a distance x from it, two negative ions at a dsitance $2x$, two positive ions at a distance $3x$ and so on as shown in Fig. 5.23.

The potential energy of negative ion at P = $(-q)$ × (potential at P due to ions)

The potential at P due to other ions at x, at $2x$... etc.

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{2q}{x} - \frac{2q}{2x} + \frac{2q}{3x} - \frac{2q}{4x} + \cdots\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2q}{x}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2q}{x}\log_e(1+1)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2q}{x}\log_e 2$$

Therefore the potential energy of negative ion at P $= -\dfrac{q^2}{2\pi\varepsilon_0 x}\log_e 2$

Similarly, if we consider a positive ion at P, then the potential at P

$$= -\frac{1}{4\pi\varepsilon_0}\frac{2q}{x}\log_e 2$$

Therefore, the potential energy of positive ion

$$= -\frac{q^2}{2\pi\varepsilon_0 x}\log_e 2$$

Thus the electric potential energy of all the ions is equal.

Exmaple 5.12 *Eight negative charges, each of the value q coul, are placed at the corners of a cube of side a metre, at the centre of which a positive charge of + 2q coul is placed. What is the electric potential energy of the system as shown in Fig. 5.24?*

Solution. The diagonal $HC = \sqrt{a^2 + a^2}$

$$= a\sqrt{2}$$

and diagonal, $HB = \sqrt{HC^2 + CB^2}$

$$= \sqrt{2a^2 + a^2}$$

$$= a\sqrt{3}$$

Therefore, $\text{HO} = \text{OB} = \frac{\text{HB}}{2} = \frac{a\sqrt{3}}{2}$

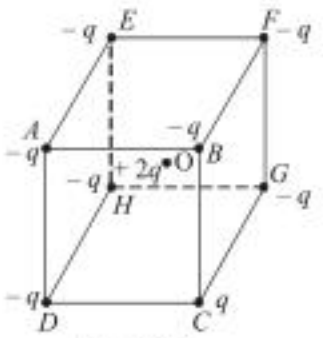


Fig. 5.24

The central charge $2q$ makes 8 similar pairs with the charges at corners. Their potential energy,

$$2q \times \frac{1}{4\pi\varepsilon_0} \frac{8(-q)}{\frac{a\sqrt{3}}{2}} = -\frac{8q^2}{\pi\varepsilon_0 a\sqrt{3}} \text{ joules} \qquad \ldots(1)$$

Further, each charge is at a distance a metre from three neighbouring charges, such as A is at a distance a metre from E. B and D and so on. However, each pair is repeated when we consider all pairs such as AE and EA.

Thus the total number of such pairs is, $\frac{8 \times 3}{2} = 12$

The potential energy of these pairs,

$$= \frac{12 \times (-q) \times (-q)}{4\pi\varepsilon_0 a}$$

$$= +\frac{3q^2}{\pi\varepsilon_0 a} \text{ joules} \qquad \ldots(2)$$

Further, each charge is at a distance $a\sqrt{2}$ metre from three diagonally opposite charges, such as A is at a distance $a\sqrt{2}$ from C, F and H and similarly H is at a distance $a\sqrt{2}$ from A, C and F and so on. Each pair is repeated here also. So, number of such pair is

$$\frac{8 \times 3}{2} = 12$$

Therefore, the potential energy of these pairs,

$$= \frac{12 \times (-q) \times (-q)}{4\pi\varepsilon_0 a\sqrt{2}}$$

$$= \frac{3}{\sqrt{2}} \frac{q}{\pi\varepsilon_0 a} \text{ joules} \qquad \ldots(3)$$

Finally, A is at a distance $a\sqrt{3}$ from G and other such pairs are EC, HB and DF. Total such pairs are 4.

$$\text{Potential energy of these pairs} = \frac{4 \times (-q) \times (-q)}{4\pi\varepsilon_0 a\sqrt{3}}$$

$$= \frac{q^2}{\pi\varepsilon_0 a\sqrt{3}} \text{ joules} \qquad \ldots(4)$$

Therefore, the total potential energy of the system

$$= -\frac{8q^2}{\pi\varepsilon_0 a\sqrt{3}} + \frac{3q^2}{\pi\varepsilon_0 a} + \frac{eq^2}{\pi\varepsilon_0 a\sqrt{2}} + \frac{q^2}{\pi\varepsilon_0 a\sqrt{3}}$$

$$= \frac{q^2}{\pi\varepsilon_0 a}\left[-\frac{8}{\sqrt{3}} + 3 + \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right]$$

$$= \frac{q^2}{\pi\varepsilon_0 a} \times 1.1$$

$$= \frac{4.4}{4\pi\varepsilon_0} \times \frac{q^2}{a}$$

$$= 9 \times 10^9 \times 4.4 \times \frac{q^2}{a} \text{ joules}$$

$$= 39.6 \times 10^9 \times \frac{q^2}{a} \text{ joules.}$$

Example 5.13 *Calculate approximately the electrical energy released when a uranium nucleus* $92U^{238}$ *breaks into two equal parts. Consider the nucleus as a sphere of uniform charge density. The radius of nucleus* $r = 1.2 \times 10^{-15} A^{1/3}$ *metres where A is the mass number of the nucleus.*

Solution. Charge on uranium nucleus = $92 \times 1.6 \times 10^{-19}$ Couls,

Radius = $1.2 \times 10^{-15} \times 238^{1/3}$ metres

Using equation (5.51) the electrostatic potential energy,

$$= \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{(92 \times 1.6 \times 10^{-19})^2}{1.2 \times 10^{-15} \times 238^{1/3}}$$

$$= \frac{3}{5}\frac{9 \times 10^9 \times 92 \times 92 \times 1.6 \times 1.6 \times 10^{-38}}{1.2 \times 10^{-15} \times 238^{1/3}}$$

$$= 0.157 \times 10^{-9} \text{ joules.}$$

5.19 ELECTRIC CAPACITY

When a charge is given to a conductor, its potential rises. There is a *fixed relationship between the two, i.e.*

$$Q \propto V \text{ or } Q = CV$$

or $$\frac{Q}{V} = C \qquad \ldots(5.52)$$

where C is a constant and is known as the capacity or capacitance of the conductor. In other words, the ratio of charge given to the potential of the conductor, is a constant and is called the capacity of the conductor.

If $V = 1$, then $C = Q$

Thus, the capacity of a conductor is the amount of charge required to raise the potential of the conductor by unity.

Unit

The practical and RMKS system unit of the capacitance is *farad which is the capacity of a conductor, the charge of one coulomb raises its potential to one volt, i.e.*

$$1 \text{ Farad} = \frac{1 \text{ Coulomb}}{1 \text{ Volt}} = \frac{3 \times 10^9 \ e.s.u \text{ of charge}}{\frac{1}{300} e.s.u. \text{ of potential}}$$

$$= 9 \times 10^{11} \ e.s.u. \text{ of capacity (cms)}$$

Capacity of a spherical conductor

Let us consider an isolated conducting sphere of radius r. Suppose it is given a positive charge q which is distributed uniformly over its surface.

The field due to charged sphere is same as the whole charge is concentrated at its centre *i.e.* the field on surface,

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2} \hat{r} \qquad \text{(See eq. (4.26))}$$

and the potential at a point on the surface

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

Hence $\frac{q}{V} = 4\pi\varepsilon_0 r = C$ (Capacitance)

or $C = 4\pi\varepsilon_0 r$ farads.

Thus the capacity of an *isolated sphere is directly proportional to its radius*.

Energy of a Charged Conductor

Suppose V is the potential at any instant, then the work done in giviing charge dq to the conductor is $V dq$. Hence the total work done in charging the conductor is given by,

$$\text{Work done} = \int_{q=0}^{q=q} V \, dq = \int_0^q \frac{q}{C} dq : \text{since } q = CV$$

$$= \frac{1}{C} \frac{q^2}{2} = \frac{1}{2} CV^2 = \frac{1}{2} q V \qquad \ldots(5.53)$$

5.20 SHARING OF CHARGES

Consider two conductors having capacitances C_1 and C_2 with a charge q_1 and q_2 uniformly distributed respectively. When they are allowed to touch each other or connected together by a wire as shown in Fig. 5.25, charge will flow from the one at higher potential V_1 to the one at lower potential V_2 till the potentials become equal. If now they are disconnected, the charges on the two will be q_1' and q_2' respectively, which can be calculated as follows.

We have,

$$V_1 = \frac{q_1}{C_1} \text{ and } V_2 = \frac{q_2}{C_2}$$

and the total charge,

$$q = q_1 + q_2 = C_1 V_1 + C_2 V_2$$

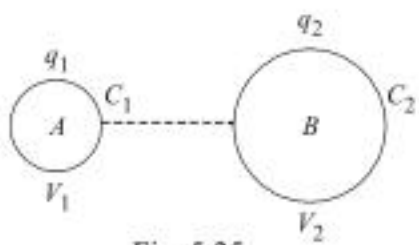


Fig. 5.25

Now, if V is the common polential of the two conductors, then,

$$q = (C_1 + C_2)\, V, \text{ since } C = C_1 + C_2$$

Therefore,

$$(C_1 + C_2)\, V = C_1 V_1 + C_2 V_2$$

or
$$V = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2} \qquad \ldots(5.54)$$

Now, $q_1' = C_1 V = \dfrac{C_1}{C_1 + C_2} (C_1 V_1 + C_2 V_2)$

and $q_2' = C_2 V = \dfrac{C_2}{C_1 + C_2} (C_1 V_1 + C_2 V_2)$

Loss of Energy

Total energy of the two conductors before they are joined,

$$= \frac{1}{2} C_1 V_1^{\,2} + \frac{1}{2} C_2 V_2^{\,2}, \qquad \text{(See Eq. (5.53))}$$

Total energy after sharing of charges,

$$= \frac{1}{2} (C_1 + C_2) V^2$$

$$= \frac{1}{2} (C_1 + C_2) \frac{(C_1 V_1 + C_2 V_2)^2}{(C_1 + C_2)^2}$$

$$= \frac{1}{2} \frac{(C_1 V_1 + C_2 V_2)^2}{(C_1 + C_2)}$$

The loss of energy due to sharing of charges,

$$= \frac{1}{2} C_1 V_1^{\,2} + \frac{1}{2} C_2 V_2^{\,2} - \frac{1}{2} \frac{(C_1 V_1 + C_2 V_2)^2}{C_1 + C_2}$$

$$= \frac{1}{2 (C_1 + C_2)} \left[C_1 V_1^{\,2} (C_1 + C_2) + C_2 V_2^{\,2} (C_1 + C_2) - (C_1 V_1 + C_2 V_2)^2 \right]$$

$$= \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} (V_1 - V_2)^2 \qquad \ldots(5.55)$$

This will always be positive whether $V_1 > V_2$ or $V_1 < V_2$.

Thus there is always loss of electrostatic energy when two conductors share their charges. The lost energy reappears as heat in the connecting wire or part of it as light and sound when a spark passes between the two conductors.

5.21 A DIPOLE IN AN ELECTRIC FIELD

(i) In uniform electric field

Fig 5.26 shows an electric dipole *i.e.* two charges $+q$ and $-q$ are separated by distance d placed in a uniform electric field $\vec{E}$. The dipole axis (or dipole moment $\vec{p}$) is making an angle θ with this field. Two equal and opposite forces $\vec{F}$ and $-\vec{F}$ act on $+q$ and $-q$ charge respectively as shown in Fig. 5.26 and $\vec{F} = q\vec{E}$.

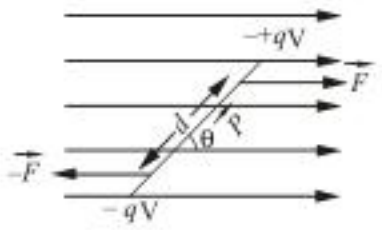


Fig. 5.26

The net force is zero, but there is a moment of couple (or torque) given by,

$$\tau = q\,E\,d\sin\theta$$

$$= p\,E\sin\theta \qquad \ldots(5.56)$$

where, p (dipole moment) $= qd$

or $$\tau = \vec{p} \times \vec{E} \qquad \ldots(5.57)$$

Thus an electric dipole placed in an external electric field experiences a torque which tends to align it with the field.

In order to change the orientation of an electric dipole in an external field, the external agent must do certain work. This work is stored as potential energy in the system.

Now work done in rotating the dipole through an additional angle $d\theta$,

$$dW = p\,E\sin\theta\,d\theta$$

Therefore, the work done in rotating the dipole from an initial position θ_1 to the final position θ_2 is

$$W = \int dW = pE\int_{\theta_1}^{\theta_2} \sin\theta\,d\theta$$

$$= -pE\,(\cos\theta_2 - \cos\theta_1)$$

Clearly this work done is stored up in the form of potential energy of the dipole in the final position. As we are interested only in change in potential energy, for convenience, we define the potential energy of a dipole making an angle 90° with the electric *field as zero*. This gives,

$$U = -pE\cos\theta \qquad \ldots(5.58)$$

or $$U = -\vec{p}\,.\,\vec{E} \qquad \ldots(5.59)$$

(ii) In Non-Uniform Field

Let us consider a small dipole in a non-uniform field, such as the field due to a point charge. Here, the forces acting on the two charges of the dipole are not equal in magnitude and also their directions are not exactly opposite to each other. For a short dipole, the torque is still given by eq. (5.57) where $\vec{E}$ is the field at the centre of the dipole because the length of the dipole is very small.

But the dipole experiences a net translatory force in addition to the torque. In the calculation of net translatory force on the dipole, the variation of electric field from one pole to other is to be considered. Let E_x, E_y and E_z be the component of E at (x, y, z) point occupied by $-q$ charge of the dipole and E'_x E'_y and E'_z be those of E at the position $(x + dx, y + dy, z + dz)$ occupied by $+q$ charge of dipole, then,

$$E'_x = E_x + dE_x$$

$$= E_x + \frac{\delta E_x\, dx}{\delta x} + \frac{\delta E_x}{\delta y} dy + \frac{\delta E_x}{\delta z} dz$$

the net force on the dipole in the x-direction,

$$F_x = q E'_x - q E_x$$

$$= q\left(\frac{\delta E_x}{\delta x} dx + \frac{\delta E_x}{\delta y} dy + \frac{\delta E_x}{\delta z} dz\right)$$

Sicne p_x the x-component of dipole moment is given by, $p_x = q\,dx$, similarly $p_y = q\,dy$ and $p_z = q\,dz$.

Hence $$F_x = p_x \frac{\delta E_x}{\delta x} + p_y \frac{\delta E_x}{\delta y} \times p_z \frac{\delta E_x}{\delta z}$$

or $$\vec{F}_x = (p_x \hat{i} + p_y \hat{j} + p_z \hat{k}) . \left(\hat{i} \frac{\delta E_x}{\delta x} + \hat{j} \frac{\delta E_x}{\delta y} + \hat{k} \frac{\delta E_x}{\delta z}\right)$$

$$= \vec{p} . \nabla \vec{E}_x$$

Similarly, the y and z-components of the net force on the dipole are given by,

$$\vec{F}_y = \vec{p} . \nabla E_y$$

and $$\vec{F}_z = \vec{p} . \nabla E_z$$

Therefore, the total translatory force experienced by a dipole in a non-uniform field is given by,

$$\vec{F} = \hat{i} F_x + \hat{j} F_y + \hat{k} F_z$$

$$= \vec{p} . \nabla E_x \hat{i} + \vec{p} . \nabla E_y \hat{j} + \vec{p} . \nabla E_z \vec{k}$$

$$= \vec{p} . \nabla \vec{E} \qquad \ldots(5.60)$$

where, $E = i \hat{E}_x + j \hat{E}_y + k \hat{E}_z$.

Example 5.14 *An electric dipole consists of two opposite charges of magnitude $q = 2.0 \times 10^{-6}$ Coul separated by 2.0 cm. The dipole is placed in an external field of 2.0×10^5 Newton/Coul. Calculate (1) the maximum torque on the dipole. (2) the work done to turn the dipole through 180°, starting from a position $\theta = 0$.*

Solution. (1) The torque is given by Eq. (5.57) *i.e.*

$$= \vec{p} \times \vec{E}$$

$$= pE \sin\theta$$

When $\theta = 90°$ the torque is maximum, therefore the maximum torque,

$$\tau_{max} = pE \sin 90°. \text{ (since } p = dq)$$

$$= (2.0 \times 10^{-6} \text{ Coul}) (0.02 \text{ metre}) (2.0 \times 10^{5} \text{ Nt/Coul})$$

$$= 0.08 \times 10^{-1} \text{ newton/ metre}$$

$$= 8.0 \times 10^{-3} \text{ newton/ metre}$$

(2) The work done to turn through 180° is the difference in potential energy between the two positions i.e. $\theta = 180°$ and $\theta = 0$.

From equation (5.58), the potential energy at $\theta = 180$.

$$U_{180} = -pE \cos 180°$$

$$= pE$$

and potential energy at $\theta = 0$

$$U_0 = -pE \cos 0°$$

$$= -pE$$

therefore, Work $= U_{180} - U_0 = 2\,pE$

$$= 2\,(2.0 \times 10^{-6} \text{ Coul}) (0.02 \text{ metre}) (2.0 \times 10^{5} \text{ Newton/Coul})$$

$$= 16 \times 10^{-3} \text{ Jouls}.$$

Example 5.15 *A dipole of strength p is placed at a distance r from the axis of an infinitely long and uniformly charged wire having a charge λ per unit length. Calculate the torque and force acting on the dipole, if it lies in the plane containing the axis of the wire and the dipole axis is making an angle θ with the radial line from the wire, as shown in Fig. 5.27.*

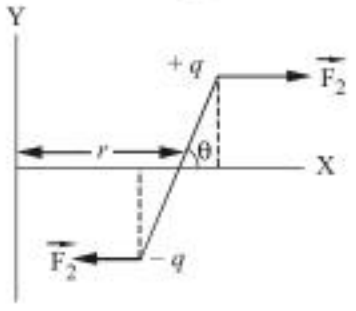


Fig. 5.27

Solution. The electric field due to uniformly charged wire is given by,

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

and the torque acting on the dipole is given by equation (5.57), *i.e.*,

$$\tau = \vec{p} \times \vec{E}$$

$$= pE \,.\, \sin\theta$$

Substituting $E = p \,.\, \frac{\lambda}{2\pi\varepsilon_0 r} \sin\theta$

The force on the positive pole *i.e.* on charge q is,

$$F_1 = qE_1$$

$$= q \cdot \frac{\lambda}{2\pi\varepsilon_0\left(r + \frac{d}{2}\cos\theta\right)}$$

Similarly, the force on the negative pole is,

$$F_2 = q\,E_2$$

$$= q \cdot \frac{\lambda}{2\pi\varepsilon_0\left(r - \frac{d}{2}\cos\theta\right)}$$

Therefore, the net force on the dipole towards the wire is,

$$F_2 - F_1 = \frac{q\lambda}{2\pi\varepsilon_0}\left[\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right]$$

$$= \frac{q\lambda}{2\pi\varepsilon_0}\frac{d\cos\theta}{r^2 - \frac{d^2}{4}\cos^2\theta}$$

Since the dipole moment $p = qd$, hence,

$$F_2 - F_1 = \frac{p\lambda\cos\theta}{2\pi\varepsilon_0\left(r^2 - \frac{d^2}{4}\cos\theta\right)}$$

$$= \frac{p\lambda\cos\theta}{2\pi\varepsilon_0 r^2}, \text{ since } r \gg d$$

Example 5.16 *Deduce the formulae for the torque and translatory force experienced by a dipole placed in a field of a point-charge.*

Solution. Fig. 5.28 shows an electric dipole of length d and dipole moment dq, placed in the field of a point-charge + Q at a distance r from Q and inclined at an angle θ with x-axis.

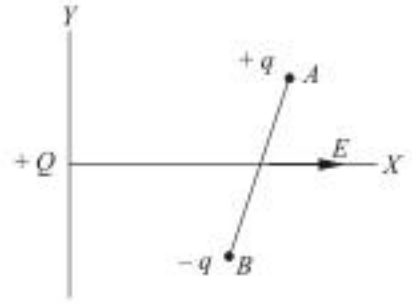


Fig., 5.28

The torque acting on dipole is given by Eq. (5.57),

$$\tau = \vec{p} \times \vec{E}$$

$$= p\,\mathrm{E}\sin\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{pQ}{r^2}\sin\theta, \text{ since } \mathrm{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

If the point-charge + Q is considered to be at origin, the field at any point in the $x-y$ plane is,

$$\mathrm{E} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{Q}}{(x^2+y^2)}, \text{ since } r^2 = x^2 + y^2$$

The x-component is,

$$\mathrm{E}_x = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{Q}\cos\theta}{(x^2+y^2)} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{Q}x}{(x^2+y^2)^{3/2}}$$

Since $\cos\theta = \frac{x}{r}$ and the y-component is,

$$\mathrm{E}_y = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{Q}\sin\theta}{(x^2+y^2)} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{Q}y}{(x^2+y^2)^{3/2}}$$

Since the dipole is placed in x-y plane, the z-component of dipole moment is zero, i.e. $p_z = 0$, and $\mathrm{E}_z = 0$, as

$$\frac{\delta \mathrm{E}_z}{\delta x} = \frac{\delta \mathrm{E}_z}{\delta y} = 0$$

The x-component of the netforce is obtained by equation (5.60), *i.e.*

$$\mathrm{F}_x = \vec{p}\cdot\nabla\,\mathrm{E}_x$$

$$= p_x\frac{\delta \mathrm{E}_x}{\delta x} + p_y\frac{\delta \mathrm{E}_x}{\delta y} + p_z\frac{\delta \mathrm{E}_x}{\delta_z}$$

Substituting E_x, E_y from above,

$$\mathrm{F}_x = \frac{1}{4\pi\varepsilon_0}p_x\frac{\delta}{\delta y}\left\{\frac{\mathrm{Q}x}{(x^2+y^2)^{3/2}}\right\} + \frac{1}{4\pi\varepsilon_0}p_y\frac{\delta}{\delta y}\left\{\frac{\mathrm{Q}x}{(x^2+y^2)^{3/2}}\right\} + 0, \text{ since } \frac{\delta \mathrm{E}_x}{\delta z} = 0$$

$$= \frac{1}{4\pi\varepsilon_0}\mathrm{Q}\,p_x\left\{\frac{y^2-2x^2}{(x^2+y^2)^{6/2}}\right\} + \frac{1}{4\pi\varepsilon_0}\mathrm{Q}\,p_y\left\{\frac{3yx}{(x^2+y^2)^{5/2}}\right\}$$

For the dipole shown in Fig. 5.28,

$$y = 0,\ x = r \text{ and } p_x = p\cos\theta$$

Hence, $$\mathrm{F}_x = -\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{Q}p\cos\theta\,2r^2}{r^5}$$

$$= -\frac{1}{2\pi\varepsilon_0}\frac{\mathrm{Q}p\cos\theta}{r^3} \qquad \ldots(1)$$

Similarly the y-component of the force is given by,

$$\mathrm{F}_y = \vec{p}\,.\nabla\,\mathrm{E}_y$$

or $$F_y = p_x \frac{\delta E_y}{\delta x} + p_y \frac{\delta E_y}{\delta y} + p_z \frac{\delta E_y}{\delta z}$$

$$= \frac{1}{4\pi\varepsilon_0} p_x \frac{\delta}{\delta x}\left\{\frac{Qy}{(x^2+y^2)^{3/2}}\right\} + \frac{1}{4\pi\varepsilon_0} p_y \frac{\delta}{\delta y}\left\{\frac{Qy}{(x^2+y^2)^{3/2}}\right\} + 0$$

$$= \frac{1}{4\pi\varepsilon_0} p_z Q\left\{-\frac{3xy}{(x^2+y^2)^{5/2}}\right\} + \frac{1}{4\pi\varepsilon_0} p_y Q\left\{\frac{x^2-2y^2}{(x^2+y^2)^{5/2}}\right\}$$

Since $y = 0, x = r$ and $p_y = p\sin\theta$

$$F_y = \frac{1}{4\pi\varepsilon_0}\frac{pQ\sin\theta}{r^3} \qquad \ldots(2)$$

Example 5.17 *A dipole of strength 1.6 × 10 $^{-29}$ Cou.. metre is situated at 5 × 10^{-10} metre from a nucleus of charge + 3e. Find the force and torque on the dipole:*

(1) When it is oriented along a radius from the nucleus, positive and further away,

(2) When it is at right angles to the radius,

(3) When it makes an angle of 45° with the radius.

Solution. Given, $p = 1.6\times10^{-29}$ coul. metre

$$Q = 3\times1.6\times10^{-19}\text{ Coul}$$

$$r = 5\times10^{-10}\text{ metre}$$

Therefore,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

$$= 9.0\times10^{-9}\text{ Newton.metre}^2/\text{Coul}^2\ \frac{3\times1.6\times10^{-19}\text{ Coul}}{(5\times10^{-10}\text{ m})^2}$$

$$= \frac{9\times10^9\times3\times1.6\times10^{-19}}{25\times10^{-20}}\text{ Newton/Coul or Volt/metre}$$

(1) When it is oriented along radius, positive and further away, in this case $\theta = 0$, hence $\cos\theta = 1$ and $\sin\theta = 0$.

Therefore, from Eq. (1) of Example 5.16,

$$F_x = -\frac{1}{2\pi\varepsilon_0}\frac{Q\,p\cos\theta}{r^3}$$

or $$F_x = -\frac{1}{4\pi\varepsilon_0}\frac{2Q\,p\cos\theta}{r^3}$$

$$= 9\times10^9\text{ Newton. metre}^2/\text{Coul}^2\times\frac{2\times3\times1.6\times10^{-19}\text{ Coul}\times1.6\times10^{-29}\text{ Coul metre}}{(5\times10^{-10}\text{ m})^3}$$

$$= -1.1\times10^{-9}\text{ newtons.}$$

The negative sign indicates that force F_x is towards the charge + 3*e i.e.*, nucleus.

$$F_y = 0 \qquad \text{(From Eq. (2) of Example 5.16)}$$

$$\text{Torque} = pE\sin\theta$$

$$= 0$$

(2) When dipole axis is at right angle to the radius *i.e.* $\theta = 90$. Then, $\cos\theta = 0$ and $\sin\theta = 1$. Therefore,

$$F_x = -\frac{1}{2\pi\varepsilon_0}\frac{Q\,p\cos\theta}{r^3} = 0,$$

and, $$F_y = \frac{9\times10^9\ \text{N.m}^2/\text{Coul}^2 \times 1.6\times10^{-29}\ \text{Coul.m} \times 3\times1.6\times10^{-19}\ \text{Coul}\times1}{(5\times10^{-10}\ \text{m})^2}$$

$$= 0.55\times10^{-9}\ \text{newtons}$$

Further, $\tau = pE\sin\theta$

$$= \frac{1.6\times10^{-29}\ \text{Coul.m}\times9\times10^9\times3\times1.6\times10^{-19}\ \text{Newton.Coul}}{25\times10^{-20}}$$

$$= 2.764\times10^{-19}\ \text{Newtons.m}$$

(3) In this case $\theta = 45°$. Therefore,

$$\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$$

$$F_x = -\frac{1}{2\pi\varepsilon_0}\frac{Q\,p\cos\theta}{r^3}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{Q\,p\cos\theta}{r^3}$$

$$= \frac{-9\times10^9\times2\times3\times1.6\times10^{-19}\times1.6\times13^{-29}}{(5\times10^{-10})^3}\frac{1}{\sqrt{2}}$$

$$= -0.777\times10^{-9}\ \text{newtons.}$$

and, $$F_y = \frac{1}{4\pi\varepsilon_0}\frac{pQ\sin\theta}{r^3}$$

$$= \frac{9\times10^9\times3\times1.6\times10^{-19}\times1.6\times10^{-19}}{(5\times10^{-10})^2}\frac{1}{\sqrt{2}}\ \text{Newton}$$

$$= 0.381\times10^{-19}\ \text{Newton}$$

Torque $\tau = pE\sin\theta$

$$= \frac{1.6\times10^{-29}\times9\times10^9\times3\times1.6\times10^{-19}}{(25\times10^{-20})}\frac{1}{\sqrt{2}}\ \text{Newton.metre}$$

$$= 1.9\times10^{-19}\ \text{Newton.metre}$$

5.22 ELECTRICAL IMAGES

We have discussed various methods for calculating electric field $\bar{E}$ and electric potential V at a point. But often, we found many complicated problems where it is not easy to find $\bar{E}$ and V. Such as the problems involving boundary surfaces on which the potential or the surface charge density is specified. *Lord Kelvin* suggested a very useful method to solve such problems on the basis of *distribution of charges*. This method is *known as the method of electrical images due to optical analogy i.e.* mirror image.

The method is as follows. Suppose, we have a distribution of charges and conductors. Search out an arrangement of imaginary charges not located within the region in which the potential is to be determined, such that this imaginary distribution of charges together with the actual specified distribution reduces potential to fixed value on all conductors. These imaginary point charges are referred as electrical image of the original charges.

The necessary conditions for using electrical charge method are followings:

(1) The potential of the conductor must remain the same.

(2) The potential at infinite distant points must remain the same.

(3) The electric flux over any closed surface in the original field must remain the same.

5.23 EXAMPLES OF ELECTRICAL IMAGES METHOD

(1) A Point Charge Near a Conducting Plane

As an example, let us consider a charge $+q$ placed at A and these be a conducting plane surface BC. Due to induction negative charge will appear over the surface of BC, but we do not know the surface density of charge over BC, hence it is very difficult to calculate the V and $\bar{E}$ at any point P.

But the method of images makes it possible because the whole of this induced charge can be replaced by a single point charge $(-q)$ at A' *i.e.* by electrical image of $+q$ at A as shown in Fig. 5.29.

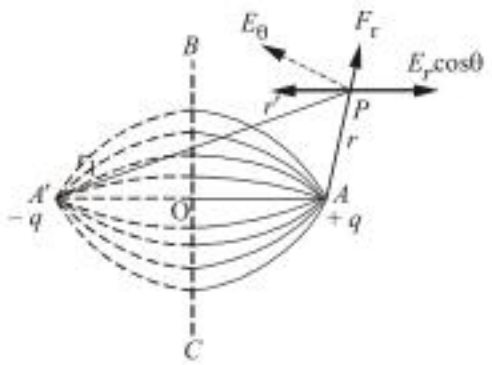


Fig. 5.29

Boundary Conditions

Let two point charges $+q$ and $-q$ be situated at a distance $2a$. Then potential at P due to these charges is given by,

$$V_p = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{q}{r'}\right)$$

in which r and r' are the distances of P from $+q$ and $-q$ respectively. If point P lies on the mid plane BC, then $r = r'$ and potential there is zero. So the *boundary condition is that the potential is zero all over the infinite plane BC.*

In the case of a point-charge near a conducting plane, the condition is satisfied by earthing the plane BC. Hence, the potential is zero. Thus we conclude that if a point-charge + q is situated at a distance a from an infinite conducting plane which has been earthed, the potential in the region on the side of BC, where + q charge is situated, is the same as if plane BC were removed and replaced by a charge – q at a distance a on the left side of BC on the perpendicular dropped from + q on BC. The charge – q at A' is called the electrical image of the charge + q at A, as shown in Fig. 5.29.

Thus an *electrical image may be defined as a point charge on one side of surface, which will produce on the other side of the surface, the same electric field as is produced by the actual electrification of the surface. The actual electrification is then ignored and the image or images are used instead of plane to find the field.*

(i) Potential at any point P

From triangle APA' $r' = (r^2 + 4a^2 + 4ar\cos\theta)^{1/2}$.

Hence potential at P

$$V_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{(r^2 + 4a^2 + 4ar\cos\theta)^{1/2}}\right] \qquad \ldots(5.61)$$

(ii) Electric Field at a Point P

The intensity of electric field at P in the direction of AP *i.e.* r is,

$$E_r = -\frac{\delta V_p}{\delta r}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{r + 2a\cos\theta}{(r^2 + 4a^2 + 4ar\cos\theta)^{3/2}}\right] \qquad \ldots(5.62)$$

And the intensity at P in a direction perpendicular to r is,

$$E_\theta = -\frac{1}{r}\frac{\delta V_p}{\delta\theta}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{2a\sin\theta}{(r^2 + 4a^2 + 4ar\cos\theta)^{3/2}} \qquad \ldots(5.63)$$

The resultant electric field at P normal to the plane BC *i.e.* parallel to AA′ according to Fig. 5.29 is given by,

$$E_{normal} = E_r\cos\theta - E_\theta\sin\theta \qquad \ldots(5.64)$$

Putting E_r and E_θ from above, we get

$$E_n = \frac{q}{4\pi\varepsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2a\cos^2\theta}{(r^2 + 4a^2 + 4ar\cos\theta)^{3/2}} - \frac{2a\sin^2\theta}{(r^2 + 4a^2 + 4ar\cos\theta)^{3/2}}\right]$$

or
$$E_n = \frac{q}{4\pi\varepsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2a}{(r^2 + 4a^2 + 4ar\cos\theta)^{3/2}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2a}{r'^3}\right] \qquad \ldots(5.65)$$

On the conducting plane BC, $r = r'$, therefore, the normal component of the electric field at the conducting plane,

$$E_n \text{ at BC} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2a}{r^3}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{2a}{r^3} \qquad \ldots(5.66)$$

(iii) Surface Density of Charge on the Conducting Plane

By Coulomb's theorem given in equation (4.34), field due to a charged conductor is,

$$E_n = \frac{\sigma}{\varepsilon_0}$$

where, σ is the surface density of charge on the conducting plane due to induced charge, hence,

$$\sigma = \varepsilon_0 E_n$$

$$= -\frac{qa}{2\pi r^2} \qquad \ldots(5.67)$$

This shows that surface density of induced charge is inversely proportional to the cube of the distance from the point charge. Therefore the amount of induced charge is greatest near the foot of perpendicular *OA*.

(iv) Total Induced Charge

The total induced charge can be estimated as follows. Consider the small element of the surface *dS* which is annular ring between circles of radii x and $x + dx$ drawn with *O* as centre, as shown in Fig. 5.30.

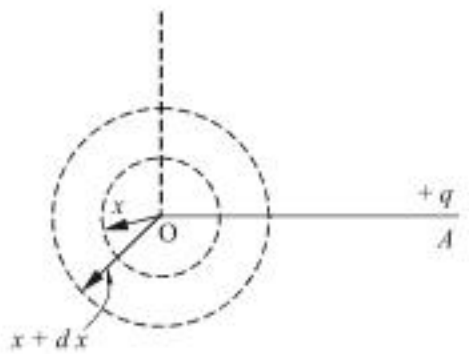


Fig. 5.30

$$dS = \pi(x+dx)^2 - \pi x^2$$

$$= 2\pi x\, dx + \pi dx^2$$

$$= 2\pi x,\, dx, \qquad \text{(since } dx \text{ is very very small)}$$

Induced charge on this element $= \sigma dS$

Hence the total induced charge

$$= \int_S \sigma dS = \int_0^\infty 2\pi x \sigma \, dx$$

$$= -\int_0^\infty \frac{2\pi qa}{2\pi r^3} x \, dx, \text{ since } \sigma = -\frac{qa}{2\pi r^3} \qquad \text{(by Eq. 5.67)}$$

$$= -qa \int_0^\infty \frac{x\,dx}{r^3}$$

$$= -qa \int_0^\infty \frac{x\,dx}{(x^2 + a^2)^{3/2}} \quad (\text{since } r = (x^2 + a^2)^{1/2})$$

$$= -q\,a \left| -\frac{1}{(x^2 + a^2)^{1/2}} \right|_0^\infty$$

$$= q\,a \left| \left(\frac{1}{(x^2 + a^2)^{1/2}} \right) \right|_0^\infty$$

$$= -qa \frac{1}{a}$$

$$= -q$$

Thus the total induced charge on the plane is $-q$.

(v) Force of Attraction between the Charge and the Plane

Further, the force of attraction between the $+q$ charge at A and the induced charge on the plane BC is the same as that between $+q$ at A and $-q$ at A′ and is given by,

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2a)^2} \qquad \ldots(5.68)$$

(2) Point Charge Near a Conducting Sphere

The method of electrical images can also be applied to calculate the potential and field due to a point charge which is placed near a sphere which has been earthed.

(i) Magnitude and Location of Image Charge

Let $+q$ charge be placed at a distance d from the centre of the sphere. Since the sphere is earthed its potential is zero. We must locate image-charge such that this charge together with positive charge at A makes the potential at any point on the surface of the sphere equal to zero. Suppose image charge q' is located within the sphere at a distance b from the centre of the sphere. The magnitude of charge q' and b is chosen such that potential at any point on the sphere is zero. Hence for point P as shown in Fig. 5.31,

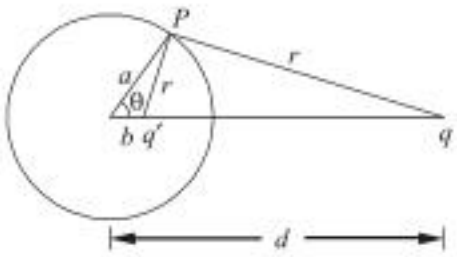


Fig. 5.31

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{q'}{r'}\right)=0$$

or $$\frac{q}{r}+\frac{q'}{r'}=0$$

where r and r' are the distances of the point P from q and q' respectively. Thus,

$$qr'=-q'r$$

On squaring we get,

$$q^2r'^2=q'^2r^2$$

Substituting r and r' from Fig. 5.31,

$$q^2(a^2+b^2-2ab\cos\theta)=q'^2(a^2+d^2-2ad\cos\theta) \qquad \ldots(1)$$

Here a is the radius of the sphere. The relation is valid for all values of θ. Therefore, equating the coefficient of $\cos\theta$, we get,

$$q^2 2ab=q'^2 2ad$$

or $$b=\frac{q'^2}{q^2}d=\alpha d, \text{ where } \alpha=\frac{q'^2}{q^2} \qquad \ldots(2)$$

Hence the remaining term of the equation (1),

$$q^2(a^2+b^2)=q'^2(a^2+d^2)$$

Putting the value of b and q'^2 from eq. (2),

$$q^2(a^2+\alpha^2d^2)=\alpha q^2(a^2+d^2)$$

or $$a^2+\alpha^2d^2=\alpha(a^2+d^2)$$

or $$d^2\alpha^2-(a^2+d^2)\alpha+a^2=0$$

Solving for α, we get,

$$\alpha=1 \text{ or } \alpha=\frac{a^2}{d^2}$$

If $\alpha=1$, then from relation $\alpha=\frac{q'^2}{q^2}$

$$q'=\pm q \text{ and } b=d$$

Therefore, b will be greater than the radius of the sphere because $d \gg a$. Thus the solution $\alpha=1$ is discarded since the iamge q' must be inside sphere.

Now, if $\alpha = \frac{a^2}{d^2}$

then the relation,

$$\alpha = \frac{q'^2}{q^2} \text{ gives,}$$

$$\frac{q'^2}{q^2} = \frac{a^2}{d^2}$$

or $$q' = \pm q \frac{a}{d}$$

As our boundary condition requires that potential due to image charge together with actual charge $+q$ should be zero at the surface of the sphere. We must choose the negative value of q' *i.e.*,

$$q' = -q \frac{a}{d} \qquad \ldots(5.69)$$

and $$b = \frac{a^2}{d}$$

(ii) Electric Potential and Electric Field at any Exterior Point P

The potential at an external point P due to the charge q and the image charge q' is

$$V_p = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$

or $$V_p = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{a}{d}\frac{1}{r'}\right), \text{ since } q' = -q\frac{a}{d}$$

From Fig. 5.32, r and r' are known by cosine law of triangle. Thus,

$$V_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(x^2 + d^2 - 2xd\cos\theta)^{1/2}} - \frac{a/d}{(x^2 + b^2 - 2xb\cos\theta)^{1/2}}\right]$$

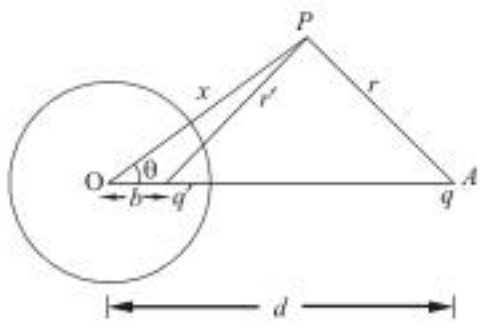


Fig. 5.32

Since $b = \frac{a^2}{d}$, (see Eq. (5.69))

$$V_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(x^2+d^2-2xd\cos\theta)^{1/2}} - \frac{a/d}{\left[x^2+\frac{a^4}{d^2}-\frac{2a^2}{d}x\cos\theta\right]^{1/2}}\right] \quad \ldots(5.70)$$

The component of electric intensity along OP is,

$$E_x = -\frac{\delta V_p}{\delta_x}$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta - x}{(x^2+d^2-2xd\cos\theta)^{3/2}} - \frac{\frac{a}{d}\left(\frac{a^2}{d}\cos\theta - x\right)}{\left(x^2+\frac{a^4}{d^2}-\frac{2a^2}{d}x\cos\theta\right)^{3/2}}\right] \quad \ldots(5.71)$$

The component of electric field perpendicular to OP,

$$E_\theta = -\frac{1}{x}\frac{\delta V}{\delta\theta}$$

$$= -\frac{q}{4\pi\varepsilon_{0x}}\left[\frac{-xd\sin\theta}{(x^2+d^2-2xd\cos\theta)^{3/2}} + \frac{\frac{a^3}{d^2}x\sin\theta}{\left(x^2+\frac{a^4}{d^2}-\frac{2a^2}{d}x\cos\theta\right)^{3/2}}\right] \quad \ldots(5.72)$$

The resultant field,

$$E = \sqrt{E_x^{\,2}+E_\theta^{\,2}} \quad \ldots(5.73)$$

(iii) Potential and Field on the Surface of the Sphere

When the point is on the surface of the sphere then, $x = a$ the radius of the sphere hence, putting $x = a$ in expression for potential, *i.e.*,

$$V_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(a^2+d^2-2ad\cos\theta)^{1/2}} - \frac{a/d}{\left(a^2+\frac{a^4}{d^2}-\frac{2a^3}{d}\cos\theta\right)^{1/2}}\right]$$

The component E_x for surface point,

$$E_x = -\frac{q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta - a}{(a^2+d^2-2ad\cos\theta)^{3/2}} - \frac{\frac{a}{d}\left(\frac{a^2}{d}\cos\theta - a\right)}{\left(a^2+\frac{a^4}{d^2}-\frac{2a^3}{d}\cos\theta\right)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta - a}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} - \frac{d\cos\theta - \frac{d^2}{a}}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0} \cdot \frac{d^2 - a^2}{a(a^2 + d^2 - ad\cos\theta)^{3/2}} \quad \ldots(5.75)$$

And the component E_θ becomes zero when $x = a$ as can be readily shown. Hence, $E_\theta = 0$ therefore the resultant intensity at the surface is,

$$E = E_x \quad \ldots(5.76)$$

Since $E = \sqrt{E_x^{\,2} + E_\theta^{\,2}}$ and E_x is normal to the surface.

(iv) Surface Density of Charge on the Surface due to Induced Charge on the Sphere

According to Coulomb's theorem as represented by Eq. (4.34), the normal component of field on the surface is given by,

$$E = -\frac{\sigma}{\varepsilon_0}$$

where $-\sigma$ is the surface density of charge (negative) induced on the sphere. Therefore,

$$\sigma = -\varepsilon_0 E$$

$$= -\frac{q}{4\pi} \cdot \frac{d^2 - a^2}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} \quad \ldots(5.77)$$

(v) The Force between the Conducting Sphere and the Point Charge at A

The force between the earthed conducting sphere and the point charge at A is the same as the force between the charge q at A and the electrical image charge q' at a distance b from the centre.

$$F = \frac{qq'}{4\pi\varepsilon_0(d-b)^2}$$

Since q and q' are separated by $(d - b)$

But $q' = -q\frac{a}{d}$

Hence, $F = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{aq^2}{d(d-b)^2}$ $\ldots(5.78)$

(vi) Unearthed Sphere

If in the above problem the sphere is not earthed, it would have certain constant potential, say, V_0 instead of zero potential. We can solve this problem immediately by assuming a second electrical image which must be placed at the centre so that the potential due to this charge may be symmetrical over the surface. Since we want V_0 to be the potential on the surface.

$$V_0 = \frac{1}{4\pi\varepsilon_0}\frac{q''}{a}$$

where q'' is the second electrical image or the imaginary charge, *i.e.* $q'' = 4\pi\varepsilon_0 a\, V_0$, thus the potential due to the actual point charge q and the sphere at any point P is given by,

$$V_p = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} - \frac{qa}{dr'} + \frac{4\pi\varepsilon_0 a\, V_0}{r''}\right] \qquad \ldots(5.79)$$

where r'' is the distance between the point P where the potential is to be determined and the centre of the sphere.

REVIEW QUESTIONS

5.1 Define electric potential. Show that in an electric field $\vec{E}$, the potential difference between two points a and b along any path is

$$V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{l}$$

5.2 What do you mean by line integral of a vector field. Show that electric field $\vec{E}$ is conservative.

5.3 What is the principle of superposition of potentials? Obtain the potential at a point due to (1) group of point charges, (2) linear distribution of charge, (3) surface distribution of charge and (4) volume distribution of charge.

5.4 Explain the term potential gradient and establish the relation $\vec{E} = -\vec{\nabla} V$.

5.5 What is an equipotential surface? Show that the potential difference between two points distant r_1 and r_2 from an infinitely long line charge of linear charge of linear charge density λ is

$$V_2 - V_1 = \frac{\lambda}{2r\varepsilon_0} \log_e \frac{r_1}{r_2}$$

5.6 Derive Poisson's and Laplace equations. What are importance of these equations?

5.7 What is dipole moment? Obtain expression for the potential and field due to an electric dipole.

5.8 Explain "electric multipole". Obtain expression for the potential and field due to a linear quadrupole.

5.9 Show that the electric field due to a dipole at a point on its axis is twice as strong as that at a point at the same distance along the perpendicular axis.

5.10 Show that the potential at a point due to an arbitrary charge distribution can be expressed as a contribution due to monopole, dipole and quadrupole moment etc.

5.11 What do you mean by electric potential energy of a system of charges? Derive an expression for it and show that energy density U is given by, $U = \frac{\varepsilon_0}{2} E^2$ joules/met^3.

5.12 Derive an expression for the electrostatic energy of a uniformly charged sphere. What is an electron volt?

5.13 An electric dipole is placed in a uniform electric field $\vec{E}$. Show that Torque $= \vec{p} \times \vec{E}$ and potential energy $= -\vec{p} \cdot \vec{E}$.

5.14 What is meant by electrical image? A point + q is placed at a distance d from an infinitely long earthed wire. Derive an expression for the surface density of charge induced on the conductor.

5.15 Explain the method of electrical images for the solution of electrostatics problems. As an example discuss the point charge near a conducting sphere.

PROBLEMS

5.1 Four point charges $q_1 = 1\times10^{-8}$ coul, $q_2 = -2\times10^{-8}$ coul, $q_3 = 3\times10^{-8}$ coul and $q_4 = 2\times10^{-8}$ coul are placed at the four corners of a square of side 1 metre. Calculate the potential at its centre.

5.2 Calculate the electric potential on the surface of a gold nucleus ($z = 79$) of radius 6.6×10^{-15} metre.

5.3 Given an electric field in a limited region, surrounding an origin in the X-Y plane for which the potential is represented as, $V = ax^2 + by^2$ where a and b are positive constants. Find
(1) the component of field intensity.
(2) where is the potential a minimum and where is the field-intensity a minimum.

5.4 The electric potential at any point in the $x-y$ plane is represented as, $V = \dfrac{x}{(x^2+y^2)^{3/2}} + \dfrac{y}{(x^2+y^2)^{5/2}}$.

Calculate the component of electric field E at that point.

5.5 The potential at a certain distance from a point charge is 600 volts and the electric field is 200 nt/coul.
(1) What is the distance of the point charge?
(2) What is the magnitude of the charge?

5.6 Calculate the potential and field due to a short dipole of dipole moment 5.0×10^{-16}. State Coul. cm. at a point distant 1 cm from it
(1) on its axis.
(2) on its perpendicular bisector.

5.7 Point charge of 3×10^{-9} coul are located at two vertices of an equilateral triangle of 20 cm, side. How much work must be done to bring a test charge 1.0×10^{-9} coul up to the third corner of the triangle from an infinite distance away?

5.8 Eight positive charges, each of magnitude q, are kept at the corners of a cube of side a. Calculate the potential energy of the system if a negative charge $-2q$ is placed at the centre of the cube.

5.9 Two plane electrodes are 0.6 cm apart within an electron tube maintained at 100 volts potential difference. An electron is required to move from one plate to the other. How long would it take to reach the other plate?

5.10 A charge q is placed with a metal sphere of radius a at a distance r from the centre. Show that there is a force attracting the charge to the inner surface of the sphere is given by,

$$F = \frac{q^2 ar}{3\pi\varepsilon_0 (a^2 - r^2)^2}$$

ANSWERS

1. 509 volts **2.** 1.72×10^7 volts **3.** $-2ax$, $2by$, 0; (0, 0) (0, 0)

5. 3 metre, (11) 2×10^{-7} Coul **6.** 10^{-15} dy/stat. coul; 5×10^{-6} dyne/stat. coul.

7. 2.7×12^{-7} joules **8.** $39.6 \times 10^9 \dfrac{q^2}{a}$ joules **9.** 2.003×10^{-9} sec

SELF-TEST OBJECTIVE QUESTIONS

5.1 The work done on a unit positive charge in bringing it from infinity to any point is called
(*a*) intensity at that point (*b*) electric potential at that point
(*c*) electric potential at infinity (*d*) electric intensity at infinity

5.2 The potential difference between two points A and B can be put as

(a) $V_a - V_b = -\dfrac{1}{4\pi\varepsilon_0}\int \dfrac{q}{x^2}dx$ (b) $V_a - V_b = -\dfrac{1}{4\pi\varepsilon_0}\int \dfrac{q}{x}dx$

(c) $V_a - V_b = \frac{1}{4\pi\varepsilon_0}\int \frac{q}{x^2}\,dx$ (d) $V_a - V_b = \frac{1}{4\pi\varepsilon_0}\int \frac{q}{x}\,dx$

5.3 The dimensions of potential are same as that of
(*a*) work (*b*) electric field per unit charge
(*c*) work per unit charge (*d*) force per unit charge

5.4 Electric field fines and equipotential lines are
(*a*) always orthogonal
(*b*) orthogonal only when electric field is uniform
(*c*) orthogonal only when potential does not change
(*d*) none of the above is correct.

5.5 For a dipole electric field varies as
(a) r^{-2} (*b*) r^{-3} (*c*) r^{-1} (d) r^{-4}

5.6 Electric field potential due to a point charge
(*a*) falls inversely proportional to the distance
(*b*) falls inversely proportional to the square of the distance
(*c*) falls inversely proportional to the square root of the distance
(*d*) it does not change with distance

5.7 The potential between two points depends only
(*a*) on the location of these points (*b*) on the path followed by the particle.

5.8 The electric dipole moment (P) is a vector; it points
(*a*) from positive to negative charge (*b*) from negative to positive charge.

5.9 In free space Poisson's equation is $(\varepsilon_0 = 8.854 \times 10^{-12}\ \text{F/m})$
(*a*) $\nabla^2 V = 8.854 \times 10^{-12}$ (*b*) $\nabla^2 V = \rho/8.854 \times 10^{-12}$
(*c*) $\nabla^2 V = 0$ (*d*) $\nabla^2 V = \infty$

5.10 When a test charge is brought in from infinity along the perpendicular bisector of the dipole, the work done is
(*a*) positive (*b*) zero
(*c*) negative (*d*) none of the above

5.11 Work done to bring a positive charge from infinity to some point in an electric field is dependent on the path.
(*a*) Ture (*b*) False

ANSWERS

1. (b)	**2.** (a)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (a)
7. (a)	**8.** (b)	**9.** (c)	**10.** (b)	**11.** (b)	

CHAPTER 6

CAPACITORS AND DIELECTRICS

6.1. CAPACITOR

A capacitor (or condenser) is a device for storing charge. In actual practice, a capacitor is an electrical device consisting of two conductors separated by an insulating or dielectric medium (including air) and carrying equal and opposite charges. The conductors are called plates and may be of any shape. The earliest capacitor was invented almost accidentally by VAN MUSSCHENBROEK of Leden in about 1746, and became known as a Leden jar.

Parallel Plate Capacitor

As an example let us take two similar flat conducting plates arranged parallel to one another, separated by distance d, as shown in Fig. 6.1. Let A be the area of each plate and suppose that there is a charge Q on one plate and – Q on the other. Away from the edge, the field $\vec{E}$ is very nearly uniform in the region between the plates and given by,

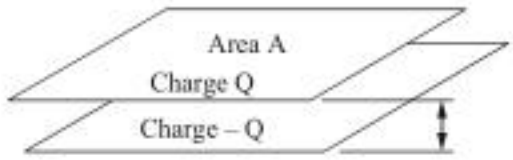


Fig. 6.1

$$E = \frac{\sigma}{\varepsilon_0} \qquad [see\ eq.\ (4.3)]$$

or $\sigma = \varepsilon_0 E$

where σ is the density of the surface charge on the inner surface of the plate. If V is the potential difference between the plates, then $E = V/d$, hence above equation reduces,

$$\sigma = \varepsilon_0 \frac{V}{d}$$

If we neglect the variation of $\vec{E}$ near the edge of the plates and therefore of σ, the total charge on one plate represented by Q is given by,

$$Q = \varepsilon_0 \frac{VA}{d}$$

or $$\frac{Q}{V} = \frac{\varepsilon_0 A}{d} \qquad \ldots(6.1)$$

The ratio $\frac{Q}{V} = \frac{\varepsilon_0 A}{d}$, *though it is only approximation in parallel plates, but it is clear that the value of R.H.S. will depend only on the size and geometrical arrangement of the plates.* That is, *for a fixed pair of conductors, the ratio of charge to potential difference will be a constant. We call this constant the capacitance of the capacitor and denoted by C, hence,*

$$\frac{Q}{V} = \text{Constant} = C$$

or $$Q = CV \qquad \ldots(6.2)$$

Thus, the capacity of the parallel plate capacitor, with edge fields neglected, is given by,

$$C = \frac{\varepsilon_0 A}{d} \text{ farad} \qquad \ldots(6.3)$$

Units

From Equation (6.2) unit of capacitance is coul/volt and is called Farad related by,

$$1 \text{ Farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} \qquad \ldots(6.4)$$

However, farad is very large unit, we use smaller unit as micro farad or micro-micro farad.

$$1\mu F = 10^{-6} \text{ F and } 1\mu\mu F = 10^{-12} \text{ F}$$

Types of Capacitors According to Shape

The value of C given in Eqn. (6.3) depends on the size and geometry of the plates, hence we can design the capacitors of different geometry. The more commonly used capacitors are,

1. a parallel plate capacitor
2. a cylindrical capacitor
3. a spherical capacitor.

Example 6.1 *The plates of parallel plate capacitor are separated by 1.00 mm. What must be the plate area if the capacitance to be 1.0 Farad assuming that air is filled between the plates?*

Solution. Using equation (6.3), we get,

$$A = \frac{dC}{\varepsilon_0} = \frac{1.0\times10^{-3} \text{ metre (1.00 Farad)}}{8.9\times10^{-12} \text{ coul}^2/\text{Newton.metre}^2}$$

as, $$\text{Farad} = \frac{\text{Coul}}{\text{Volt}} - \frac{\text{Coul}^2}{\text{newton.metre}}$$

Hence, $A = 1.1\times10^{8} \text{ metre}^2$

This is approximately the area of a square sheet of more than 9 km of side. Indeed farad is vary large unit. Therefore, we use smaller unit as micro farad or micro-micro farad.

6.2 A CYLINDRICAL CAPACITOR

Such a capacitor consists of two coaxial cylinders of radius *a* and *b* and length *l*. Calculate capacitance of device, sectional diagram is given in Fig. 6.2. Construct a Gaussian surface shown by dotted line having radius *r*. By using Gauss's law we get,

$$\varepsilon_0 \oint \vec{E}.d\vec{S} = Q$$

where Q is the charge on the surface.

Fig. 6.2

$$\varepsilon_0 \, \mathrm{E}.2\pi rl = \mathrm{Q}$$

$$\mathrm{E} = \frac{\mathrm{Q}}{2\pi\varepsilon_0 rl} \text{ at P}$$

The potential difference between the plates is given by,

$$\mathrm{V} = -\int_b^a \vec{\mathrm{E}}.d\vec{r}$$

As the angle between $\vec{\mathrm{E}}$ and $\vec{dr}$ is 0, since $\vec{\mathrm{E}}$ is radial *i.e.* along $\vec{r}$, so $\vec{\mathrm{E}}.d\vec{r} = \mathrm{E}\,dr$. Hence,

$$\mathrm{V} = -\int_b^a \mathrm{E}\,dr = -\int_b^a \frac{\mathrm{Q}}{2\pi\varepsilon_0 rl} dr$$

$$= \frac{\mathrm{Q}}{2\pi\varepsilon_0 l} \log_e \frac{b}{a}$$

Therefore, the capacitance,

$$\mathrm{C} = \frac{\mathrm{Q}}{\mathrm{V}} = \frac{2\pi\varepsilon_0 l}{\log_e (b/a)} \qquad \ldots(6.5)$$

Like (6.3) equation, the capacitance C given in equation (6.5) also depends on geometrical factors.

6.3 SPHERICAL CAPACITORS

Let us calculate the capacitance of two concentric spheres shown in Fig. 6.3. The potential difference between the two spheres V, given by,

$$\mathrm{V} = -\int_b^a \vec{\mathrm{E}}.d\vec{r}$$

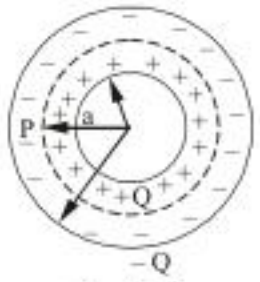


Fig. 6.3

$\vec{\mathrm{E}}$ at a distance r is given by,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2}$$

$\vec{\mathrm{E}}$ is radial *i.e.* along r, hence angle between $\vec{\mathrm{E}}$ and $\vec{r}$ is zero. Therefore, $\vec{\mathrm{E}}.d\vec{r} = \mathrm{E}\,dr$. Hence,

$$\mathrm{V} = -\int_b^a \frac{\mathrm{Q}}{4\pi\varepsilon_0} \frac{dr}{r^2} = -\frac{\mathrm{Q}}{4\pi\varepsilon_0} \int_b^a \frac{dr}{r^2}$$

$$= \frac{\mathrm{Q}}{4\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right) \text{ and } \mathrm{C} = 4\pi\varepsilon_0 \frac{ab}{b-a} \qquad \ldots(6.6)$$

Here again the capacitance C depends only on their radii.

6.4 GUARD RING CAPACITOR

To overcome the edge effect in parallel plate capacitor the two rings, one on right and the other on left side, are attached as shown in Fig. 6.4. These rings are called guard rings to ensure uniformity in field and such capacitor is caled guard ring capacitor.

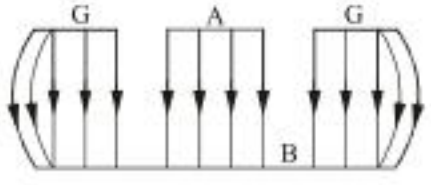


Fig 6.4

6.5 ENERGY STORED IN A CAPACITOR

*The energy of a charged capacitor is the amount of work donw in c*harging it. Suppose V is the potential difference between the plates of the capacitor of capacitance C. The charge Q is given by,

$$Q = CV.$$

There is charge Q on one plate and – Q on the other. Let us increase the charge from Q to Q + dQ by transferring a positive charge dQ from the negative to the positive plate, working against the potential difference V. The work that has to be done is given by,

$$dW = V\,dQ = \frac{Q}{C}dQ$$

Therefore, to charge the capacitor starting from the uncharged state to the final charge, say, Q,

$$W = \int_{Q=0}^{Q=Q} \frac{Q}{C}\,dQ = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \qquad \ldots(6.7)$$

This work done is stored up in the form of potential energy and supplies the energy necessary to drive the current when the capacitor is discharged.

For a parallel plate capacitor the C is given by,

$$C = \frac{\varepsilon_0 A}{d} \text{ and } V = Ed$$

Hence the energy stored is given by,

$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d} E^2 . d^2 = \frac{1}{2}\varepsilon_0 E^2 . A.d \text{ joules} \qquad \ldots(6.8)$$

Energy stored per unit volume

$$= \frac{1}{2}\varepsilon_0 E^2 \text{ joules/metre}^2 \qquad \ldots(6.9)$$

(A.d is the volume between the plates.) This expression for energy is same as the general formula for the energy in an electric field given in Eqn. (4.41).

6.6 FORCE OF ATTRACTION BETWEEN CAPACITORS PLATES

Consider a simple parallel plate capacitor such as illustrated in Fig. 6.5. It appears reasonable, since one plate is positively charged and the second plate is negatively charged, that there should be a force of attraction between the plates tending to pull them together.

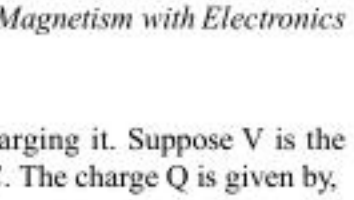


Fig. 6.5

An expression for the force can be readily obtained by the application of the *principle of virtual work*. Suppose E is intensity of electric field between the plates. The total energy in the field is given by,

$$U = \frac{1}{2}\varepsilon_0 E^2 Ad \text{ joules}$$

Let us now calculate the change in the total energy when the capacitor plate to be separated by an additional amount Δd. *The work that would be required to effect the additional separation is equal to the change in energy in the field*, and this must be equal to the product of the force and distance. The energy per unit volume is given by,

$$U = \frac{1}{2}\varepsilon_0 E^2 \text{ joules/metre}^3$$

The change in volume due to Δd change in separation = A Δd. Hence, the change in total energy due to change in volume

$$= \frac{1}{2} \varepsilon_0 E^2 A \Delta d \qquad \ldots(6.10)$$

The mechanical work done from the outside against the force of attraction between the plates in moving them further apart through a distance Δd will be,

$$W = \vec{F}.\Delta\vec{d} \qquad \ldots(6.11)$$

or $$W = -F\Delta d \qquad \ldots(6.12)$$

This work done = change in energy

$$\frac{1}{2} \varepsilon_0 E^2 A \Delta d = \vec{F}.\Delta\vec{d}$$

or $$\frac{1}{2} \varepsilon_0 (\vec{E}.\vec{E}) A \Delta d = \vec{F}.\Delta\vec{d} = -F\Delta d$$

The *negative sign indicates that the force acting over the area A is normal to the plates in the direction of E, i.e. F* and Δd are 180° apart. The magnitude is given

$$F = \frac{1}{2} \varepsilon_0 E^2 A \qquad \ldots(6.13)$$

But the force per unit area is pressure, and hence the pressure is given by,

$$P \text{ (Pressure)} = \frac{F}{A} = \frac{1}{2} \varepsilon_0 \vec{E}\,\vec{E}$$

or $$P = \frac{1}{2} \varepsilon_0 E^2 \text{ newton/metre}^2 \qquad \ldots(6.14)$$

This expression for pressure is the same as the general formula for pressure on a conductor due to electric field $\vec{E}$ discussed in Equation (4.37).

6.7 THE DIELECTRIC CONSTANT

Now *we will discuss insulator or dielectric materials which do not conduct electricity.* However, using a simple electroscope and two parallel plate capacitors Faraday discovered that this was not so. He constructed two identical capacitors in one of which he placed a dielectric. When both capacitors were charged to the same potential difference as shown in Fig. 6.6 (a).

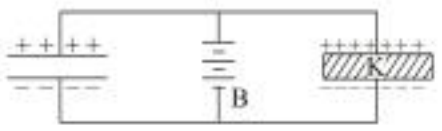


Fig. 6.6 (*a*)

Faraday found by experiment that the charge on the capacitor with dielectric is greater than than that without. *Since q is larger, for the same V, it follows from the relation C = q/V that the capacitance of a capacitor increases if dielectric is placed between the plates.* If dielectric material fills completely, the space between the plates the *ratio of the capacitance C with the dielectric to that without C_0 is called dielectric constant K,* which is the property of dielectric material *i.e.*,

$$\frac{C}{C_0} = K \qquad \ldots(6.15)$$

K is also *called relative permittivity* of the medium.

Faraday obtained the same relation Eq. (6.15) by placing same charge to the capacitors as in Fig. 6.6 (b) and measured the potential difference. If V_d is the potential difference with dielectric and V_0 without dielectric capacitor, then $V_d = \frac{V_0}{K}$. So the potential

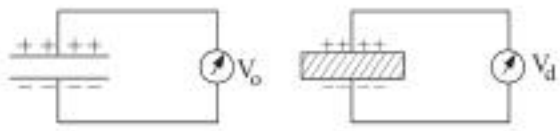


Fig. 6.6 (*b*)

difference with dielectric decreases by a factor 1/K. Hence from relation C = q/V as q is same, here again the same conclusion.

6.8 CLASSIFICATION OF DIELECTRIC ON THE BASIS OF MOLECULAR STRUCTURE (POLAR AND NON POLAR MOLECULES)

The molecules or atoms are composed of electrons and nuclei in a state of vibration about certain fixed points in space. Though the molecule or atom as a whole is neutral *i.e.* the magnitude of charge on electrons is equal to the positive charge on nuclei due to protons. But the spacial arrangement of charges in a molecule may be different in various substances. All positive and negative charges of a molecule can be replaced by an equivalent single positive and single negative charge located at the centres of gravity of separate positive and separate negative charges respectively. These centres of gravity can be found with the aid of the same rules which are used to determine the position of the centres of gravity of systems of point masses in statics. These centres of gravity may either coincide or not coincide in space.

At any given instant, if the centre of gravity of the positive and negative electrical charges "do not coincide", then the molecule is "polar" and the matter composed of such molecules is also called polar. When the two centres of gravity are separated, the molecules may momentarily possess a dipole moment equal to the product of charge and distance of separation. While if the two centres of gravity "coinncide" then the molecule is called a "non-polar molecule", having negligible dipole moment and the matter composed of such molecules is also called non-polar. Matter comprising different molecules both non-polar and polar should be regarded as polar matter.

Examples of Non-Polar Molecules

The rigid molecules which have symmetrical structure are non-polar and have always a negligible small dipole moment. They are H_2, N_2, Cl_2, CH_4, CCl_4, C_6H_6, C_2H_2 and C_2H_4.

Examples of Polar Molecules

Most molecules are, however, polar, their average dipole moment is finite. If *symmetry is disturbed*, the molecules becomes polar. If a hydrogen atom of methane molecule is replaced by chlorine atom, it loses symmetry and becomes polar.

6.9 A DIELECTRIC IN AN ELECTROSTATIC FIELD (POLARISATION OF DIELECTRIC)

We shall discuss this in the next section in detail, here is an *elementary atomic view. A dielectric whether polar or non-polar will acquire dipole moment by induction when placed in an electric field.* Then these elementary dipoles are oriented along the field direction, dielectric becomes polarised without affecting the neutrality of dielectric. In other words, by *induction the one side gets positive charge while the other same negative charge.* So the electric field setup by them (E′) opposes the external electric field say E_0. Fig. 6.7 shows the induction process. *One important point to be noted is that induced effect is present only when the electric field is present.* The resultant field $\vec{E} = \vec{E}_0 + \vec{E}'$. The field E′ due to induced charge points opposite to E_0, hence the field inside dielectric decreases same as potential. So we can write, using the relation, E = Vd

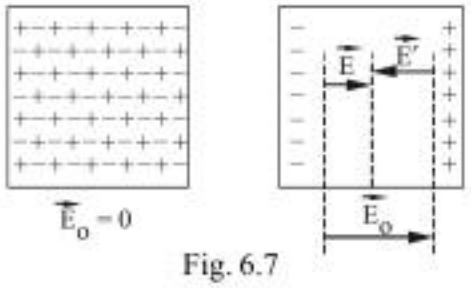


Fig. 6.7

$$\frac{V_0}{Vd} = \frac{E_0}{E} = K \qquad \ldots(6.16)$$

6.10 CAPACITY OF A PARALLEL PLATE CAPACITOR PARTIALLY FILLED WITH DIELECTRIC

Fig. 6.8 shows a capacitor partially filled with a dielectric material of thickness t. If Q is the charge on each plate of area A, then value of electric field E_0 due to charge Q in the free-space between plates and dielectric material is given by Gauss's law

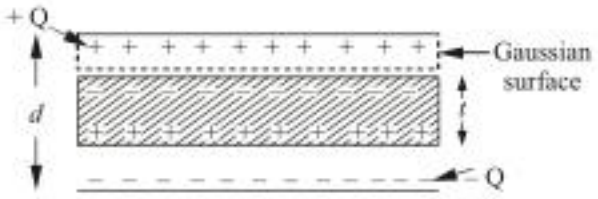


Fig. 6.8

$$\varepsilon_0 \oint \vec{E}.d\vec{S} = \varepsilon_0 E_0 A = Q$$

or
$$E_0 = \frac{Q}{\varepsilon_0 A} \qquad \ldots(6.17)$$

From Eq. (6.16) the field E inside the dielectric is given by,

$$E = \frac{E_0}{K} = \frac{Q}{K\varepsilon_0 A} \qquad \ldots(6.18)$$

E_0 is the value of electric field in the free space (*i.e.* without dielectric) which is d-t, and E is the value of electric field inside the dielectric of thickness t. The potential difference between the two points is given by,

$$\left(V = V_1 - V_2 = -\int_2^1 \vec{E}.d\vec{l} \right)$$

Hence the total potential difference between the plates A and B,

$$V_a - V_b = -\underset{\substack{\text{for free space}\\ d-t}}{\int} \vec{E}_0.d\vec{l} - \underset{\substack{\text{for thickness}\\ t}}{\int} \vec{E}.d\vec{l}$$

Since the total field is $(\vec{E}_0 + \vec{E})$.

Therefore, $V_a - V_b = Et + E_0(d - t)$

$(\vec{E}.d\vec{l} = -E\,dl$; similarly, $\vec{E}_0.d\vec{l} = -E_0\,dl)$

Substituting E_0 and E from equations (6.17) and (6.18), we get,

$$V_a - V_b = \frac{Q}{A\varepsilon_0}\left(d - t + \frac{t}{K}\right)$$

Therefore, from equation (6.1) the capacitance C is given by,

$$C = \frac{Q}{V_a - V_b} = \frac{\varepsilon_0 A}{\left[d - t\left(1 - \frac{1}{K}\right)\right]} \qquad \ldots(6.19)$$

6.11 DIELECTRICS AND GAUSS'S LAW

Now apply Gauss's law to a parallel plate capacitor filled with a dielectric of dielectric constant K. Fig. 6.9 (a) shows an usual parallel plate capacitor without dielectrics. Let charge be Q on the plates with proper sign and A be the area of each plate. E_0 is the electric field without dielectric.

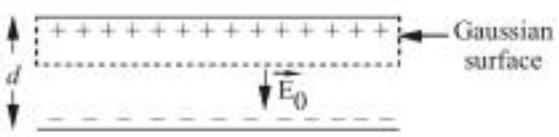


Fig. 6.9 (*a*)

Therefore, the Gauss's law gives,

$$\varepsilon_0 \oint \vec{E} . d\vec{S} = \varepsilon_0 E_0 A = Q$$

or $$E_0 = \frac{Q}{\varepsilon_0 A}$$

Now place the dielectric as shown in Fig. 6.9 (b) then the net charge enclosed by Gaussian surface is Q – Q′. Therefore with dielectric the Gauss's law is,

Fig. 6.9 (*b*)

$$\varepsilon_0 \oint \vec{E} . d\vec{S} = Q - Q' \qquad \ldots(6.20)$$

or $$\varepsilon_0 EA = Q - Q'$$

or $$E = \frac{Q}{\varepsilon_0 A_0} - \frac{Q'}{\varepsilon_0 A} \qquad \ldots(6.21)$$

in which – Q′ is the induced charge due to polarisation. Using the Equation (6.16), we have,

$$E = \frac{E_0}{K}$$

where E is field inside the dielectric. Substituting E_0, we have,

$$E = \frac{Q}{K \varepsilon_0 A} \qquad \ldots(6.22)$$

Substituting this value of E in Eq. (6.21), we have,

$$\frac{Q}{K \varepsilon_0 A} = \frac{Q}{\varepsilon_0 A} - \frac{Q'}{\varepsilon_0 A} \qquad \ldots(6.23)$$

or $$Q' = Q\left(1 - \frac{1}{K}\right) \qquad \ldots(6.24)$$

If there is no dielectric between the plates then K = 1, and Q′ = Q, *i.e. there will be no induced charge.* The value of K is always greater than one (K > 1), hence Q′ (induced charge) is always less than Q (free charge).

From Equation (6.24) the value of $Q - Q'$ is given by,

$$Q - Q' = \frac{Q}{K}$$

Now we can write the Gauss's law with dielectric as given in equation (6.20) after substituting $Q - Q'$ in the following form *i.e.*,

$$\varepsilon_0 \oint \vec{E} . d\vec{S} = Q - Q'$$

or $$= \frac{Q}{K}$$

or $$\varepsilon_0 \oint K\vec{E} . d\vec{S} = Q \qquad \ldots(6.25)$$

Though we have derived equation (6.25) for a parallel plate capacitor, but it is applicable in all cases when the dielectrics are present.

6.12 CAPACITY OF A PARALLEL PLATE CAPACITOR WHEN COMPLETELY FILLED WITH DIELECTRIC

Fig. 6.9 (b) represents such capacitor. In such circumstances, the value of intensity of electric field is E between the plates is given by,

$$E = \frac{Q}{K \varepsilon_0 A} \qquad [\textit{see equation } (6.21)]$$

and thickness of the slab is same as the separation between the plates *i.e., d*. Hence,

$$V = E\,d$$

$$V = \frac{Qd}{K \varepsilon_0 A}$$

therefore, $C = \frac{Q}{V} = \frac{K \varepsilon_0 A}{d}$ Farad ...(6.26)

Example 6.2 *A parallel plate capacitor is filled with two dielectrics of same dimensions but different dielectric constants K_1 and K_2 respectively as shown in Fig. 6.10 (a) and (b). Calculate its capacitance.*

Solution. From Fig. 6.10 (a) the arrangement is simply two capacitors in parallel, and each capacitor will have the area A/2.

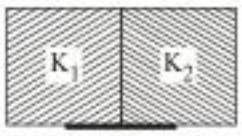


Fig. 6.10 (*a*)

Total capacitance $C = C_1 + C_2$

$$= \frac{\varepsilon_0 A/2\, K_1}{d} + \frac{\varepsilon_0 A/2\, K_2}{d}$$

$$= \frac{\varepsilon_0 A}{d}\left(\frac{K_1 + K_2}{2}\right)$$

The arrangement of Fig. 6.10 (b) can be regarded as two capacitors in series. Each capacitor will be of area A and separation *d*/2. Therefore, the total capacitance is given by the relation,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d}{2}\left(\frac{1}{\varepsilon_0 AK_1} + \frac{1}{\varepsilon_0 AK_2}\right)$$

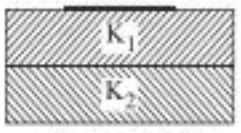


Fig. 6.10 (*b*)

or $$C = \frac{2\varepsilon_0 A}{d}\left(\frac{K_1 K_2}{K_1 + K_2}\right)$$

Example 6.3 *The separation between the plates of parallel plate capacitor is d, one plate is at zero potential. The space between the plates is filled with a material whose dielectric constant varies uniformly from one plate to the other. If K_1 and K_2 are the values of the dielectric constant at the two plates, show that the capacity per unit area is*

$$C = \frac{\varepsilon_0}{d}\,\frac{K_2 - K_1}{\log_e \frac{K_2}{K_1}}$$

Solution. The electric intensity E is given by,

$$E = \frac{\sigma}{\varepsilon_0 K}, \text{ since } E = \frac{E_0}{K}$$

As K is not constant, but varies from K_1 to K_2, increase in dielectric constant per unit length $= \frac{K_2 - K_1}{d} = \alpha$. The dielectric constant at point at a distance *x* from the first plate is $K = K_1 + \alpha x$.

Hence the electric intensity at this point $= \frac{\sigma}{\varepsilon_0 (K_1 + \alpha x)}$.

Potential difference between the plates,

$$V = -\int_{d}^{0} \vec{E} \cdot d\vec{x} = \int_{0}^{d} E\, dx$$

$$= \frac{\sigma}{\alpha\varepsilon_0} \log \frac{K_1 + \alpha d}{K_1}, \text{ substituting } \alpha$$

$$= \frac{\sigma d}{\varepsilon_0 (K_2 - K_1)} \log \frac{K_1 + \frac{(K_2 - K_1) d}{d}}{K_1}$$

$$= \frac{\sigma d}{\varepsilon_0 (K_1 - K_2)} \log_e \frac{K_2}{K_1}$$

Hence, $C = \dfrac{\text{total charge}}{V} = \dfrac{\sigma A}{V}$

$$= \frac{A \varepsilon_0 (K_2 - K_1)}{d \log_e K_2 / K_1}$$

Example 6.4 *A parallel plate capacitor consists of two square metal plates 5.0 cm of side and separated by 1 cm. A sulphur slab 6 mm thick is placed on the lower plate. Calculate the capacitance of the capacitor, dielectric constant of sulphur is 4.*

Solution. When sulphur slab of 6 mm is placed on the lower plate, it leaves an airgap of 4 mm thickness. The capacitor can be regarded as two capacitors in series.

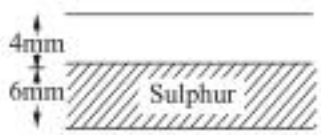


Fig. 6.11

An air capacitor of 4 mm plate spacing and a sulphur field capacitor of 6 mm plate space are shown in Fig. 6.11. From equation (6.3) the capacitance of air capacitor.

$$C_a = \frac{\varepsilon_0 A}{d} \text{ farad when A and } d \text{ are in metres}$$

$$= 8.85 \times 10^{-12} \frac{A}{d} \text{ farad}$$

$$= 8.85 \frac{A}{d} \mu\mu F, \text{ since } A = 0.5 \times 0.5 \text{ and } d = 4 \text{ mm} = 0.004 \text{ metre}$$

Hence, $$= \frac{8.85 \times 0.5^2}{0.004}$$

$$= 553\ \mu\mu F$$

The capacitance of the sulphur filled capacitor by Equation (6.26),

$$C_3 = \frac{K \varepsilon_0 A}{d'}$$

$$= \frac{8.85 \times 4 \times 0.5^2}{0.006} \mu\mu F$$

$$= 1,475\ \mu\mu F$$

Then total capacitance,

$$\frac{1}{C} = \frac{1}{C_a} + \frac{1}{C_3}$$

or $$C = \frac{C_a C_3}{C_a + C_3} = 402\ \mu\mu F$$

Example 6.5 *A concentric spherical capacitor is formed by two concentric spheres of radii a and b. If the medium between the spherical shells has the dielectric constant K_1 from a to r and K_2 from r to b show that*

$$C = -\frac{4\pi\varepsilon_0}{\left[\frac{1}{K_1 a} - \frac{1}{K_2 b} + \frac{1}{r}\left(\frac{1}{K_2} - \frac{I}{K_1}\right)\right]}$$

Solution. See Fig. 6.3. The potential difference between the inner and outer spheres is given by,

$$V_a - V_b = -\int_b^a E\,dr = +\int_a^b \frac{Q}{4\pi\varepsilon_0 K r^2}\,dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{K r^2}$$

As $K = K_1$ from a to r and $K = K_2$ from r to b,

$$V_a - V_b = \frac{Q}{4\pi\varepsilon_0}\left[\int_a^r \frac{dr}{K_1 r^2} + \int_r^b \frac{dr}{K_2 r^2}\right]$$

$$= \frac{Q}{r\pi\varepsilon_0}\left[\left|-\frac{1}{K_1 r}\right|_a^r + \left|-\frac{1}{K_2 r}\right|_r^b\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(\frac{1}{K_2} - \frac{1}{K_1}\right) + \left(\frac{1}{K_1 a} - \frac{1}{K_2 b}\right)\right]$$

Therefore the capacitance

$$C = \frac{Q}{V_a - V_b}$$

$$= \frac{4\pi\varepsilon_0}{\frac{1}{r}\left(\frac{1}{K_2} - \frac{1}{K_1}\right) + \left(\frac{1}{K_1 a} - \frac{1}{K_2 b}\right)}$$

Example 6.6 *For a parallel plate capacitor following data are given—area of each plate = 100 cm^2; separation of plates = 1.00 cm. A potential difference of 100 volt is applied with no dielectric present. A dielectric slab of dielectric constant 7.00 and of thickness 0.50 cm is introduced. Calculate (1) the free charge, (2) electric field strength in the gap, (3) electric field strength in the dielectric, (4) potential difference between the plates, (5) capacitance with dielectric.*

Solution. Capacitance before the dielectric is introduced,

$$= C_0 = \frac{\varepsilon_0 A}{d} = 8.9\times10^{-12}/10^{-2} = 8.9\ \mu\mu F$$

(1) Free charge $q = C_0 V_0$

$$= 8.9\times10^{-12}\ \text{Farad} \times 100\ \text{Volt}$$

$$= 8.9\times10^{-10}\ \text{coul}$$

(2) Electric field strength in the gap. Using the relation,

$$E_0 = \frac{Q}{\varepsilon_0 A} \qquad \text{[}\textit{see}\text{ equation (6.17)]}$$

$$= \frac{8.9\times10^{-10}}{8.9\times10^{-12}\times10^{-2}} = 1\times10^4\ \text{volt/metre}$$

(3) Electric field in the dielectric,

$$E = \frac{E_0}{K}, \qquad \text{[see equation (6.16)]}$$

$$= \frac{1\times10^4}{7} = 0.14\times10^4 \text{ volt/metre}$$

(4) Potential difference between the plates:

$$V = E_0(d-t) + Et$$

Substituting E and E_0 from above,

$$V = 1\times10^4(0.5\times10^{-2}) + 0.14\times10^4(0.5\times10^{-2})$$

$$= 57 \text{ volts}$$

(5) Capacitance with slab:

$$\frac{Q}{A} = \frac{8.9\times10^{-12} \text{ coul}}{57 \text{ volt}} = 16 \text{ } \mu\mu\text{F}$$

6.13 ELECTRIC POLARISATION VECTOR P

The polarisation P is defined as the induced surface charge per unit area i.e., surface density of bound charges in dielectric,

$$P = \frac{Q'}{A} = \sigma_{pol} \qquad \ldots(6.27)$$

The induced charge Q′ appears when the dielectric is polarised that is the reason we call Q′/A as electric polarisation P.

Now multiply the numerator and denominator of Eq. (6.27), by d the thickness of the dielectric slab we get,

$$P = \frac{Q'd}{Ad}$$

The product Q′d is the induced electric dipole moment because by polarisation the dielectric slab develops the charge Q′ on one surface and – Q′ on the opposite separated by thickness of the slab, hence a dipole of dipole moment Q′d. The product A.d is the volume (v) of the slab therefore, we can write,

$$P = \frac{\text{Induced dipole moment}}{\text{Volume}} = \frac{Q'd}{v} \qquad \ldots(6.28)$$

The polarisation P is defined as the induced dipole moment per unit volume. Since the electric dipole moment is a vector (as discussed in last chapter) the electric polarisation is also a vector and its magnitude is P.

$$\vec{P} = \frac{Q'.\vec{d}}{v} \qquad \ldots(6.29)$$

Thus the magnitude P is the polarisation charge per unit area on the surface, *i.e.* the surface density of charges (σ_{pol}) appearing at right angles to the direction of the applied field. Hence,

$$\sigma_{pol} = \frac{Q'}{A} = P \qquad \ldots(6.30)$$

The equation (6.30) can be generalised by considering the case when dielectric surface is not perpendicular to P.

Let the normal to the surface OO′ make an angle θ with the direction of *P* as shown in Fig. 6.12. The component of P normal to the surface is equal to $\vec{P}.\hat{n}$, where $\hat{n}$ is a unit vector along the normal to the surface, we have,

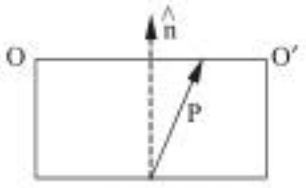


Fig. 6.12

$$\sigma_{pol} = \vec{P}.\hat{n} \qquad \ldots(6.31)$$

$\vec{P}$ at a Point

The value of P in equation (6.28) is an average for the volume A*d*. To define the meaning of P at a point, it is convenient to assume that a dielectric in an electric field has a *continuous distribution of infinitesimal dipoles*, that is, a continuous polarisation. Let us consider a continuously polarised dielectric, *the value of* $\vec{P}$ *at a point can be defined as the net dipole moment* $\vec{P}$ *of a small volume* Δv *divided by the volume, with the limit taken as* Δv *shrinks to zero around the point, thus,*

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\vec{P}}{\Delta v} \qquad \ldots(6.32)$$

6.14 UNIFORM AND NON-UNIFORM POLARISATION

When isotropic and homogeneous materials (*i.e.* the materials having constant value of *K* hence the constant value of dipole moment or polarisation vector $\vec{P}$) are placed in an electric field developed opposite charges of equal magnitude on the two sides of the surface because of polarisation. *Hence the net charge or average charge density is zero.* Such polarisation is called "uniform polarisation". The charge density remains the positive and negative charges are displaced realtive to each other. Their displacement does not produce any net charge inside the volume.

On the other hand, if the material is *non-isotropic, and inhomogeneous*, *K* and $\vec{P}$ are not constant, say $\vec{P}$ is larger at one place and smaller at another. This means that more charge be moved into some region than away from it. We then get a *volume density* of charge.

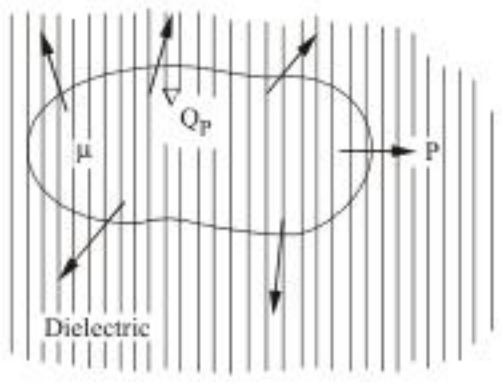


Fig. 6.13

The total charge displaced out of any volume *v* by the polarisation is the integral of the outward drawn normal component of P over the surface that bounds the volume. See Fig. 6.13.

An equal excess charge of the opposite sign is left behind when charge crosses the surface. Let the net charge enclosed by volume be ΔQ_p. We can write,

$$\Delta Q_p = -\int_S \vec{\sigma}_p . d\vec{S}$$

Now using relation,

$\sigma_{pol} = \vec{P}.\hat{n}$, see equation (6.31)

Hence, $\Delta Q_p = -\int_S \vec{P}.\hat{n}\, dS$

If ρ_p is the volume density of polarised charge in the dielectric, then $\Delta Q_p = \int_V \rho_p \, dv$. Equating these two values, we get

$$\int_V \rho_p \, dv = -\int_S \vec{P}.\hat{n} \, dS$$

using Gauss's divergence theorem, *i.e.*

$$\int_V \vec{P}.\hat{n} \, dS = \int_V \text{div}.\vec{P} \, dv = \int_V \vec{\nabla}.\vec{P} \, dv$$

Hence above relation reduces,

$$\int_V \rho_p \, dv = \int_V \vec{\nabla}.\vec{p}.dv$$

or $$\vec{\nabla}.\vec{p} = -P_p \qquad \ldots(6.32\ a)$$

In uniform polarisation process which occurs in isotropic materials the value of $\vec{P}$ *is constant throughout the material, hence* $\nabla \vec{p}$ *is zero. Now let us consider the non-uniform polarisation.* $\vec{P}$ is not constant, it varies from point to point inside the dielectric material, then $\vec{P}$ is function of position and $\nabla.\vec{p}$ *is given by equation (6.32 a).*

6.15 THREE ELECTRIC VECTORS

Let us consider the polarisation of a dielectric slab placed between the plates of parallel plate capacitor and consider equation (6.23) *i.e.*,

$$\frac{Q}{K\varepsilon_0 A} = \frac{Q}{\varepsilon_0 A} - \frac{Q'}{\varepsilon_0 A}$$

or $$\frac{Q}{A} = \varepsilon_0\left(\frac{Q}{K\varepsilon_0 A}\right) + \frac{Q'}{A} \qquad \ldots(6.33)$$

According to equation (6.21), $\frac{Q}{K\varepsilon_0 A} = E$ and from equation (6.27), $\frac{Q'}{A} = P$.

After substitution, equation (6.33) reduces,

$$\frac{Q}{A} = \varepsilon_0 E + P \qquad \ldots(6.34)$$

Put the ratio, $\frac{Q}{A} = D$. $\qquad \ldots(6.35)$

We call it the electric displacement an equivalent term for flux density. As E and P on the right hand side of equation (6.34) are vectors, D is also a vector, hence we can write,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \text{ everywhere} \qquad \ldots(6.36)$$

or $$\vec{D} = \left(\varepsilon_0 + \frac{\vec{P}}{\vec{E}}\right)\vec{E} \qquad \ldots(6.37)$$

The displacement vector is an important addition which is of great use in Maxwell electro-magnetic equation, to explain the displacement current. We shall discuss it in detail in Chapter 15. However,

from equation (6.35), it is clear that $\vec{D}$ is associated only with free charges.

Fig. 6.14 shows the representation of $\vec{D}$, $\varepsilon_0 \vec{E}$ and $\vec{P}$ for a parallel plate capacitor with dielectric.

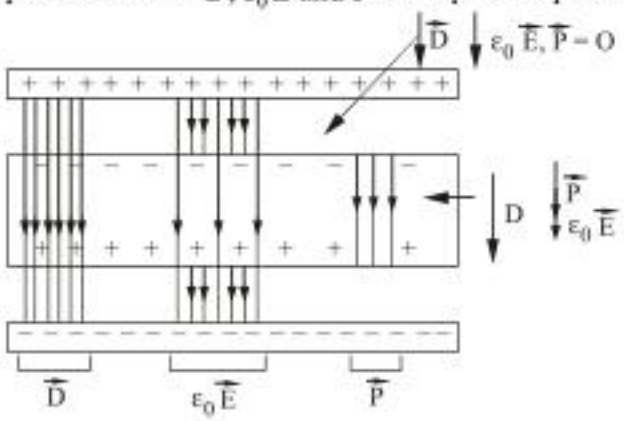


Fig. 6.14

Be careful to note the following points.

1. *$\vec{D}$ is connected with the free charge only.* The lines of $\vec{D}$ begin and end on free charges. $\vec{D}$ is constant throughout the capacitor.
2. *$\vec{P}$ is connected with polarisation charge only.* The lines of $\vec{P}$ begin and end on polarisation charges. Clearly $\vec{P}$ is zero except inside the dielectric.
3. *$\vec{E}$ is connected with all charges.*
4. *In isotropic media $\vec{P}$* and *$\vec{E}$* are in the same direction. However, for *non-isotropic media $\vec{P}$ and $\vec{E}$ are, in general, not in the same direction.*
5. $\vec{E}$ is reduced inside the dielectric, where there are fewer lines.
6. Unlike the electric field $\vec{E}$ (which is the force acting on unit charge) or the polarisation $\vec{P}$ (the dipole moment per unit volume), the electric displacement $\vec{D}$ has no clear physical meaning. The only reason for introducing it is that it enables one to calculate fields in the presence of dielectrics without first having to know the distribution of polarisation charges.

Unit of $\vec{D}$ and $\vec{P}$

From equation (6.27) and (6.36) the units of $\vec{P}$ and $\vec{D}$ are couls/metre^2 whereas $\vec{E}$ is measured in Newtons/coul.

$\vec{D}$ and $\vec{P}$ in terms of $\vec{E}$

The vectors $\vec{D}$ and $\vec{P}$ are separately connected to $\vec{E}$. Let us start from the definition of $\vec{D}$, the magnitude of $\vec{D}$ is given by,

$$D = \frac{Q}{A} = K\varepsilon_0 \left(\frac{Q}{K\varepsilon_0 A} \right)$$

But $\dfrac{Q}{K\varepsilon_0 A} = E$, the expression for $\vec{D}$ can be written as,

$$\vec{D} = K\varepsilon_0 \vec{E} = \varepsilon \vec{E}, \text{ (as } \varepsilon = K\varepsilon_0) \qquad \ldots(6.38)$$

Hence, relation $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ can be written as,

$$\varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$$

or $$\varepsilon = \left(\varepsilon_0 + \frac{\vec{P}}{\vec{E}}\right) \qquad \ldots(6.39)$$

Similarly, we can express $\vec{P}$ in terms of $\vec{E}$ the magnitude of P is given by equation (6.27) *i.e.*,

$$P = \frac{Q'}{A}$$

The induced charge Q' due to polarisation is given in equation (6.24) *i.e.*,

$$Q' = Q\left(1 - \frac{1}{K}\right)$$

Substituting the value in the above equation, we have,

$$P = \frac{Q}{A}\left(1 - \frac{1}{K}\right)$$

Since $\frac{Q}{A} = D$, using eq. (6.38), we get

$$\vec{P} = K\varepsilon_0 \vec{E}\left(1 - \frac{1}{K}\right)$$

or $$\vec{P} = \varepsilon_0 (K-1)\vec{E} \qquad \ldots(6.40)$$

In vacuum K = 1, therefore the polarisation vector $\vec{P}$ is zero.

6.16 RESTATEMENT OF GAUSS'S LAW OR (DIVERGENCE OF $\vec{D}$)

The definition of $\vec{D}$ given by equation (6.38) when substituted in equation (6.25), then the simple modified form of the Gauss'a law in the presence of dielectric is written as,

$$\oint_S \vec{D}.d\vec{S} = Q \qquad \ldots(6.41)$$

or in other words, flux of $\vec{D}$ out of a closed surface S = Total free charge enclosed within S.

That is the surface integral of the normal component of electric displacement vector $\vec{D}$ over any closed surface equals the charge enclosed, where Q is the free charge and the polarisation charge is not included. Equation (6.41) can be written in divergence form as follows. Since according to divergence theorem,

$$\int_S \vec{D}.d\vec{S} = \int_V \text{div}\ \vec{D}\, dv$$

and, $$Q = \int_V \rho_f\, dv$$

hence, $$\vec{\nabla}.\vec{D} = \rho_f \qquad \ldots(6.42)$$

where ρ_f is the volume density of free charge Q. For physical significance see the Gauss's law. Equation (6.42) is still really Gauss's law, now modified in such a way that the effects of polarisation charge are automatically included.

6.17 VERFICATION OF RELATION $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

The variation of electric displacement $\vec{D}$, electric field intensity $\vec{E}$, polarisation $\vec{P}$ along the axis between the plates of an air capacitor is shown in Fig. 6.15 (a). The variation after a dielectric (paraffin) slab has been introduced is illustrated by Fig. 6.15 (b). For a given applied potential *i.e.* battery is

disconnected when the paraffin is introduced. *In air capacitor equation (6.36) reduces,*

$$\vec{D} = \varepsilon_0 \vec{E} \qquad \ldots(1)$$

because there is no polarisation, hence,

$$\vec{P} = 0, \text{ and } \vec{E} = \frac{\vec{D}}{\varepsilon_0} \qquad \ldots(2)$$

In paraffin capacitor the relation is given in equation (6.36)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

The relation between $\vec{D}$ and $\vec{E}$ for this case is given in equation (6.38).

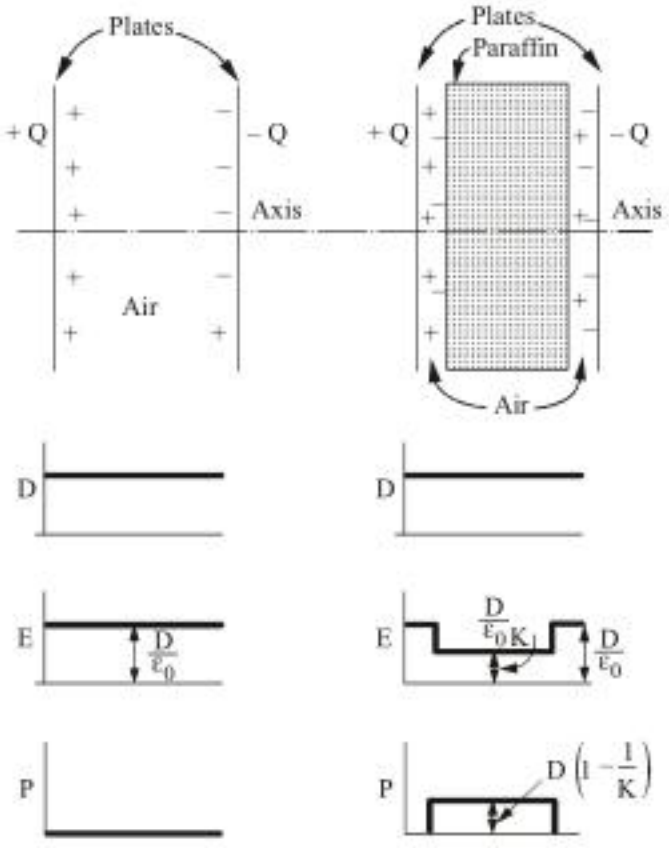


Fig. 6.15 (*a*) Fig 6.15 (*b*)

$$\vec{D} = K \varepsilon_0 \vec{E}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0 K} \qquad \ldots(3)$$

and the polarisation $\vec{P}$ by equation (6.40),

$$\vec{P} = \varepsilon_0 (K-1) \vec{E}$$

or $$\vec{P} = \varepsilon_0 K \left(1 - \frac{1}{K}\right) \vec{E}$$

$$= \vec{D} \left(1 - \frac{1}{K}\right) \qquad \ldots(4)$$

K for paraffin is 2.1, hence the capacitance with paraffin increases by 2.1 times. That is with paraffin dielectric the capacitor plates can store 2.1 times charge for a given potential. *Since the battery was disconnected before the paraffin was introduced, the charge on the plates is same as with air.* Therefore, $\vec{D}$ is the same as before. However, now electric intensity,

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0 K} = 0.475 \frac{\vec{D}}{\varepsilon_0}, \text{ with air it is } \frac{\vec{D}}{\varepsilon_0} \qquad \ldots(5)$$

$$\text{the polarisation } \vec{P} = \vec{D}\left(1 - \frac{1}{K}\right) = 0.525\vec{D}, \text{ with air } \vec{P} = 0 \qquad \ldots(6)$$

However, the polarisation $\vec{P}$ has such value that

$$\vec{D} \text{ with dielectric} = \vec{P} + \varepsilon_0 \vec{E} = 0.525\vec{D} + 0.475\vec{D} = \vec{D} \qquad \ldots(7)$$

$$\vec{D} \text{ with air} = \varepsilon_0 \vec{E} = \varepsilon_0 \frac{\vec{D}}{\varepsilon_0} = \vec{D}$$

Hence, equation (6,.36) is verified.

Example 6.7 *Using data of example 6.6 calculate* $\vec{E}$, $\vec{D}$ and $\vec{P}$, *then verify the equation,* $\vec{D} = \varepsilon_p \vec{E} + \vec{P}$ *is correct both in the gap and in the dielectric.*

Solution. In the dielectric:

$D = K \varepsilon_0 E$, putting the value of E and K from example 6.6,

$\vec{D} = 7 \times 8.9 \times 10^{-12}$ couls^2/Newton.metre$^2 \times 1.4 \times 10^3$ volts/ metre

$= 8.9 \times 10^{-3}$ couls/ metre^2, volt = Newtons.metre

and $\vec{P} = \varepsilon_0 (K-1) \vec{E}$

$= 8.9 \times 10^{-12}(7-1)(1.43 \times 10^3)$

$= 7.5 \times 10^{-3}$ coul/ metre^2

The D_0 in air gap is given by,

$D_0 = \varepsilon_0 E_0$ as K = 1 for air

$= 8.9 \times 10^{-12} \times 1 \times 10^4$

$= 8.9 \times 10^{-3}$ coul/ metre^3

In air gap K = 1, hence,

$P_0 = \varepsilon_0 (K-1) E_0 = 0$

in the dielectric,

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = 8.9 \times 10^{-12} \times 1.4 \times 10^3 + 7.5 \times 10^{-8}$ coul/ metre^3

$= 8.9 \times 10^{-8}$ coul/ metre^2

hence verified.

In the air gap $\vec{P} = 0$

$$D_0 = \varepsilon_0 E_0$$

$$= 8.9\times10^{-12}\times1\times10^{4}$$

$$= 8.9\times10^{-8}\ \text{coul/ metre}^2$$

We can say that the relation $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ holds in general, or $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, everywhere.

6.18 DIELECTRIC SUSCEPTIBILITY AND PERMITTIVITY

In most cases (in the so-called linear dielectrics) the magnitude of polarisation is directly proportional to the intensity of the electric field at a given point of a dielectric *i.e.*, we can have,

$$P \propto E$$

or $$\vec{P} = \varepsilon_0 \chi_{er} \vec{E} \qquad \ldots(6.43)$$

Substituting the dimensions in equation (6.43), as,

$$P = \frac{\text{couls}}{\text{metre}^2},\ E = \frac{\text{Newtons}}{\text{Couls}},$$

$$\varepsilon_0 = \frac{\text{couls}^2}{\text{Newton/ metre}^2}$$

we see that χ_{er} is a dimensionless parameter of dielectric called relative dielectric susceptibility. The total proportionality factor $\chi_{er}\,\varepsilon_0 = \chi_r$ has the dimensions Farad/metre is known as absolute *susceptibility or dielectric susceptibility i.e.,*

$$\chi_{er} = \frac{\chi_e}{\varepsilon_0} \qquad \ldots(6.44)$$

$$\left[\chi_{er}\,\varepsilon_0 = \frac{P}{E} = \text{coul}^2/\text{Newton.metre}^2 \text{ since volt} = \frac{\text{Newton metre}}{\text{coul}} \text{ and Farad} = \text{coul/volt}\right.$$

$$\left.\text{Farad} = \frac{\text{coul}^2}{\text{Newton.metre}} \text{ hence, } \chi_{er}\,\varepsilon_0 = \frac{\text{Farad}}{\text{metre}}\right]$$

The polarisation vector $\vec{P}$ is also related to $\vec{E}$ by equation (6.40), *i.e.*,

$$\vec{P} = \varepsilon_0 (K-1) \vec{E}$$

Hence by comparison of equations (6.40) and (6.43), we can write,

$$\chi_{er} = K-1 \qquad \ldots(6.45)$$

The dielectric susceptibility is measure of the capability of the dielectric to take polarisation, by equation (6.45).

$$K = 1+\chi_{er} \qquad \ldots(6.46)$$

Let us now rewrite equation (6.38),

$$\vec{D} = K\,\varepsilon_0 \vec{E} = \varepsilon \vec{E} \qquad \ldots(6.47)$$

where ε is the permittivity of the medium,

$$\varepsilon = K\,\varepsilon_0$$

or $$K = \frac{\varepsilon}{\varepsilon_0} \qquad \ldots(6.48)$$

K is relative permittivity or dielectric constant (sometimes relative permittivity is denoted by ε_r) *when a dielectric material is placed in electric field, the distribution of field changes to a degree depending upon relative permittivity.* The χ_{er} is given by using equation (6.43),

$$\chi_{er} = \frac{P}{\varepsilon_0 E}$$

$$= \frac{\frac{Q'}{A}}{\frac{\varepsilon_0 Q}{K \varepsilon_0 A}} \qquad \left(\text{Since } E = \frac{Q}{K \varepsilon_0 A} \text{ see equation (6.21) and } P = Q/A \text{ by definition given in equation (6.27)} \right)$$

$$= K \frac{\frac{Q'}{A}}{\frac{Q}{A}}$$

$$= \frac{\frac{\varepsilon Q'}{A}}{\frac{\varepsilon_0 Q}{A}} = \frac{\text{Charge density in dielectric}}{\text{Charge density in free space}} \qquad \ldots(6.49)$$

Now from equation (6.46),

$$K = 1 + \chi_{er}$$

This relation clearly shows that the relative permittivity of any matter is larger than unity. The value of χ_{er} of any matter is positive so that K should be larger than one and for vacuum $\chi_{er} = 0$ and therefore K = 1.

In the case of *non-linear dielectrics including ferroelctrics to be discussed later on, there is no proportionality between vectors* $\overline{P}$ *and* $\overline{E}$ *as discussed here.*

6.19 BOUNDARY RELATIONS

1. Boundary conditions at the surface separating two substances

Let us examine, how the electric field changes at the boundary between two different media. The electric field change that occurs in going from one medium to another is determined by the two basic ideas in electrostatics, namely,

(i) the first is Gauss's law *i.e.*, $\oint_s \overline{D}.d\overline{S} = Q$ and

(ii) the second is that an electrostatics field is a conservative field, in other words, no work is done in transporting a change around a closed path in an electrostatics field *i.e.*,

$$\oint \overline{E}.d\overline{l} = 0$$

We shall apply Gauss's law to the cylindrical surface of height *h* and base area Δs as shown in Fig. 6.16. The cylindrical box is so constructed that it lies half in each medium. Let D_{1n} be the average normal component of displacement vector $\overline{D}$ to the bottom of the box in medium 1 and D_{2n} the average normal component of displacement vector $\overline{D}$ to the face of the box in medium 2. D_{1n} *is an inward normal.* By making the height of the cylinder *h* approaching zero, the contribution of the curved surface to the flux is taken as zero. Thus by Gauss's law, the total flux,

$$D_{2n} \Delta S - D_{1n} \Delta S = Q$$

where Q is total charge enclosed by the surface.

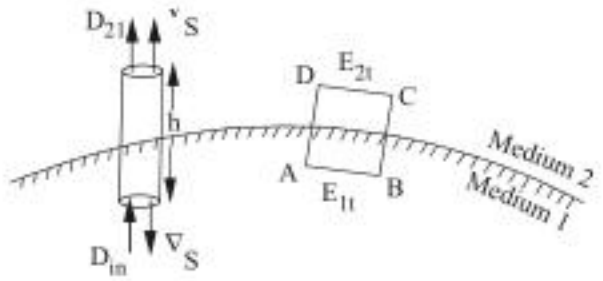


Fig. 6.16

The second term of left hand side of this equation is negative because D_{1n} *and* ΔS *are oppositely directed.* We can write,

$$D_{2n} - D_{1n} = \frac{Q}{\Delta S} = \sigma \qquad \ldots(6.50)$$

where σ is the charge per unit area on the boundary of the two substances. *According to equation (6.50) the normal component of the displacement vector* $\vec{D}$ *changes at a charged boundary between two dielectrics by an amount equal to the surface charge density.*

If the boundary is free from charge $\sigma = 0$, then equation (6.50) reduces,

$$D_{2n} = D_{1n} \qquad \ldots(6.51)$$

Thus, the normal component of displacement vector is continuous across the charge free boundary between two dielectrics.

We shall now use the idea that the electrostatics field is conservative, the integral of $\vec{E}$ around a closed path,

$$\int_{ABCD} \vec{E}.d\vec{l} = \int_{AB} \vec{E}.d\vec{l} + \int_{BC} \vec{E}.d\vec{l} + \int_{CD} \vec{E}.d\vec{l} + \int_{DA} \vec{E}.d\vec{l} = 0$$

By making the path length BC very small approaching zero, the work along the segments BC and DA of the path normal to the boundary is zero even though a finite electric field may exist normal to the boundary. Therefore, the line integral of $\vec{E}$ around ABCD rectangle is,

$$E_{1t}\ \Delta x - E_{2t}\ \Delta x = 0, \quad AB = CD = \Delta x$$

or $$E_{1t} = E_{2t} \qquad \ldots(6.52)$$

Thus the tangential components of the electric field are the same on both sides of a boundary between two dielectrics. In other words, the *tangential electric field is continuous across such a boundary.*

2. Boundary conditions at the surface of a charged conductor

If one of the substances (medium or dielectric) considered above is a conductor, for example, if medium 1 is a conductor, then the electric field inside the conductor is zero, consequently, since there are no permanent dipoles in the conductors, the polarisation $\vec{P}$ must be zero, then according to relation $(\vec{D} = \varepsilon \vec{E} + \vec{P})\ \vec{D}$ is also zero,

$$D_{1n} = 0$$

Thus equation (6.50) reduces,

$$D_{2n} = \sigma \qquad \ldots(6.53)$$

This expresses the important result that the normal component of the displacement just outside the conductor is equal to the surface charge density on the conductor. As medium 1 is a conductor ($\sigma_1 \neq 0$) the field E_{1t} in medium 1 must be zero under static condition. Equation (6.52) reduces,

$$E_{2t} = 0 \qquad \ldots(6.54)$$

That is, the tangential component of the electric field just outside conductor (a dielectric conductor boundary) must be zero.

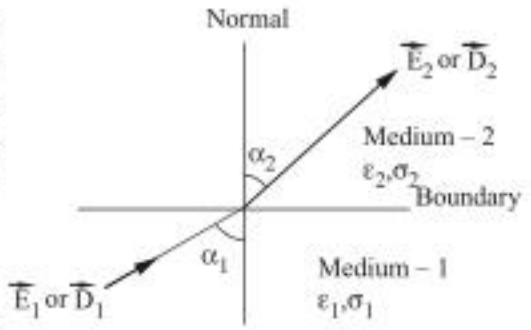


Fig.6.17

Example 6.8 *Boundary between two dielectrics. Let two isotropic dielectrics media 1 and 2 be separated by a charge free plane boundary as in Fig. 6.17. Let permittivities be ε_1 and ε_2 then show that,*

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{K_1}{K_2}$$

where K_1 and K_2 are the dielectric constants of the two mediums.

Solution. As given media 1 and 2 are separated by a charge free plane boundary, *i.e.*, surface density of charge on boundary is zero, then boundary relations are:

$D_{1n} = D_{2n}$ and $E_{1t} = E_{2t}$ as discussed above in eq. (6.51) and (6.52)

From figure,

D_{1n} *i.e.* normal component of $D_{1n} = D_1 \cos\alpha_1$

and $D_{2n} = D_2 \cos\alpha_2$

while, $E_{1t} = E_1 \sin\alpha_1$ and $E_{2t} = E_2 \sin\alpha_2$

therefore,

$D_1 \cos\alpha_1 = D_2 \cos\alpha_2$, (since $D_{1n} = D_{2n}$)

and $E_1 \sin\alpha_1 = E_2 \sin\alpha_2$, (since $E_{1n} = E_{2n}$)

Ratio of these,

$$\frac{D_1 \cos\alpha_1}{E_1 \sin\alpha_1} = \frac{D_2 \cos\alpha_2}{E_2 \sin\alpha_2}$$

But $D_1 = \varepsilon_1 E_1$ and $D_2 = \varepsilon_2 E_2$

$$\frac{\varepsilon_1 E_1 \cos\alpha_1}{E_1 \sin\alpha_1} = \frac{\varepsilon_2 E_2 \cos\alpha_2}{E_2 \sin\alpha_2}$$

or $$\frac{\varepsilon_1}{\tan\alpha_1} = \frac{\varepsilon_2}{\tan\alpha_2}$$

or $$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

If K_1 and K_2 are the relative permittivities of two medium, then

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\frac{K_1}{\varepsilon_0}}{\frac{K_2}{\varepsilon_0}} = \frac{K_1}{K_2} \qquad \ldots(6.55)$$

6.20 FLUX TUBES

Maxwell and Faraday extended the idea of *flux lines to the flux tubes which* have a definite quantitative significance. Let us consider the field intensity at a constant radius from a point charge, the intensity is same for all points on the circumference of this radius.

So that the $\overline{E}$ *and the flux density* $\overline{D}$ *are continuous functions of position around the point charge, and both are constant for a fixed radius.*

Thus, if no flux lines pass through a certain area surrounding the point charge this means flux density is zero there which is not true becasue $\overline{\mathrm{D}}$ and $\overline{\mathrm{E}}$ have constant value for all points at equal distances from the charge. It may be that some lines would pass through the area if a larger number of flux lines had been assumed. This difficulty can be avoided by simple extension of the concept of flux lines to flux tubes.

A *flux tube is defined as an imaginary tube with walls that are everywhere parallel to* $\overline{D}$ *and with a constant electric flux over any cross-section. The requirement that the flux over any cross-section be a constant actually is a consequence of the fact that* $\overline{D}$ *is parallel to the sides of the tube and, therefore, that the flux over the side walls is zero.*

Using the flux-tube idea, we have the flux tubes as shown in Fig. 6.18 instead of flux lines as discussed in Art. 4.20.

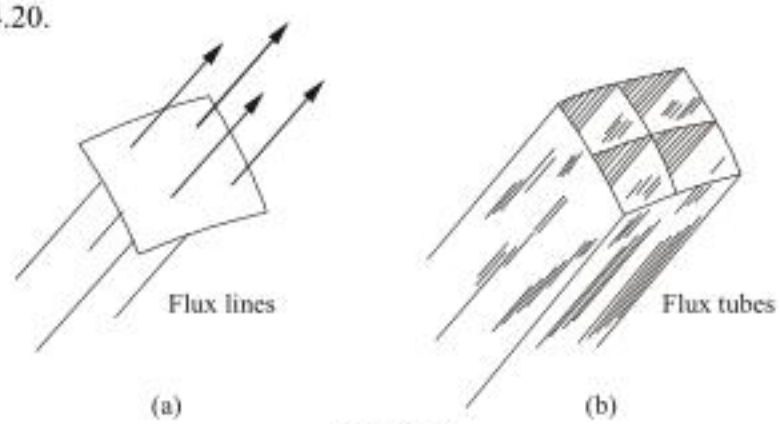


Fig. 6.18

The cross-section of the tube may be of any shape. Conveniently tubes of triangular, squares, or hexagonal cross-section are preferred. In Fig. 6.19 a single flux tube of square cross-section is shown which originates at +Q and ends on – Q.

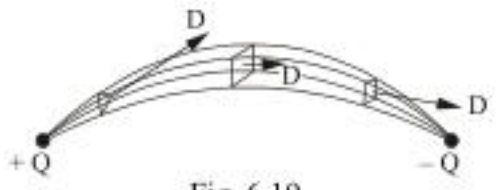


Fig. 6.19

The magnitude of $\overline{D}$ *varies along the tube but the integral of the normal component of* $\overline{D}$ *over any cross-section of the tube is constant.*

In Fig. 6.19 *the flux density* $\overline{D}$ *is indicated at three points along the tube, where the magnitude of* $\overline{D}$ *is large, the cross-sectional area is small and vice versa. The integral of* $\overline{D}$ *over each cross-sectional area is constant.* All the space can be divided up into tubes of equal flux originating on positive charge.

Number of Tubes

In rationlised RMKS system, we have,

$$\oint_{S} \overline{\mathrm{D}}.d\overline{\mathrm{S}} = \mathrm{Q}$$

Thus *each unit charge gives rise to a unit tube (in place of* 4π *tubes in unrationalised system).*

Faraday Tubes (Unit Tubes of Induction)

In RMKS, each unit charge gives rise to a unit tube irrespective of the surrounding medium. The tubes are then referred as unit tubes of induction. The number of unit tubes of induction due to a charge q placed in vacuum per unit area

$$= \frac{q}{4\pi r^2}$$

$$= \varepsilon_0 \frac{q}{4\pi\varepsilon_0 r^2}$$

$$= \varepsilon_0 \vec{E} \qquad \ldots(6.56)$$

i.e., flux density $\vec{D}$.

Maxwell Tubes

However, Maxwell assumed that $1/\varepsilon$ unit tubes originate from each unit charge in a medium of permittivity ε. Thus the charge q gives rise to q/ε Maxwell's tubes. If we draw a sphere of radius r surrounding the charge, then,

$$\text{No. of Maxwell's tubes per unit area} = \frac{q}{\varepsilon 4\pi r^2}$$

$$= \vec{E} \qquad \ldots(6.57)$$

Hence the electric intensity at the point is the number of Maxwell unit tubes of force over a unit area surrounding the point.

Maxwell assumed that these tubes are under longitudinal tension of $\varepsilon_0 \frac{E^2}{2}$ Newton/metre^2 along the direction of the tube which tends to contract and the *lateral pressure of* the same value $\left(\varepsilon_0 \frac{E^2}{2} \text{ Newton/ metre}^2\right)$ normal to the length, and showed that such a system of tension and pressures in the medium are in equilibrium. He explained that the *attraction between two opposite charges is due to longitudinal tension tending to shorten the tube. The repulsion between the charges could similarly be due to the pressure of the tubes upon each other at right angles to their length and which tend to push the two charges further apart.*

Suppose a tube of force as shown in Fig. 6.20. Apply Gauss's law at P and Q planes of areas dS_1 and dS_2.

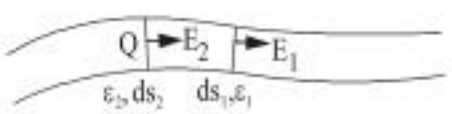


Fig. 6.20

$$\varepsilon_1 E_1 dS_1 = \varepsilon_2 E_2 dS_2 = 0$$

Since there is no charge enclosed by the tube where ε_1 and ε_2 are the permittivity constant at these two surfaces.

Thus, $\varepsilon_1 E_1 dS_1 = \varepsilon_2 E_2 dS_2$

or $\oint_S \vec{D}_1 . d\vec{S}_1 = \oint_S \vec{D}_2 . d\vec{S}_2$

Thus electric displacement vectors (or flux densities) are different but flux is same.

6.21 PHYSICAL MEANING OF POLARISATION OF DIELECTRIC

When a dielectric is placed in an external electric field, the electrically charged particles (atoms, ions, molecules) with their displacement arrange in such order that *dielectric acquires a certain electric moment, such a process is called polaristion.*

Linear dielectrics show a *direct proportionality* between the electric moment P (induced moment) acquired by the particle during the process of polarisation and the intensity of the electric field acting on the particle *i.e.*, the local electric field inside the dielectric E_{int}. Mathematically we represent,

$$P \propto E_{int}$$

$$P = \alpha E_{int} \qquad \ldots(6.58)$$

The coefficient of proportionality α in equation (6.58) is known as polarizability of the given dipole. Polarisability reflects the properties of an individual particle of matter and not of large valume of it hence it is called microscopic electrical parametre of a dielectric.

If, $E_{int} = 1$, in equation (6.58)

then, $$P = \alpha \qquad \ldots(6.59)$$

So the *polarisability a is dipole moment per unit field strength.* If there are N dipoles in unit volume of the matter then equation (6.58) can be written as,

$$P = N \alpha E_{int} \qquad \ldots(6.60)$$

Here P is the average dipole moment of N dipoles.

Unit for Polarisability

When dipole moment P is measured in coul. metre, (P = charge × distance) and E is volt/metre, we get SI unit for a, from α, equation (6.58),

$$\alpha = \frac{\text{Coul. metre}}{\text{Volt/ metre}} = \frac{\text{Coul. metre}^2}{\text{volt}}, \text{(Coul = volt} \times \text{Farad)}$$

$$\alpha = \text{Farad. metre}^2$$

6.22 MECHANISM OF POLARISATION

Since α is defined in terms of dipole moment, its magnitude will clearly be a measure of the extent to which electric dipoles are formed by ions, atoms and molecules. The electric *dipoles are formed through a variety of mechanisms*, each one contributing to the value of α, hence α is regarded as total polarisability to be the sum of individual polarisabilities each arising from a particular mechanism.

In general, the molecules or atoms (ions) may be affected in the following ways:

1. Electronic Polarisability

Centres of gravity of the electrons and protons may change resulting in induced dipole moment *i.e.*,

$$P_e = \alpha_e E_{int} \qquad \ldots(6.61)$$

P_e has all the characteristics of an assembly of dipoles produced by displacement of electrons which have natural frequency equal to, or higher than those of visible light. So that α_e is called the *optical or electronic polarisability*.

2. Molecular Polarisability

Molecules having permanent dipole moment *i.e.*, polar molecules are affected in two possible ways:

(a) **Atomic Polarisability:** Firstly, the field may cause the atoms to be displaced altering the distance between them and hence changing the dipole moment of the molecule. Such polarisability is called atomic represented by α_a.

(b) **Orientational or Dipolar Polarisability:** Secondly, the molecule as a whole may rotate about its axis of symmetry, so that the dipoles align with the field. This is referred as orientational polarisation or dipolar polarisation and polarisability is called orientational or dipolar, denoted by α_d.

3. Interfacial Polarisability

Such type of mechanism is due to large number of defects in the structure of crystals, such as lattice vacancies, impurity centre, dislocation, etc. Polarisability is denoted by α_i, however, this is generally neglected. Thus we can write the total polarisability α as follows:

$$\alpha = a_e + \alpha_a + \alpha_d + \alpha_i \qquad \ldots(6.62)$$

6.23 LORENTZLOCAL FIELD

In equation (6.60) *i.e.*, $P = N\alpha E_{int}$, it should be noted that this internal intensity of the field, where molecules exist surrounded on all sides by other polarised molecules of matter, is not equal to the external intensity of the field at a given point of a dielectric which is calculated by relation $E = V/d$. The internal intensity E_{int} on a molecule is a geometrical sum of the external intensity E and the component of the intensity caused by the action of the other polarised molecules of the matter on the molecules in question. The following method suggested by Lorentz can be used to estimate the relation between E and E_{int}.

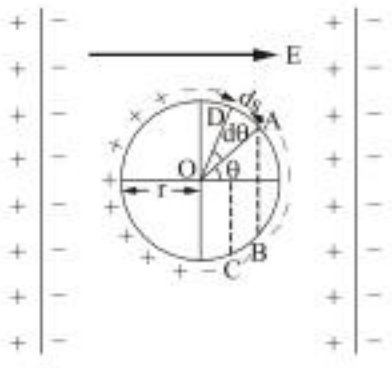


Fig. 6.21

Let us describe a sphere in a dielectric as shown in Fig. 6.21, containing the molecule considered above in its centre. The molecules present inside dielectric outside of the sphere will build up an internsity component E_1 in the centre of the sphere and those inside the sphere an intensity component E_2. Thus internal intensity will be equal to:

$$\vec{E}_{int} = \vec{E} + \vec{E}_1 + \vec{E}_2 \qquad \ldots(6.63)$$

To calculate E_1, let us imagine that the dielectric is removed from the sphere. For the actual pattern of the electric field not to be distorted, a surface electric charge should be placed on the sphere.

At each point of the sphere the surface density of the charge is given by equation (6.31), *i.e.* $\sigma = \vec{P}.\hat{n}\,dS$ that is the definition of $\vec{P}$ so that,

$$\sigma = P\cos\theta \qquad \ldots(6.64)$$

where θ is the angle between the direction of $\vec{P}$ *i.e.*, the direction of $\vec{E}$, and the radius of the sphere. The charge dq on the small element of the sphere surface dS will be given by,

$dq = \sigma\, dS$, substituting from Eq. (6.64), we get

$$dq = P\cos\theta\, dS \qquad \ldots(6.65)$$

The field created by this charge dq inside the sphere,

$$dE_1 = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$

Putting dq from equation (6.65),

$$dE_1 = \frac{P}{4\pi\varepsilon_0 r^2}\cos\theta\, dS \qquad \ldots(6.66)$$

The field dE_1 has the component one is parallel to the direction of E, *i.e.*,

$$dE_1 \cos\theta = \frac{P}{4\pi\varepsilon_0 r^2}\cos^2\theta \, dS \qquad \ldots(6.67)$$

And the other perpendicular to the direction of E *i.e.*,

$$dE_1 \sin\theta = \frac{P}{4\pi\varepsilon_0 r^2}\cos\theta \sin\theta \, dS \qquad \ldots(6.68)$$

The total intensity E_1 of the entire charge on the sphere is determined by integrating dE_1 over the whole surface of the sphere. But the components of the upper and lower halves in Fig. 6.21 of the sphere perpendicular to E will cancel. So there will be only parallel components. Therefore, the component of E_1 at O due to the right half sphere.

$$= dE_1 \cos\theta = \int_0^{\pi/2} \frac{P}{4\pi\varepsilon_0}\frac{\cos^2\theta}{r^2} dS$$

dS is the area of ring ABCD *i.e.*,

$$dS = 2\pi r^2 \sin\theta \, d\theta$$

Therefore, $dE_1 \cos\theta = \int_0^{\pi/2} \frac{P}{4\pi\varepsilon_0}\frac{\cos^2\theta}{r^2} 2\pi r^2 \sin\theta \, d\theta$

$$= \frac{P}{2\varepsilon_0}\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{P}{2\varepsilon_0}\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

$$= \frac{P}{6\varepsilon_0} \qquad \ldots(6.69)$$

As shown in Fig. 6.21, the charges on the surfaces of the two halves of the sphere are equal in magnitude but opposite in sign an equal force acting in the same direction will be exerted on unit positive charge at O. So, the component of E_1 due to left half sphere is also given by,

$$= \frac{P}{6\varepsilon_0}$$

So, the total intensity E_1 is sum of the two *i.e.*,

$$\vec{E}_1 = \frac{\vec{P}}{3\varepsilon_0} \qquad \ldots(6.70)$$

The direction of this intensity coincides with the direction E, *the value of E_1 given in equation (6.70) is called Lorentz field.*

6.24 CLAUSIUS MOSOTTI EQUATION

For the simplest case when there is no prevailing effect of any adjacent charges on the molecules in question *i.e.* $\vec{E}_2$ components of Eq. (6.63) is taken zero, the equation (6.63) after substituting (6.70), gives,

$$\vec{E}_{int} = \vec{E} + \frac{\vec{P}}{3\varepsilon_0} \qquad \ldots(6.71)$$

We have also a relation between $\vec{P}$ and $\vec{E}$ given in equation (6.40), *i.e.*,

$$P = \varepsilon_0 (K-1)\,\vec{E}$$

or $$\frac{\vec{P}}{\varepsilon_0} = (K-1)\,\vec{E}$$

Inserting $\frac{\vec{P}}{\varepsilon_0}$ in equation (6.71), we get

$$\vec{E}_{int} = \vec{E}\left(1+\frac{(K-1)}{3}\right)$$

or $$\vec{E}_{int} = \vec{E}\left[\frac{K+2}{3}\right] \qquad \ldots(6.72)$$

The $\vec{E}_{int}$ given in equation (6.72) is called Mosotti field. Since the expression for $\vec{E}_{int}$ contains K, hence it depends on dielectric material in case of vacuum capacitor with K = 1, that the ratio $\frac{K+2}{3} = 1$ and $\vec{E}_{int} = \vec{E}$.

The value of polarisation P as an electric moment of a unit volume of dielectric is given in equation (6.60) *i.e.*,

$$\vec{P} = N \propto \vec{E}_{int}$$

Putting the value $\vec{E}_{int}$ from equation (6.72), in the above equation, we have,

$$\vec{P} = N \propto \frac{(K+2)}{3}\vec{E}$$

and equate this expression to P given below:

$$\vec{P} = \varepsilon_0 (K-1)\,\vec{E}, \qquad \text{[From equation (6.40)]},$$

i.e. $$(K-1)\,\varepsilon_0\, E = N \propto E \frac{K+2}{3}$$

or $$\frac{K+1}{K+2} = \frac{N\alpha}{3\varepsilon_0} \qquad \ldots(6.73)$$

where $\alpha = \alpha_e + \alpha_a$. Equation (6.73) is *remarkable for the fact that it shows the connection between the parametres K of a dielectric and its parametres* α *and N.* The equation (6.73) *is known as Clausius Mosottia equation* As $K = \varepsilon/\varepsilon_0$ the Clausius Mosotti equation in another simple form is,

$$\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0} = \frac{N\alpha}{3\varepsilon_0} \qquad \ldots(6.74)$$

The magnitude $\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}$ or $\frac{K-1}{K+2}$ is known as *specific polarisation of a dielectric.* This is dimensionless because K is dimensionless. It is easy to see that the R.H.S. of equation (6.73) is also dimensionless because N is measured in metre^{-3}, α in farad metre^2 and ε_0 in farad/metre, hence $N\alpha/3\varepsilon_0$ is dimensionless.

Now using relation $K = 1 + \chi_{er}$. *The Clausius Mosotti equation reduces in terms of the susceptibility of the dielectric.*

$$\frac{\chi_{er}}{\chi_{er}+3} = \frac{N\alpha}{3\varepsilon_0} \qquad \ldots(6.75)$$

A version of Clausius Mosotti equation convenient for practical calculation is obtained by multiplying equation (6.73) on both sides by the ratio of the molecular mass M of the substance to its density D.

$$\frac{K-1}{K+2}\frac{M}{D} = \frac{N\alpha}{3\varepsilon_0}\frac{M}{D}$$

Since $\frac{NM}{D}$ = Avogadro number N_a

$$\frac{K-1}{K+2}\frac{M}{D} = \frac{1}{3\varepsilon_0} N_a \alpha \qquad \ldots(6.76)$$

but $\frac{K-1}{K+2}$ = specific polarisation (P_{sp})

Hence, equation (6.76) reduces,

$$\frac{K-1}{K+2}\frac{M}{D} = P_{sp}\frac{M}{D} \qquad \ldots(6.77)$$

The magnitude given in equation (6.77) is called the *molar polarisation represented by* π,

$$\pi = \frac{K-1}{K+2}\frac{M}{D} = P_{sp}\frac{M}{D} = \frac{1}{3\varepsilon_0} N_a \alpha \qquad \ldots(6.78)$$

The dimension of molar polarisation is equal to that of ratio of volume to mass, *i.e.*, the dimension inverse to that of density. For neutral dielectric the value of $K = \eta^2$ where η is the refractive index of the medium then we get after substituting in eqns. (6.77) and (6.78),

$$\frac{\eta^2-1}{\eta^2+2} = \frac{N\alpha}{3\varepsilon_0} \qquad \ldots(6.79)$$

and $$\frac{\eta^2-1}{\eta^2+2}\frac{M}{D} = \frac{N_a\alpha}{3\varepsilon_0} \qquad \ldots(6.80)$$

Equations (6.79) and (6.80) are referred as *Lorentz-Lorentz equation*.

Limitation of the Clausius Mosotti equation

This equation is derived by taking $E_2 = 0$ (See equation (6.63) for details of E_2) because of the following assumptions:

1. Since polarisation is considered as proportional to the field, it means the polarisation of the molecules by elastic displacement only.
2. Absence of short-range interaction.
3. Isotropy of the polarisability of the molecules.

All these conditions are satisfied with neutral molecules having no constant dipoles *i.e. non polar* (molecules of symmetrical structure). Thus the equation is applicable to *neutral liquids and specially to gases in which the molecules are far apart.*

If a graph is plotted between $\frac{K-1}{K+2}\frac{1}{D}$ on one axis and $\eta = \sqrt{K}$ on the other axis with different values of D, for D = 1, gives the function $\frac{K-1}{K+2}$ (K) approximating unity when K infinitely increases, *i.e.*,

$$\frac{K-1}{K+2} \cong 1 \qquad \ldots(6.81)$$

Since K in gases is close to unity, we can write,

$$\frac{K-1}{K+2} \cong \frac{K-1}{3}$$

Equation (6.73) reduces,

$$\frac{K-1}{K+2} \cong \frac{K-1}{3} = \frac{N\alpha}{3\varepsilon_0}$$

or $$K = 1 + \frac{N\alpha}{\varepsilon_0} \qquad \ldots(6.82)$$

6.25 DEBYE EQUATION

When a polar molecule is placed in the electric field, it is affected in two ways. *Firstly*, it displaces the centre of gravity of protons and electron, so that an extra dipole moment is induced, giving the *electronic* (atomic) polarisability. For which the *Clausius Mosotti equation is valid. Secondly, the dipole tend to orient* itself so that its potential energy is minimum *i.e.*, it will *tend to align its moment* against the field. The process is called *orientational polarisation. Debye transformed the Clausius-Mosotti equation for polar molecules* by replacing the electronic and ionic (or atomic) polarisability α of the molecule which is taken in Clausius-Mosotti equation with the sum of deformational polarisability α and the polarisability due to dipole polarisation α_d.

The Clausius-Mosotti equation has no indication of a *temperature variation of permittivity.* However, *for polar dielectrics the polarisability is inversely proportional to temperature due to thermal agitation.* So that orientational polarisation mechanism is temperature dependent. In fact the orientative action of the field is hindered by the thermal agitation. In consequence, a statistical equilibrium is set up in which a slight excess of molecules have their permanent dipoles antiparallel to the field *i.e.* the molecules are aligned.

Let us calculate the average component of dipole moment per molecule at temperature T due to alignment. If molecule of permanent dipole moment $\overline{P}_0$ has its axis inclined at an angle θ with the field, then its potential energy is,

$$U = -\overline{P}_0 \,.\, \overline{E}, \qquad \text{[See equation (5.59)]} \qquad \ldots(6.83)$$

According to Maxwell Boltzmann law, the number of molecules distributed with axis of their dipoles pointing in all directions within a solid angle $d\Omega$ at θ is,

$$dn = A \exp\left[-\frac{U}{K_b T}\right] d\Omega \qquad \ldots(6.84)$$

where K_b is Boltzmann's constant, and A is constant depending on the number of molecules considered.

Integrating this overall possible direction, total number of molecules after putting the value of U from equation (6.83),

$$n = \int_{\theta=0}^{\theta=\pi} A \exp\left[\frac{P_0 E \cos\theta}{K_b T}\right] d\Omega \qquad \ldots(6.85)$$

The effective dipole moment *i.e.*, the dipole moment of a dipole along the field direction $= P_0 \cos\theta$. Therefore, the dipole moment of dn molecules in the field direction,

$$= dn\, P_0 \cos\theta$$

$$= A \exp\left[\frac{P_0 E \cos\theta}{K_b T}\right] P_0 \cos\theta\, d\Omega, \qquad \text{(after substituting dn from eqn. (6.84))}$$

Integrating this within limit $\theta = 0$ to $\theta = \pi$, we get the total dipole moment of all molecules,

$$\sum P_0 = \int_{\theta=0}^{\theta=\pi} A\, e^{x\cos\theta} P_0 \cos\theta\, d\Omega$$

where, $x = \dfrac{P_0 E}{K_b T}$

The average dipole moment $(\overline{P_0})$ along the field,

$$\overline{P_0} = \frac{\Sigma P_0}{n} = \frac{\int_0^{\pi} A\, e^{x\cos\theta} P_0 \cos\theta\, d\Omega}{\int_0^{\pi} A\, e^{x\cos\theta}\, d\Omega} \qquad \ldots(6.86)$$

Now the value of $d\Omega$ between θ and $\theta + d\theta$ is calculated by Fig. 6.22.

$$d\Omega = \frac{\text{area of ring between } \theta \text{ and } \theta + d\theta}{r^2}$$

$$= \frac{2\pi r \sin\theta\, r\, d\theta}{r^2}$$

$$= 2\pi \sin\theta\, d\theta \qquad \ldots(6.86a)$$

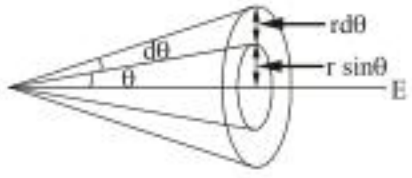


Fig. 6.22

To evaluate the integral of equation (6.86) we put,

$\cos\theta = a$ as θ varies from 0 to π, a will vary +1 to – 1

Hence, $\sin\theta\, d\theta = -da$

Therefore, substituting in equation (6.86) this value, we get,

$$\overline{P_0} = \frac{\int_{-1}^{+1} e^{xa} P_0\, a\, da}{\int_{-1}^{+1} e^{xa}\, da}$$

or
$$\frac{\bar{P}_0}{P_0} = \frac{\left[\frac{a}{x}e^{xa} - \frac{1}{x^2}e^{xa}\right]_{-1}^{+1}}{\left[\frac{1}{x}e^{xa}\right]_{-1}^{+1}}$$

$$= \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x}$$

$$= \coth x - \frac{1}{x}$$

Therefore,

$$\frac{\bar{P}_0}{P_0} = L(x) \qquad \ldots(6.87)$$

where $L(x) = \coth x - \frac{1}{x}$ is called *Langevin function*, because it was derived by Langevin. The relation (6.87) is called *Langevin formula*. The variation of $L(x)$ with x is shown in Fig. 6.23.

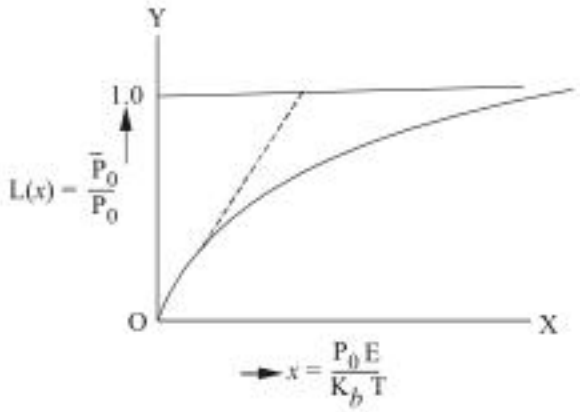


Fig. 6.23

The curve is linear for small values of x, *i.e.* for very small value of E and large values of T, where for large value of x *i.e.* $\frac{P_0 E}{K_b T}$, it approaches unity. Therefore, for *large value of E saturation is obtained i.e., practically all the molecules are aligned with field direction and further increase in E has no effect on average dipole moment.*

However, the linear region corresponding to small values of x is important at ordinary temperature. The dipole moment P_0 of most polar materials is such taht $x << 1$ for a full range of field strength, even for those approaching the dielectric strength of the material, so long as the temperature is above about 250°K. Thus a polar dielectric is, in general, linear. For small values of x, $L(x)$ can be expanded as,

$$L(x) = \frac{x}{3} - \frac{x^3}{4 \times 5} + \ldots$$

Practically x is very small, hence,

$$L(x) = \frac{x}{3}$$

Hence, equation (6.87) reduces,

$$\frac{\bar{P}_0}{P_0} = \frac{x}{3}$$

$$= \frac{P_0 E}{3 K_b T}$$

or
$$\bar{P}_0 = \frac{P_0^2 T}{3 K_b T} \qquad \ldots(6.88)$$

E is the polarising field *i.e.* the field acting on the dipoles. Comparison of this equation (6.88) with equation (6.48), *i.e.* $P = \alpha E_{int}$ which is for linear dielectric, E_{int} here is E, we get polarisability α_d (i.e. molecular dipole moment per unit polarising field) is,

$$a_d = \frac{P_0^2}{3K_b T} \qquad \ldots(6.89)$$

This represents the orientational polarisability. Therefore, the total polarisability,

$$\alpha_{orf} = \alpha + \alpha_d = \alpha + \frac{P_0^2}{3K_b T} \qquad \ldots(6.90)$$

where α is deformational polarisability as discussed in the beginning ($\alpha = \alpha_e + \alpha_a$). Hence,

$$\alpha_{eff} = \alpha_e + \alpha_a + \frac{P_0^2}{3K_b T}$$

Equation (6.90) is known as the *Langevin-Debye equation*. Therefore in the case of polar molecule equations (6.73) and (6.78) are given by replacing α by α_{eff} as below:

$$P_{sp} = \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} = \frac{K-1}{K+2} = \frac{N}{3\varepsilon_0}\left(\alpha + \frac{P_0^2}{3K_b T}\right) \qquad \ldots(6.91)$$

and $$\pi = \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\frac{M}{D} = \frac{K-1}{K+2}\frac{M}{D} = \frac{N_a}{3\varepsilon_0}\left(\alpha + \frac{P_0^2}{3K_b T}\right) \qquad \ldots(6.92)$$

6.26 TEMPERATURE DEPENDENCE

Equation (6.90) shows that α_{eff} depends on temp T. Therefore, for *polar molecule the polarisability is inversely proportional to the temperature so we can say that K and permittivity also depends on temperature in the same manner as* α. (See Eqn. (6.92)).

Since $\frac{P_0^2}{3K_b T} \gg \alpha$, and for *this reason, the permittivity of polar dielectric may be much higher than that of non-polar.* The temperature dependence of dielectric constant K is shown in Fig. 6.24. The intercept of the straight line thus obtained on the *y*-axis is $\frac{N\alpha}{3\varepsilon_0}$ and its slope is

$$\frac{N_a P_0^2}{9\varepsilon_0 K_b}$$

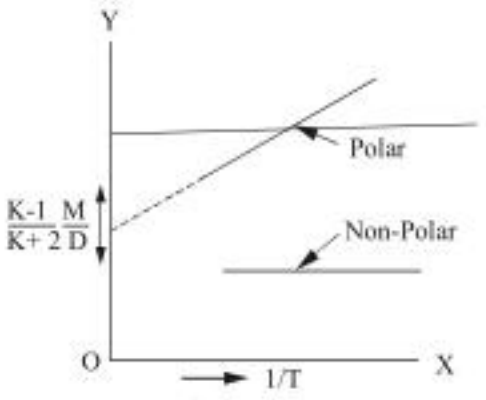


Fig. 6.24

Example 6.9 *The polarisability of NH_3 molecule in gaseous state, from the measurement of dielectric constant is found 2.5 × 10^{-39} farad. metre2 and 2.0 × 10^{-39} farad-metre² at temperature 300°K and 400°K respectively. Calculate the contribution to the polarisability because of deformation of molecules and the contribution because of permanent dipole at these temperatures.*

Solution. Using equation (6.90) of Langevin theory,

$$\alpha_{eff} = \alpha \text{ (due to deformation)} + \frac{P_0^2}{3K_b T} \text{ (due to dipole moment } P_0)$$

On putting the values,

$$2.5\times10^{-39} = \alpha + \frac{P_0^{\,2}}{3K_b\,300}$$

or $$2.5\times10^{-39} = \alpha + \frac{\beta}{300}, \text{ where } \beta = \frac{P_0^{\,2}}{3K_b}$$

and $$2.5\times10^{-39} = \alpha + \frac{\beta}{400}$$

Solving these two equations,

$$\beta = 6\times10^{-37} \text{ farad.metre}^2$$

hence, $\alpha = 0.5\times10^{-39}$ farad.metre^2

The deformational polarisability $= 0.5 \times 10^{-39}$ farad.metre^2 and is independent of temperature.

Due to orientational polarisation at temperature 300°K, the polarisability,

$$= \frac{\beta}{T} = \frac{6\times10^{-37}}{300} \text{ Farad.metre}^2$$

$$= 2\times10^{-39} \text{ Farad.metre}^2$$

at temperature $400°\,K = \frac{6\times10^{-37}}{400}$

$$= 1.5\times10^{-39} \text{ Farad.metre}^2$$

6.27 CLASSIFICATION OF DIELECTRICS (FERROELECTRIC AND PARAELECTRIC DIELECTRIC)

The earlier classification of dielectrics into *polar and non-polar is not sufficient.* They are *classified according to their behaviour in electric field.* Take the equations $\vec{P} = \varepsilon_0(K-1)\,\vec{E}$, $\vec{E}$ is the resultant of polarising (external field) E_0 and field due to polarisation charges E', *i.e.*, $\vec{E} = \vec{E}_0 + \vec{E}'$ and $\vec{E}$ is related to E_0 by relation,

$$\vec{E} = \frac{\vec{E}_0}{K}$$

In most cases the polarisation is proportional to $\vec{E}$.

When polarising field $\vec{E}_0$ is removed then $\vec{E}$ vanishes and in turn $\vec{P}$ also vanishes. But under certain conditions given by equation (6.71),

$$E_{int} = E + \frac{P}{3\varepsilon_0}$$

Now when external field is removed, E is zero, then

$$E_{int} = \frac{P}{3\varepsilon_0}$$

[P_0 is polarisation without external field called permanent polarisation]

Ferroelectric

So we can say that *certain materials retain a permanent polarisation even though the polarising field is removed, such materials are called ferro-electric by analogy with ferromagnetism.* In other words, if a polarisation P_0 exists, it will create an electric field at the molecule which tends to polarise the molecules. Such polarisation is called *spontaneous polarisation*. Such materials are able to develop large polarisation when placed in an electric field for them $K \gg 1$ and $\chi_{er} > 1$.

Examples: Rochelle salt, $BaTiO_3$ (Barium Titenate), $SrTiO_3$ (Stroncium titenate), $KNbO_3$ (Potensiam niobate).

Paraelectric

Materials with *small positive susceptibility* due to polarisation in an electric field are termed as para-electric. K and χ_{er} are positive but having small value than ferroelectric. In fact all dielectrics are para with the exception of vacuum, which has precisely zero susceptibility, $\chi_{er} = K - 1$, for vacuum $K = 1$.

Dia-electric do not Exist

There is no material which has *negative susceptibility*. There is *no analogy with diamagnetism*. In other words, the polarisation in the direction opposite to the polarising field is not possible.

6.28 HYSTERESIS LOOP FOR FERROELECTRIC

Fig. 6.25 shows complete curve of *polarisation vs. electric field*. As it is the property of ferro-electric to retain polarisation when polarising field is removed, a curve such as in Fig. 6.25 is called *hysteresis loop*.

Hysteresis means to lag behind, and it is clear that $\vec{P}$ *lays the electric field vector* $\vec{E}$ when $\vec{E} = 0$, the retained polarisation is *ob*, it is *spontaneous polarisation (analogous to residual magnetism in magnetisation)*. And point C is the electric field which is needed to *completely reverse the polarisation, it is analogous to coercive force in magnetisation*.

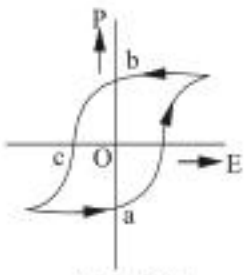


Fig. 6.25

Curie Temperature

However, the *spontaneous polarisation exists only below the certain critical temperature called Curie temperature. Above* the Curie temperature, the *spontaneous polarisation disappears and* substance becomes *para electric*. Below the critical temperature, the variation of susceptibility with temperature is governed by Curie law and above critical temperature by Curie Weiss law.

So, we can say that polarisation study of a dielectric is more or less analogus to magnetisation. The behaviour of materials in electric and magnetic field are identical.

6.29 ENERGY AND FORCE RELATIONS FOR ELECTROSTATICS FIELD WITH DIELECTRIC PRESENT

The basic energy and pressure laws are modified with dielectrics. When a dielectric is placed in an external field, it is polarised. Let the induced charge $+ q$ appears on the surface and $- q$ on the opposite surface, $+ q$ and $- q$ are separated by distance d is the case of parallel plate capacitor with dielectric *i.e.* the thickness of the dielectric slab, Dipole moment $\vec{P} = qd$. Suppose that while the field $\vec{E}$ is present the charges are further displaced by a distance Δd. The change in dipole moment is given by,

$$dp = q(d + \Delta d) - qd$$

or $$dp = q \, \Delta \, d.$$

The force on the polarisation charge is qE. So the work done for displacement Δd is given by,

$$d\,\mathrm{W} = q\,\mathrm{E}\,\Delta d$$

Δd is the displacement in the direction of $\vec{\mathrm{E}}$ inserting the value of dP given above, we get

$$d\mathrm{W} = \vec{\mathrm{E}}.d\vec{\mathrm{P}} \text{ as } \vec{\mathrm{E}} \text{ and } \vec{\mathrm{P}} \text{ point in the same direction.}$$

This work done is stored in the form of energy in the dielectric. The supplier of the work is the source of the field, the battery that maintained constant potential difference, between the plates. This stored energy in the dielectric increases the internal energy of the molecules.

But since $\vec{\mathrm{P}} = \varepsilon_0(\mathrm{K}-1)\,\vec{\mathrm{E}}$, for linear dielectric, where $\vec{\mathrm{P}}$ is dipole moment per unit volume. Therefore,

$$d\vec{\mathrm{P}} = \varepsilon_0\,(\mathrm{K}-1)\,d\vec{\mathrm{E}}$$

where $d\vec{\mathrm{P}}$ is the change in dipole moment per unit volume. So work done is also given by,

$$\vec{\mathrm{E}}.d\vec{\mathrm{P}} = \vec{\mathrm{E}}.\varepsilon_0\,(\mathrm{K}-1)\,d\vec{\mathrm{E}}$$

Since, $\vec{\mathrm{E}}.d\vec{\mathrm{E}} = \mathrm{E}\,d\,\mathrm{E} = \dfrac{1}{2}\,d\,(\mathrm{E}^2)$, therefore,

$$\vec{\mathrm{E}}.d\vec{\mathrm{P}} = \frac{\varepsilon_0}{2}\,(\mathrm{K}-1)\,d\,(\mathrm{E}^2)$$

or $$\int_0^{\mathrm{P}} \vec{\mathrm{E}}.d\vec{\mathrm{P}} = \frac{\varepsilon_0}{2}\,(\mathrm{K}-1)\,\mathrm{E}^2$$

This equation gives the energy change due to polarisation P. Therefore the *total energy density equal to energy density in free space plus the energy density in dielectric.* Therefore, the total energy density,

$$= \frac{1}{2}\,\varepsilon_0\,\mathrm{E}^2 + \frac{\varepsilon_0}{2}\,(\mathrm{K}-1)\,\mathrm{E}^2$$

$$= \frac{1}{2}\,\varepsilon_0\,\mathrm{K}\,\mathrm{E}^2 \qquad \ldots(6.93)$$

That is *energy increase by K times than that of without dielectric.* Energy per unit volume represented by U is given by,

$$\mathrm{U} = \frac{\varepsilon_0}{2}\,\mathrm{K}\,\mathrm{E}^2 \text{ jouls/ metre}^3$$

Since $\vec{\mathrm{D}} = \mathrm{K}\,\varepsilon_0\,\vec{\mathrm{E}}$ and $\vec{\mathrm{D}}$ is in the direction of $\vec{\mathrm{E}}$, therefore we can write,

$$\mathrm{U} = \frac{1}{2}\,\vec{\mathrm{E}}.\vec{\mathrm{D}} \text{ joules/ metre}^3 \qquad \ldots(6.94)$$

For any volume,

$$\mathrm{U} = \frac{1}{2}\,\varepsilon_0 \int_v \mathrm{K}\,\mathrm{E}^2\,dv \qquad \ldots(6.95)$$

or $$\mathrm{U} = \frac{1}{2}\int_v \vec{\mathrm{E}}.\vec{\mathrm{D}}\,dv \qquad \ldots(6.96)$$

If dielectric is isotropic then,

$$U = \frac{1}{2}\varepsilon_0 K \int_v E^2 \, dv \text{ joules} \qquad \ldots(6.97)$$

Though we have *derived it for parallel plate capacitor but it is applicable in all cases.*

Force

An expression for force can be obtained by *principle of virtual work.* If the work is done by an external agent against the force of attraction between the plates in moving them further apart through a distance Δd,

$$W = \vec{F}.\Delta\vec{d}$$

or $$W = -F\Delta d$$

The change in volume due to this additional separation, $\Delta d = A\Delta d$, where A is the area of the plate, change in energy due to this change in volume equals $\frac{1}{2}\varepsilon_0 K E^2 A\Delta d$

work done = change in energy

$$\frac{1}{2}\varepsilon_0 K E^2 A\Delta d = \vec{F}.\Delta\vec{d}$$

$$\frac{1}{2}(\vec{E}.\vec{D})\,A\Delta d = \vec{F}.\Delta\vec{d}$$

Since $\vec{D} = \varepsilon_0 K\vec{E}$ and in the direction of $\vec{E}$

or $$F = \frac{1}{2}\vec{E}.\vec{D}\,A \text{ Newtons}$$

(Since, $\vec{E}$ = Newtons/coul, $\vec{D}$ = Couls/ metre^2 and A = metre^2)

$$\text{Pressure} = \frac{\vec{F}}{A} = \frac{1}{2}\vec{E}.\vec{D} \text{ Newtons/ metre}^2 \qquad \ldots(6.98)$$

This is force of attraction between the plates tending to pull them together. Though the pressure exists at any ponit in an electric field.

Pressure at a Conductor (Plates)–Dielectric Boundary

The pressure is in the direction to draw the conductor into the dielectric. The result follows because the two plates attract each other. The pressure of the conductor on the dielectric is given by equation (6.98) which may be written in several forms,

$$P = \frac{1}{2}\vec{E}.\vec{D} = \frac{1}{2}\frac{D^2}{\varepsilon} \text{ since } \vec{D} = \varepsilon_0 K\vec{E} \text{ and } K = \frac{\varepsilon_0}{\varepsilon}$$

$$= \frac{1}{2}\varepsilon E^2$$

$$= \frac{1}{2}\frac{\sigma^2}{\varepsilon} \qquad \ldots(6.99)$$

Since σ is surface density of charge and given by $\sigma = K\varepsilon_0 E = D$.

Alternate Method

As we discussed in Chpater 5, that the potential energy due to a charge distribution is given by,

$$U = \frac{1}{2}\int_v \rho V \, dv \qquad \text{(See Eq. (5.46))}$$

where v is the volume.

Now according to equation (6.42),

$$\vec{\nabla} . \vec{D} = \rho$$

Therefore, $$U = \frac{1}{2}\int_v V (\nabla . \vec{D}) \, dv \qquad \ldots(6.100)$$

But, $$\nabla . (\nabla . \vec{D}) = V(\nabla . \vec{D}) + \vec{D} . \nabla V, \qquad \text{(See Eq. 3.54)}$$

Substituting in above equation (6.100), we get,

$$U = \frac{1}{2}\int_v \nabla . (V \vec{D}) \, dv - \frac{1}{2}\int_v \vec{D} . \text{grad } V \, dv,$$

(Since $(\nabla V) = \text{grad } V$). But according to divergence theorem,

$$\int_v \nabla . V \vec{D} \, dv = \int_s V \vec{D} . d\vec{S}$$

therefore, $$U = \frac{1}{2}\int_S V \vec{D} . d\vec{S} - \frac{1}{2}\int_v \vec{D} . \text{grad } V \, dv \cdot$$

As we know that $V \propto \frac{1}{r}$

$$D \propto \frac{1}{r^2}$$

$$dS \propto r^2$$

So, $$\vec{\nabla} . D \, dS \propto \frac{1}{r}$$

Hence, $\int_S VD \, dS$ *will vanish if the surface be expanded infinitely, therefore,*

$$U = \frac{1}{2}\int_v \vec{D} . \vec{E} \, dv, \quad (as \; E = -\text{grad } V) \qquad \ldots(6.101)$$

same as given in equation (6.96).

Example 6.10 (a) *In example 6.7 if we maintain a constant potential difference 100 volts after the dielectric is introduced, calculate the energy.*

Solution. The electrostatics energy without dielectric per unit volume,

$$= \frac{1}{2}\varepsilon_0 E_0^{\,2} \text{ jouls/ metre}^2$$

and the energy density with dielectric $= \frac{1}{2} K \varepsilon_0 E_0^{\ 2}$

Substituting K and E_0 from example 6.6, we get,

$$U = \frac{1}{2} \times 7 \times 8.9 \times 10^{-12} \times 1 \times 10^{8} \text{ jouls/ metre}^3$$

$$= 31.15 \times 10^{-4} \text{ jouls/ metre}^3$$

So energy density increases by K items.

(b) However if source of emf is withdrawn as soon as dielectric is introduced, then the example 6.10 (a) reduces.

Solution. Potential difference after dielectric is introduced = V_0/K
where V_0 is potential difference before introducing the dielectric,

$$C = K C_0,\ U = \frac{1}{2} CV^2 = \frac{1}{2} K C_0 \left(\frac{V_0}{K} \right)^2 = \frac{1}{2} \frac{C_0 V_0^{\ 2}}{K}$$

$\frac{1}{2} C_0 V_0^{\ 2}$ *is the energy before the dielectric is introduced. Hence energy decreases by factor 1/K.*

6.30 POTENTIAL AND FIELD DUE TO A POLARISED DIELECTRIC

When a dielectric is polarised the polarisation charge appears.

The polarisation vector $\vec{P}$ is defined by the equation,

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E}$$

Let us take the divergence of this equation. We have,

$$\vec{\nabla} . \vec{P} = \vec{\nabla} . \vec{D} - \varepsilon_0 \vec{\nabla} . \vec{E}$$

or $$\varepsilon_0 \vec{\nabla} . \vec{E} = \vec{\nabla} . \vec{D} - \vec{\nabla} . \vec{P}$$

or $$\vec{\nabla} . \vec{E} = \frac{1}{\varepsilon_0} (\rho_f - \vec{\nabla} . \vec{P}), \text{ since } \vec{\nabla} . \vec{D} = \rho_f \text{ by eq. (6.42)}$$

As we know that electrostatic potential due to charge distribution ρ is given by,

$$V = \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho}{r} dv', \qquad \text{[See eq. 5.14]}$$

where r = distance from element containing charge to point at which V is to be calculated, and ρ = true charge volume density (couls/metre)^2.

When dielectric is polarised the polarisation charge appears and volume charge density due to polarisation is given by equation (6.32 a) *i.e.*

$$\rho_p = -\vec{\nabla}' . \vec{P}$$

Here prime over ∇ is put to avoid the ambiguity with source point and point of observation because divergence is w.r.t. source ponit i.e. charge distribution.

Thus, the resultant volume charge density is $\rho - \vec{\nabla}' . \vec{P}$

Therefore, the total potential due to a polarised dielectric is given by,

$$V_T = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho - \vec{\nabla}'.\vec{P}}{r}\,dv'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho dv'}{r} - \frac{1}{4\pi\varepsilon_0}\int_v \frac{\vec{\nabla}'.\vec{P}}{r}\,dv' \qquad \ldots(6.102)$$

Hence, the *potential V_P due to polarisation distribution is given by,*

$$V_P = -\frac{1}{4\pi\varepsilon_0}\int_v \frac{(\vec{\nabla}.\vec{P})}{r}\,dv' \qquad \ldots(6.103)$$

and the potential due to true charge $= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r}\,dv'$ $\qquad \ldots(6.104)$

where, P = polarisation (couls/metre^2)

r = distance from element containing charge or polarisation to point at which V_T is to be calculated.

The volume integration is taken over all regions containing charge or polarisation.

Dipole Approximation to Polarised Dielectrics

The potential of a dipole is given by,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{p.r}{r^2}, \qquad \text{[See Eq. (5.23)]}$$

or $$V(r) = \frac{1}{4\pi\varepsilon_0}\left[-\vec{p}.\vec{\nabla}\left(\frac{1}{r}\right)\right], \quad \left[\text{since } \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\right]$$

where $\vec{p}$ is the dipole moment.

If there are n dipoles per unit volume, then potential due to unit volume.

$$V(r) = -\frac{1}{4\pi\varepsilon_0} n\vec{p}.\vec{\nabla}\left(\frac{1}{r}\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\vec{P}.\vec{\nabla}\left(\frac{1}{r}\right) \qquad \ldots(6.105)$$

where $\vec{P} = n\vec{p}$ is dipole moment per unit volume. Let us compare equations (6.103) and (6.105), then we can interpret the integrand in equation (6.103) as the potential produced by fictitious polarisation by the formation of dipoles over the volume dv'. Thus, the potential V_p can then be regarded as the total potential produced by all these dipoles spread through the volume of dielectric. This means that the polarised dielectric is equivalent to an assembly of dipoles in an electric field.

Now in equation (6.103), the divergence operation (*i.e.* $\nabla.\vec{P}$) is with respect to the charge distribution *i.e.*, source point whereas the potential is determined at any external point say point of observation. Therefore, *we introduce the operator ∇' operating on the source coordinates.* The prime on the ∇ is only necessary in integral expressions which relate a field quantity to an integral over a source quantity.

Let (x, y, z) and (x', y', z') be the coordinates of the observation and source points respectively. And the scalar U is, $U = U(r)$

where, $r = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$

then, $$\nabla U = \frac{\delta U}{\delta x}\hat{i} + \frac{\delta U}{\delta y}\hat{j} + \frac{\delta U}{\delta z}\hat{k} \qquad \ldots(6.106)$$

and, $$\nabla' U = \frac{\delta U}{\delta x'}\hat{i} + \frac{\delta U}{\delta y'}\hat{j} + \frac{\delta U}{\delta z'}\hat{k} \qquad \ldots(6.107)$$

But, $$\frac{\delta U}{\delta x} = \frac{\delta U}{\delta r} \cdot \frac{\delta r}{\delta x}$$

and, $$\frac{\delta U}{\delta x'} = \frac{\delta U}{\delta r} \cdot \frac{\delta r}{\delta x'}$$

But, $$\frac{\delta r}{\delta x} = \frac{1}{2}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2} . 2(x-x')$$

$$= \frac{x-x'}{r}$$

and, $$\frac{\delta r}{\delta x'} = \frac{1}{2}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1.2} \times (-2)(x-x')$$

$$= \frac{x-x'}{r}$$

therefore, $\dfrac{\delta r}{\delta x'} = -\dfrac{\delta r}{\delta x}$

Hence, $\dfrac{\delta U}{\delta x'} = -\dfrac{\delta U}{\delta x}$

Similarly, $\dfrac{\delta U}{\delta y'} = -\dfrac{\delta U}{\delta y}$ and $\dfrac{\delta U}{\delta z'} = -\dfrac{\delta U}{\delta z}$.

Putting these values in equation (6.107), we have,

$$\nabla U(r) = -\nabla' U(r) \qquad \ldots(6.108)$$

Therefore, the ∇ in equation (6.105) which is with respect to the observation point using equation (6.108), becomes,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\,\bar{P}.\bar{\nabla}'\left(\frac{1}{r}\right)$$

Now ∇' is with respect to source point. *The above relation can be changed into a form that is physically more meaningful by means of Gauss's divergence theorem* and the vector relation.

$$\bar{\nabla}\left(\frac{\bar{P}}{r}\right) = \frac{1}{r}\bar{\nabla}'\,\bar{P} + \bar{P}.\bar{\nabla}'\left(\frac{1}{r}\right)$$

Substituting $\overline{\nabla}\left(\frac{1}{r}\right)$ in the expression for V(r), we get,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left[\overline{\nabla}'\left(\frac{\overline{P}}{r}\right) - \frac{1}{r}\overline{\nabla}' \cdot \overline{P}\right]$$

The potential due to the volume distribution of dipoles is found by integrating the expression over the volume which is potential due to polarised dielectrics,

$$V_P = \frac{1}{4\pi\varepsilon_0}\int\left[\overline{\nabla}' \cdot \left(\frac{\overline{P}}{r}\right) - \frac{1}{r}\overline{\nabla}' \cdot \overline{P}\right] dv'$$

or
$$V_P = \frac{1}{4\pi\varepsilon_0}\left[\int \overline{\nabla}' \cdot \left(\frac{\overline{P}}{r}\right) dv' - \int \frac{\overline{\nabla}' \cdot \overline{P}}{r} dv'\right]$$

The first term by Gauss's divergence theorem,

$$\int_v \overline{\nabla}'\left(\frac{P}{r}\right) dv = \int_S \frac{\overline{P}.\hat{n}}{r} dS'$$

Thus,
$$V_P = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\overline{P}.\hat{n}}{r} dS' - \frac{1}{4\pi\varepsilon_0}\int \frac{\overline{\nabla}' \cdot P}{r} dv'$$

We know that $\sigma_P = \overline{P}.\hat{n}$ [See equation (6.31)] and $\rho_P = -\overline{\nabla}' \cdot \overline{P}$ [See equation (6.32 a)]

The expression can be interpreted as follows. *The first term, a surface integral, is a potential equivalent to that of a surface charged density while the second term is a potential equivalent to that of a volume charge density. Thus,*

$$V_p = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma_P\, dS'}{r} + \int \frac{\rho_P\, dv'}{r} \qquad \ldots(6.109)$$

Therefore the total potential produced by a charge distribution ρ in the presence of dielectric or the potential due to a polarised dielectrics given by eq. (6.102) is,

$$V_T = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r} dv' + \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_P dv'}{r} + \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma_P\, dS'}{r} \qquad \ldots(6.110)$$

The electric field E produced by ρ in the presence of a dielectric can be expressed by,

$$\overline{E} = -\overline{\nabla}\, V_T$$

$$= -\frac{\delta V_T}{\delta r}$$

or,
$$\overline{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho dv'}{r^2}\hat{r} + \int \frac{\rho_p\, dv'}{r^2}\hat{r} + \int \frac{\sigma_p\, dS'}{r^2}\hat{r} \qquad \ldots(6.111)$$

where $\hat{r}$ is unit vector directed from the source point to point of observation. The electrostatics field lines begin and end on electric charges. According to Eq. (6.111), the sources of the electrostatics field $\overline{E}$ in the presence of dielectrics are not only the real charges ρ, but also the fictitious, polarisation charges ρ_P and σ_P. Therefore, the field lines of $\overline{E}$ begin and end not only at the points where ρ is present, but also on dielectric surfaces (σ_P) as well as, the points within the dielectrics (ρ_P). The

field lines of $\vec{D}$ at on the other hand, begin and end only on the real charge ρ . This follows from the divergence law $\nabla.\vec{D} = \rho$, which shows that $\vec{D}$ is always associated with the real charges only, so that real charges are sources of $\vec{D}$, therefore an $\vec{E}$ and a $\vec{D}$ field may have entirely different structure when dielectrics are present.

6.31 TYPES OF CONDENSERS

The condensers are also classified according to dielectric material used. The more common are given below:

1. Variable air condensers.
2. Condenser with solid dielectric. The solid dielectric commonly used are:
   (a) Mica–then we call mica condenser.
   (b) Paper–paper condenser are made by taking two strips of tin or aluminium foil insulated by paper and rolling it into a bundle. They can be safely used up to 500 volts.
   (c) Ceramic–for large capacitance using ceramic as dielectric.
3. Electrolytic condenner: It is made by putting two aluminium plates in an electrolyte.

6.32 DIELECTRIC STRENGTH

The *dielectric material loses its property of insulation for a particular field strength.* It is *referred as breakdown strength*. Let us examine what occurs in a material when breakdown occurs. When a dielectric is placed in an electric field, a strain is produced on the elastic forces (by which the electrons are bound to the nucleus or the atoms are bound together in a molecule) which permits the polarisation of the molecule. Under normal condition, the electrons are not able to leave the molecule, should the electric field be made so high that electrons can move away from their normal position on the molecule. The electrons will drift to the positive electrode and the positive ion that results when electron is removed will drift to the negative electrode. *Because of this drift of charged carriers, the dielectric is no longer an insulator and dielectric breakdown is said to have occurred.* The magnitude of the electric field at which the dielectric breakdown occurs in an insultaing material is the dielectric strength of the material.

Dielectric strength of some common materials are given belwo:

Material	*Dielectric strength KV/m*
Air	3,000
Mica	1,00,000
Rubber	40,000
Porcelain	10,000
Transformer Coil	12,000

Example 6.11 *A potential of 50 KV is applied between a pair of parallel plates which are 0.02 m apart the medium between them being air. A sheet of mica of thickness 0.0075 m is introduced. Examine whether the breakdown will occur or not. Given the dielectric constant of mica is 4 times than that of air. The breakdown strength of air is 3000 KV/m and mica 1,00,000 KV/M.*

Solution. See the geometry of parallel plate capacitor. Before the addition of mica

$$E = \frac{V}{d} = \frac{50}{0.02} = 2500 \text{ KV/m}.$$

Since the dielectric strength of air is 3,000 KV/m, which is higher than the potential gradient in the field, there will be no breakdown. When mica plate is introduced the potential difference,

$$V = Ed + E_1 d_1$$

But as electric displacement vector is constant,

$$D = \varepsilon_0 KE, \text{ everywhere}$$

and $\varepsilon_0 KE = \varepsilon_0 K_1 E_1$, E_1 is field inside mica.

or $KE = K_1 E_1$

As given $K_1 = 4K$

we get, $E = 4E_1$

or $E_1 = \frac{E}{4}$.

Substituting in the above equation and noting that air space,

$= 0.02 - 0.0075$

$= 0.0125$ metre,

hence equation $V = Ed + E_1 d_1$ after substitution is,

$$50 \text{ KV/m} = 0.0125\,E + 0.0075\,E/4$$

Solving for E, we get after substituting E given above *i.e.*, $E_1 = E/4$, $E = 3,440$ KV/m.

Since this field strength exceeds the dielectric strength of air, the air will break down, since the air after breakdown is conducting then the field intensity in the mica becomes,

$$E_1 = \frac{50\,KV}{0.0075}$$

$= 66,700$ KV/m.

But the breakdown strength of mica is greater than this value, and mica will not break-down.

So, we cannot introduce the mica because air will breakdown and there will be sparking in the system.

Table 6.1 Three Electric Vectors

Name	*Symbol*	*Associated with*	*Boundary condition*
Electric field Intensity	$\vec{E}$	All charges	Tangential component continuous Eq. (6.52)
Electric displacement or Flux Density	$\vec{D}$	Free charges	Normal component continuous Eq. (6.51)
Polarisation (Electric dipole moment per unit volume)	$\vec{P}$	Polarisation charge only	Vanishes in a vacuum Eq. (6.40)
Defining equation for		$\vec{E}\ \vec{F} = q\vec{E}$, Eq. (4,10)	
General relation among the three vectors,		$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, Eq. (6.36)	
Gauss's law in presence of Dielectric media		$\oint \vec{D} . d\vec{S} = Q$, (Q is free charge Eq. (6.41)	
		or $\nabla . \vec{D} = \rho_f$, Eq. (6.42)	
Empirical relations for certain dielectric materials		$\vec{D} = K\varepsilon_0 \vec{E} = \varepsilon \vec{E}$, Eq. (6.38)	
		$\vec{P} = (K-1)\varepsilon_0 \vec{E}$, Eq. (6.40)	

REVIEW QUESTIONS

6.1 Define capacitance and capacitor. On what factors the capacitance depends?

6.2 Calculate the capacity of a parallel plate condenser. What will be the capacity if the space between the plates is partially filled with a slab of thickness t and dielectric constant K?

6.3 Calculate the capacity of a condenser, consisting of two sphereas of radii a and b, separated by a dielectic of S.I.C.K.

6.4 Find an expression for the capacity per unit length of two coaxial conducting cylinders placed in air.

6.5 Obtain an expression for energy and force of attraction between parallel plate capacitor.

6.6 What are polar and non-polar molecules? Give examples.

6.7 What do you mean by dielectric polarisation? What is uniform and non-uniform polarisation? Show that for non-uniform polarisation $\nabla . \mathrm{P} = -\rho_p$.

6.8 Define $\vec{\mathrm{D}}, \vec{\mathrm{E}}$ and P. Establish the relation

(i) $\vec{\mathrm{D}} = \varepsilon_0 \vec{\mathrm{E}} + \vec{\mathrm{P}}$ (ii) $\nabla . \vec{\mathrm{D}} = \rho_f$

Symbols have usual meaning.

6.9 Consider a surface separating two dielectric media of permittivity ε_1 and ε_2. If the dielectric field intensity in medium 1 makes an angle θ_1 with the normal to the surface, show that the angle θ_2 which the field intensity in median 2 makes with the normal to the surface is given by

$$\varepsilon_1 \cos\theta_1 = \varepsilon_2 \cos\theta_2$$

6.10 Define a flux tube. How the force of attraction and repulsion between charges is explained by the idea of Maxwell tubes?

6.11 Explain what you mean by electronic, ionic and dipolar polarisabilities.

6.12 If E denotes the electric intensity in a homogeneous and isotropic dielectric, show that the actual field acting on one of the molecules of the dielectric is

$$\vec{\mathrm{E}}_{eff} = \vec{\mathrm{E}} + \frac{\vec{\mathrm{P}}}{3\varepsilon_0}$$

6.13 Derive the Classius-Mosotti's equation. How this equation is modified by Debye?

6.14 What is meant by polarisability of a molecule? Give in brief Langevin Debye theory of dielectric polarisation.

6.15 Derive a relation between electric susceptibility and atomic polarisability.

6.16 Show that the potential due to a polarised dielectric is the same as the potential due to a surface charge density over the surface of volume which has been polarised plus that due to a volume charge density.

6.17 What do you mean by dielectric strength?

PROBLEMS

6.1 (a) Let n capacitors of capacitance $c_1, c_2, \ldots c_n$ be connected in series. Show that effective capacitance c is given by,

$$\frac{1}{c} = \sum_{i=1}^{n} \frac{1}{c_i}$$

(b) Let n capacitors of capacitance $c_1, c_2, \ldots c_n$ be connected in parallel. Show that effective capacitance

$$c = \sum_{i=1}^{n} c_i$$

6.2 The capacitance of a parallel plate capacitor is 400 picofarad and its plates are separated by 2 mm of air. (1) What will be the energy when it is charged to 1500 volts? (2) What will be the potential difference with same charge if plate separation is doubled? (3) How much energy is needed to double the distance between its plates?

6.3 A one core lead sheeted cable has a conductor core of 0.5 cm diametre and the lead sheet has inside diametre 1.5 cm. The insulating material is rubber. At what voltage will the insulation break down. Given that rubber has a dielectric strength of 400 KV/cm.

6.4 A co-axial cable consists of a copper core of 1 mm radius within outer metal sheet of 1 cm radius separated by an insulating material of dielectric constant 5. What is the capacitance per metre length of the cable in picofarad?

How much energy is stored in 10 km length of the cable when 10,000 volts is applied between the core and the sheet?

6.5 If half the space between two concentric conducting spheres be filled with dielectric of dielectric constant K and the rest is filled with air, show that capacitance of the capacitor thus formed will be same as if the whole part is filled with the dielectric of dielectric constant $\frac{1}{2}(1+\mathrm{K})$.

6.6 Calculate the induced dipole moment per unit volume of He gas if placed in a field of 6000 volts/cm. The atomic polarisability of He is 0.21×10^{-24} cm^3 and density of He is 20.6×10^{19} atoms/cc.

6.7 An atom of oxygen on being polarised produces a dipole moment of 1.5×10^{-23} stat coul × cm. If the distance of the centre of the negative charge cloud from the nucleus is 4×10^{-15} cm, calculate the polarisability of oxygen atom.

ANSWERS

2. 4.5×10^{-4} joules, 3000 volts, 4.5×10^{-4} joules

3. 329.4 KV

4. 120.3 pico-farad, 60.35 joules

6. 3.64×10^{-7} coul/cm^2

7. 6.25×10^{-44}cm^3

SELF-TEST OBJECTIVE QUESTIONS

6.1 Which statement influences the capacity of a capacitor?

(*a*) area of the plates, thickness of the plates and the rate of charge

(*b*) area of the plates, dielectric and the rate of charge

(*c*) distance between the plates, dielectric and thickness of the plate

(*d*) distance between the plates, area of the plates and dielectric

6-2 To increase the capacitance of capacitor, the plates must be placed

(a) further apart (b) closer together

(c) in series (d) made smaller

6.3 The effect of the dielectric is to

(*a*) increase the capacitance (*b*) decrease the capacitance

(*c*) reduce the working voltage (*d*) increase the distance between the plates

6.4 Electric flux density is the ratio of

(*a*) charge to area of dielectric at right angles to direction of electric flux

(*b*) charge to distance between the plates

(*c*) electric flux to distance between the plates
(*d*) electric flux to area of the plates

6.5 Which of the following is correct for a capacitor?

(a) $W = \frac{1Q^2}{2\ C}$ (*b*) $W = \frac{1}{2} CI^2$

(*c*) $W = \frac{1}{2} VC^2$ (*d*) $W = \frac{1}{2} CV^2$

6.6 $\nabla . \vec{D} = \rho$ is based on

(*a*) Ampere's law (*b*) Faraday's law
(*c*) Ohm's law (*d*) Gauss's law

6.7 The effective capacitance is reduced when capacitors are connected in

(*a*) series (*b*) parallel
(*c*) series-parallel combination (*d*) none of the above

6.8 The unit of $\vec{D}$ is

(*a*) V/m^2 (*b*) $Coul/m^2$
(*c*) V/m (*d*) Q/m

6.9 The relation $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ holds good

(*a*) only in vacuum (*b*) only inside dielectric
(*c*) only outside dielectric (*d*) everywhere

6.10 Poisson's equation states that

(*a*) $\nabla^2 V = -\frac{\rho}{\varepsilon}$ (*b*) $\nabla^2 V = \frac{\rho}{\varepsilon}$

(*c*) $\nabla V = \frac{\rho}{\varepsilon}$ (*d*) $\nabla V = \frac{\rho^2}{\varepsilon}$

6.11 By inserting a plate of a dielectric material between the plates of a parallel plate capacitor, the energy (stored) is increased five times. The dielectric constant of the material is

(*a*) 1/25 (*b*) 1/5
(*c*) 5 (*d*) 25

6.12 The unit of polarisation $\vec{P}$ is

(*a*) same as that of $\vec{E}$ (*b*) same as that of $\vec{D}$
(*c*) same as that of $\vec{E}$/coulomb (*d*) same as that of charge

ANSWERS

1. (d)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (d)	**6.** (d)
7. (a)	**8.** (b)	**9.** (d)	**10.** (a)	**11.** (c)	**12.** (b)

CHAPTER 7

ELECTROMETERS AND ELECTROSTATICS MACHINES

7.1. ELECTROMETERS

In common practice we use the *electromagnetic voltmeters* for measuring the potential difference, but they are *less accurate* because they *require some current,* though very small, *for their working*. In electrostatics phenomena, certain instruments have been devised, in which the *curernt is to be completely avoided* by perfect insulation which is difficult to achieve. However, the efforts have been made to reduce, the leakage of charge due to imperfect insulation, to minimum. Thus, *accuracy is increased and we measure correctly the potential difference*, charge and capacitance. *Such device are called electrometers*. The simplest electrometer was first used by Faraday for measuring potential difference and that was gold leaf electroscope.

7.2 KELVIN'S ABSOLUTE OR ATTRACTED DISC ELECTROMETER

Principle. The principle of attraction between the plates of charged capacitor was utilised by Kelvin for constructing the attracted disc-electrometer.

Construction. It is essentially a guard ring air condenser consisting of two metallic plates A and B and guard ring G to avoid edge effect as shown in Fig. 7.1. The central plate A which can freely pass concentrically through G serves as the attracted disc and is held in position with the help of spring S. Spring S is connected to a rod which can be moved up or down with micrometer screw N. Hence plate A can be moved by screw N.

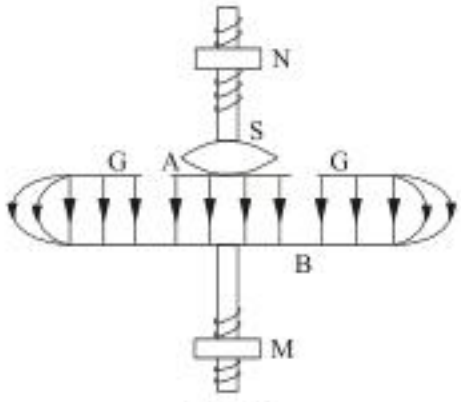


Fig. 7.1

The plate A and guard ring G are always charged to the same potential and the plate B is either earthed or charged to a different potential. In order to obtain uniform electric field between A and B, the plate A is connected to G by light flexible wire, then the lines of force will be parallel and equidistant.

Theory. Let V_a and V_b be the potentials of plates A and B respectively and d be the distance between them, then the electric field,

$$E = \frac{V_a - V_b}{d} \qquad \ldots(1)$$

Now, the plate A will experience a force of attraction per unit area *i.e.*, pressure given by

$$P = \frac{\varepsilon_0 E^2}{2}$$ [See equation (4.37)]

Therefore, the plate A will experience a force of attraction.

$$F = \frac{\varepsilon_0 E^2 A}{2} \qquad \ldots(2)$$

where A is the "effective area" of the plate which is equal to the actual area of the plate A plus half of the area of the gap between A and G. Substituting E from Eq. (1) in Eq. (2), we get

$$F = \frac{\varepsilon_0}{2}\left(\frac{V_a - V_b}{d}\right)^2 A$$

or $$V_a - V_b = d\sqrt{\frac{2F}{\varepsilon_0 A}} \text{ Volts} \qquad \ldots(7.1)$$

Procedure for Measuring the Potential Difference

1. At first the plates A, B and C are connected to the *earth* so that the potential difference between the plates is zero. A small mass *m* is then *placed* on the plate A. Due to weight *mg*, A gets depressed *below* G. With the help of the screw nut N, the plate A is raised to the same plane as GG.
2. Now, the small mass *m* is *removed* so that the plate A is pulled up by the spring S and is *above the plane of G*. The *earth connection* of all the plates is now *broken*.
3. Next the plate *A is connected to a potential* V_0 *i.e.* the plate is charged to the poential V_0 by a machine (Kelvin Replenisher). The plate B is then *connected to one of the points*, the potential difference between which we wish to *measure*. The position of B is adjusted with the help of screw M until the plate A comes in *level with the guard ring*. The reading on the scale is noted, say, R_1 corresponding to the distance between the plates d_1. Here, the force of attraction between A and B is *mg*. If the potential of the first point be V_1, then from Eq. (7.1)

$$V_0 - V_1 = d_1\sqrt{\frac{2mg}{\varepsilon_0 A}} \qquad \ldots(7.2)$$

4. Finally, the plate B is *connected to the second point* whose potential is V_2. The position of B is again adjusted so that A comes in level with the guard ring. Let the reading on scale be R_2 and the distance between the plates be d_2, then

$$V_0 - V_2 = d_2\sqrt{\frac{2mg}{\varepsilon_0 A}} \qquad \ldots(7.3)$$

Subtracting (7.3) from (7.2), we get

$$V_2 - V_1 = (d_1 - d_2)\sqrt{\frac{2mg}{\varepsilon_0 A}} \qquad \ldots(7.4)$$

Since $d_2 - d_1 = R_2 - R_1$, we get

$$V_2 - V_1 = (R_1 - R_2)\sqrt{\frac{2mg}{\varepsilon_0 A}} \qquad \ldots(7.5)$$

Absolute Electrometer

It is called the *"absolute electrometer"* because the potential difference $V_2 - V_1$ in equation (7.5) is known in terms of the difference between the two micrometer readings, the mass *m* and the area A. That is, the *potential difference* $V_2 - V_1$ *is obtained in terms of "fundamental quantities" such* as

mass and length, which can be measured. No calibration is required in this electrometer, hence it is called an absolute electrometer.

It is, however, *less sensitive* than other electrometers. Further, it requires much setting, therefore, it is more *cumbersome* to use than a voltmeter.

7.3 MEASUREMENT OF DIELECTRIC CONSTANT USING ATTRACTED DISC ELECTROMETER

If the material is given in the form of a slab of thickness, say, t, which can be introduced between the plates A and B. The effective air distance between the plates decreases and arrangement is now simply the parallel plate capacitor partially filled with dielectric, its capacity,

$$C = \frac{\varepsilon_0 A}{\left[d - t\left(1 - \frac{1}{K}\right)\right]}, \quad \text{See Eq. (6.19)}$$

where K is the dielectric constant of the slab. Clearly, the effect of introducing of a dielectric slab of thickness t is *effectively decreasing the distance between plates A and B by an amount* $t\left(1 - \frac{1}{K}\right)$.

Hence, if after introducing the slab the separation between the plates is increased by an amount $h = t\left(1 - \frac{1}{K}\right)$, the capacity of the capacitor reduces to its original value, *i.e.* $C = \frac{\varepsilon_0 A}{d}$ without the dielectric slab [see Eq. (6.3)].

Hence, $h = t\left(1 - \frac{1}{K}\right)$

or $\frac{h}{t} = 1 - \frac{1}{K}$

or $\frac{1}{K} = 1 - \frac{h}{t} = \frac{t - h}{t}$

or $K = \frac{t}{t - h}$ ...(7.6)

Therefore the value of K can be determined in terms of t and h. The value of t and h can be easily determined with the help of attracted disc electrometer when the slab is introduced between the plates A and B. The potential is obtained by replacing d in Eq. (7.1) by $d - t\left(1 - \frac{1}{K}\right)$, *i.e.*,

$$V_a - V_b = \left[d - t\left(1 - \frac{1}{K}\right)\right]\sqrt{\frac{2F}{\varepsilon_0 A}} \quad \ldots(7.7)$$

since $F = \frac{\varepsilon_0 E^2}{2}$ and $E = \frac{V_a - V_b}{d - t\left(1 - \frac{1}{K}\right)}$

Now if the potential difference between the plates is maintained constant and the dielectric slab introduced, the distance between the plates effectively decreases and electric field E increases so consequently F (force of attraction) increases. Hence the plate A is pulled down by the plate B by an amount $h = t\left(1 - \frac{1}{\mathrm{K}}\right)$ with the help of micrometer screw, the force of attraction will decrease to its original value and the plate A will return to its original position. Hence h is known from the two readings of micrometer screw.

For measuring the thickness of the slab, the plate B is moved up so that the top and the bottom surfaces of the slab touch the plates A and B respectively. The reading of the position of B is noted.

Next, the slab is removed and the plate B is again moved such that B touches A and reading of B is noted again. The difference between two readings is thickness t. Therefore, K is known by Eq. (7.6).

Example 7.1 *Calculate the force of attraction between the plates of an attracted disc electromter when a potential of 3,000 volts is applied between them, they are separated by distance of 0.5 cm and area of each plate is 10 sq. cm.*

Solution. Potential difference = 3000 volts

$$\text{Electric field } \mathrm{E} = \frac{3,000}{0.5 \times 10^{-2}} \text{ volts/ metre}$$

$$\text{Force} = \frac{\varepsilon_0 \mathrm{E}^2 \mathrm{A}}{2}$$

$$= \frac{8.85 \times 10^{-12}}{2} \frac{\text{Coul}^2}{\text{Nt} \times \text{m}^2} \times 10 \times 10^{-4}\ \text{m}^2 \left(\frac{3000}{0.5 \times 10^{-2}} \frac{\text{Volts}}{\text{m}}\right)^2$$

$$= \frac{8.85 \times 10^{-12} \times 3 \times 3 \times 10^6 \times 10 \times 10^{-4}}{2 \times 25 \times 10^{-6}} \frac{\text{Coul}^2}{\text{Nt}} \left(\frac{\text{Volt}}{\text{m}}\right)^2$$

$= 157.3 \times 10^{-5}$ Newtons, since Volt = Nt. m/Coul.

Example 7.2 *Find the mass required to give a force equal to the force of attraction between the plates of attracted disc electrometer when they are 4 mm apart, having area 10 sq. cm and at a potential difference of 1000 Volts.*

Solution. $\mathrm{V}_2 - \mathrm{V}_1 = 1000$ volts

and $d_1 - d_2 = 4 \times 10^{-3}$ m

and $\mathrm{A} = 10 \times 10^{-4}\ \mathrm{m}^2$

Using Eq. (7.4), *i.e.*,

$$\mathrm{V}_2 - \mathrm{V}_1 = d_2 - d_1 \sqrt{\frac{2\,\mathrm{m}\,g}{\varepsilon_0 \mathrm{A}}}$$

On putting the values,

$$1000 = 4 \times 10^{-3} \sqrt{\frac{2 \times 9.8 \times m}{8.85 \times 10^{-12} \times 10^{-3}}}$$

or $\mathrm{m} = 2.8 \times 10^{-5}$ kg.

Example 7.3 *Two large metal plates are fixed harizontally at a distance 0.5 cm from each other. What potential in volts should be applied between the plates, if a droplet of oil of mass 1.5 × 10^{-11} gms and carrying a charge 4.9 × 10^{-10} e.s.u. is to be held at rest between the plates.*

Solution. Let the potential to be applied be V volts, then electric field,

$$E = \frac{V}{0.5\times10^{-2}} \text{ Volts/m}$$

Upward force on the droplet

$$qE = \frac{V}{0.5\times10^{-2}} \times \frac{4.9\times10^{-10}}{3\times10^{9}} \text{ Newtons}$$

and this is balanced by weight, mg,

$$= 1.5\times10^{-14}\times9.8 \text{ Newtons}$$

Hence, $$\frac{V}{0.5\times10^{-2}} \cdot \frac{4.9\times10^{-10}}{3\times10^{9}} = 1.5\times10^{-14}\times9.8$$

Or $$V = 4500 \text{ volts}$$

7.4 QUADRANT ELECTROMETER

The attracted disc electrometer, though it gives absolute values, is not very *sensitive*. Lord Kelvin devised another more sensitive electrometer known as quadrant electrometer. It was further modified by Dolezalek. The Dolezalek quadrant electrometer possesses *high sensitivity and accuracy*.

Construction. The quadrant electrometer consists of a hollow cylindrical box of brass, 5 cms. in diameter and 1 cm in height, divided into four hollow quadrants. The quadrants are mounted separately on amber insulating stand. The opposite pairs of quadrants AA and BB (Fig. 7.2) are joined together to the terminals T_1 and T_2 respectively and can be connected to a source potential V_a and V_b. N is a light paddle-shaped van called the needle which is made of thin paper coated with silver or aluminium. The needle is suspended by a quartz fibre so as to swing freely in a horizontal plane without touching the quadrants. A small mirror M is attached to the suspension fibre to read the deflection by lamp scale arrangement.

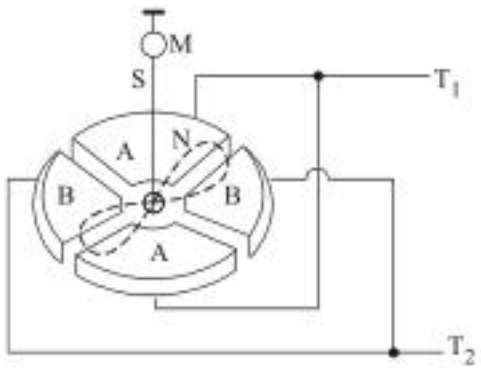


Fig. 7.2

Working. When the pairs of quadrants AA and BB are earthed or connected to the same potential, the needle rests symmetrically between them. But when these are charged to different potentials, the needle deflects from the higher to the lower potential. This twists the suspension fibre S and a restoring couple acts which brings the needle to rest. *When the deflecting couple due to the applied potential difference becomes equal to the restoring couple* due to the twist, the resulting deflection of the needle, say, θ, is proportional to the difference of potential $V_a - V_b$. This is shown in Fig. 7.3 where dotted line indicates the new position of the needle N and θ is deflection. Thus,

$$V_a - V_b \propto \theta$$

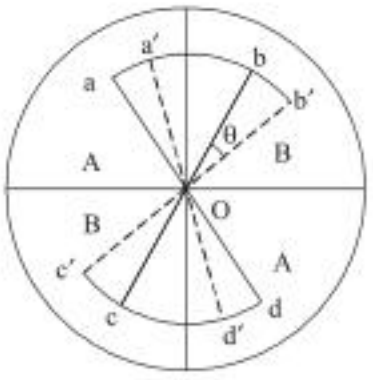


Fig. 7.3

The portion of needle, which lies between one pair of the quadrants, forms a parallel plate condenser, and that which lies between the other pair forms another condenser. When the needle moves through an angle θ from AA quadrants to BB quadrants, the area of needle shown dotted in Fig. 7.3 is transferred from AA to BB quadrants. Thus when the needle is deflected, its area in quadrants BB increases and that in quadrants AA decreases.

The increase in area of one face of the needle in BB is area Obb' or Occ'. Let are Obb' or Occ' is S and r be the radius of the needle, then an angle 2π represents an area πr^2 the angle θ will represent an area obtained by the condition, *i.e.,*

$$\frac{\text{area of } Obb' \text{ (S)}}{\text{area of the circle } \pi r^2} = \frac{\text{angle subtended by the } Obb' \text{ at centre } \theta}{\text{angle subtended by the circle at centre } 2\pi}$$

Hence, $S = \frac{\theta}{2\pi} . \pi r^2 = \frac{1}{2} r^2 \theta$

This is same as the corresponding decrease in the area of one face of needle in AA. As the needle has two arms and two faces, therefore, the total area of the needle transferred from AA to BB

$$= 4 . \frac{1}{2} r^2 \theta = 2r^2 \theta$$

If d' is the distance between the needle and the quadrant, the *increase in the capacity of the* condenser formed by the needle and quadrants BB due to this increase in area is given by

$$\Delta C = \frac{\varepsilon_0 2r^2 \theta}{d} \qquad \ldots(7.8)$$

since, $C = \varepsilon_0 \frac{A}{d}$, see eq. (6.3).

However, *the capacity of the condenser formed by the needle and quadrants AA will decrease by an "equal amount".* Since the capacity of BN condenser increases while the potential difference $(V_0 - V_b)$ between N (needle) and B remains constant, it "*gains charge*" whereas the capacity of AN condenser decreases while the potential difference $(V_0 - V_a)$ between N and A remains constant, it "*loses charge*".

Gain of charge by BN condenser = Change in capacitance × p.d.

$$= \Delta C (V_0 - V_b)$$

"*Gain of electrical energy*" by BN condenser = charge × potential difference

$$= \Delta C (V_0 - V_b) . (V_0 - V_b)$$

$$= \Delta C (V_0 - V_b)^2$$

Similarly, the "*loss of electrical energy*" by AN condenser $= \Delta C (V_0 - V_a)^2$

Therefore, the "*net gain of energy*",

$$= \Delta C \left[(V_0 - V_b)^2 - (V_0 - V_a)^2 \right]$$

$$= \Delta C \left[\{V_0 - V_b + V_0 - V_a\} \{V_0 - V_b - V_0 + V_a\} \right]$$

$$= \Delta C \left[2V_0 - (V_a + V_b) \right] (V_a - V_b)$$

$$= 2\Delta C \left[V_0 - \frac{V_a + V_b}{2} \right] (V_a - V_b) \qquad \ldots(7.9)$$

This is to be supplied by the source of electrical energy which is responsible for maintaining the difference of potential $(V_a - V_b)$.

The energy gain given by Eq. (7.9) is used in two parts:

1. in increasing the electrostatics potential energy of the system, and
2. in doing mechanical work in twisting the suspension fibre when the needle is deflected.

The increase in the electrostatics potential energy of the BN condenser

$$= \frac{1}{2} \Delta C (V_0 - V_b)^2$$

and the loss of potential energy of the AN condenser is

$$= \frac{1}{2} \Delta C (V_0 - V_a)^2$$

Therefore, the net gain of potential energy,

$$= \frac{1}{2} \Delta C \left[(V_0 - V_b)^2 - (V_0 - V_a)^2 \right]$$

$$= \frac{1}{2} \Delta C \left[2V_0 - (V_a + V_b) \right] (V_a - V_a)$$

$$= \Delta C \left[V_0 - \frac{V_a + V_b}{2} \right] (V_a - V_b) \qquad \ldots(7.10)$$

Comparison of equations (7.9) and (7.10) shows that "half" of the energy supplied by the source is used in "*increasing*" the "*electrical potential*" energy of the system. Hence the "*remaining half*" of the energy supplied by the source is used in "*twisting the suspension fibre*" during the deflection of the needle. Hence the energy available for twisting the suspension fibre is

$$= \Delta C \left[V_0 - \frac{V_a + V_a}{2} \right] (V_a - V_b)$$

Putting the value of ΔC from Eq. (7.8), we get

Energy spent (or work done) in twisting the fibre

$$= \frac{2\varepsilon_0 r^2 \theta}{d} \left[V_0 - \frac{V_a + V_b}{2} \right] (V_a - V_b)$$

But the work done = twisting couple × angle of twist

Therefore, twisting or "*deflecting couple*"

$$= \frac{2\varepsilon_0 r^2}{d} \left[V_0 - \frac{V_a + V_b}{2} \right] (V_a - V_b)$$

If τ is the restoring couple per unit twist, then the couple for the deflection θ is

$$\tau\theta = \frac{2\varepsilon_0 r^2}{d} \left[V_0 - \frac{V_a + V_b}{2} \right] (V_a - V_b)$$

or

$$\theta = \frac{2\varepsilon_0 r^2}{d\tau} \left[V_0 - \frac{V_a + V_b}{2} \right] (V_a - V_b) \qquad \ldots(7.11)$$

Thus the deflection θ is proportional to $\frac{2\varepsilon_0 r^2}{d\tau}$ which is constant, to the potential difference $(V_a - V_b)$ between AA and BB and to the average difference of potential between the needle and quadrant $\left(i.e.\ V_0 - \frac{V_a + V_b}{2} \right)$.

The quadrant electrometer can be used in two ways:

(1) Heterostatic use, and (2) Idiostatic use

(1) Heterostatic Use

In this case the needle is charged to a very high potential as compared to the potentials V_a and V_b, which are given to the quadrants AA and BB respectively as shown in Fig. 7.4 (a). Thus, when $V_0 \gg V_a$ or V_b, then $\frac{V_a + V_b}{2}$ may be neglected in Eq. (7.11) and hence,

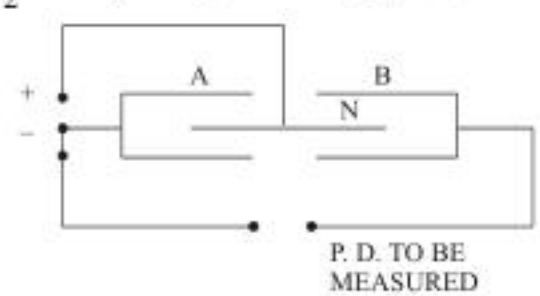


Fig. 7.4 (*a*)

$$\theta = \frac{2\varepsilon_0 r^2}{d\tau} V_0 (V_a - V_b)$$

or $$\theta = K (V_a - V_b) \quad \ldots(7.12)$$

where, $K = \frac{2\varepsilon_0 r^2}{d\tau} V_0$

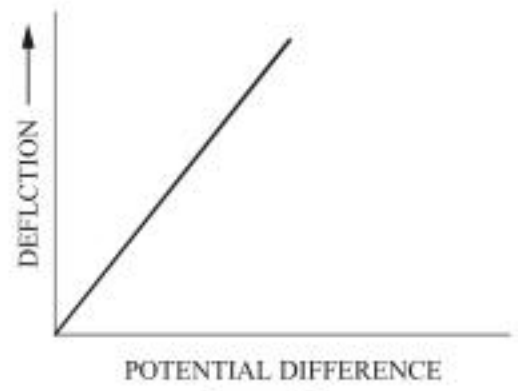


Fig. 7.4 (*b*)

Therefore, *the deflection in the heterostatic use is proportional to the potential difference*. If a graph is plotted between deflection and potential difference, a straight line is obtained as shown inFig. 7.4 (b).

Limitation of this Method

Since $V_0 \gg V_a$ or V_b, *this method is applicable for the measurement of the "small potential difference" only.* If the potentials applied to two pairs of quadrants are reversed the deflection will also be reversed, *hence in this use electrometer can measure only the "direct voltage".*

(2) Idiostatic Use

In this case, the needle and one pair of quadrants, say AA, are connected together as shown in Fig. 7.5 (a), hence,

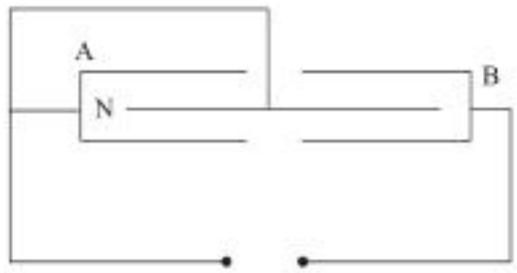


Fig. 7.5 (*a*)

$V_0 = V_a$ and Eq. (7.11) reduces to

$$\theta = \frac{\varepsilon_0 r^2}{d\tau}(V_a - V_b)^2 = K'(V_a - V_b)^2 \qquad \ldots(7.13)$$

where, $K' = \dfrac{\varepsilon_0 r^2}{d\tau}$

Thus, *the deflection in the idiostatic use is proportional to the "square of the potential difference".* Hence, *this method is used for determining "alternating potential difference"* and *"large DC potentials"* when the needle and quadrants AA are connected to the higher terminal. The variation of deflection with p.d is shown in Fig. 7.5 (b).

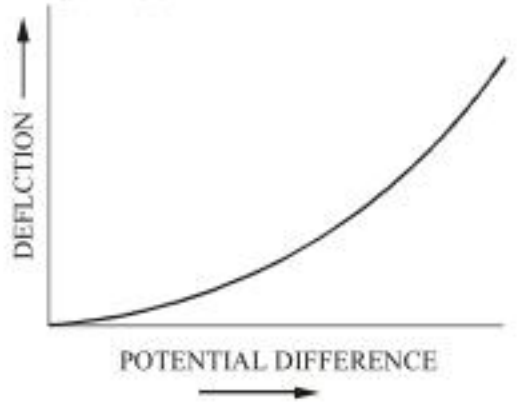


Fig. 7.5 (*b*)

Relative Merits of Heterostatic and Idiostatic Uses

We have, $\dfrac{K}{K'} = \dfrac{2\varepsilon_0 r^2}{d\tau} V_0 / \dfrac{\varepsilon_0 r^2}{d\tau}$

$$= 2V_0 \qquad \ldots(7.14)$$

Hence $K \gg K'$. Thus, *the electrometer when used heterostatically is more accurate than when used idiostatically because in the heterostatic use, we have comparatively larger deflection for the same difference of potential.* But in idiostatic use the deflection is proportional to the square of the potential, therefore, it is possible to measure alternating potential also. Again for a given deflection, the value $V_a - V_b$ is much greater on the idiostatic use than in the heterostatic use, since $K' \ll K$, hence the *range of the instrument is much more in the* idiostatic use, although the *accuracy is considerably less* in this case than to heterostatic use.

7.5 USES OF A QUADRANT ELECTROMETER

A quadrant electrometer can be used for the following purposes:

1. To measure small direct potential difference (or voltage). For details, see heterostatic use discussed above.
2. To measure alternating potential difference. For details, see idiostatic use discussed above.
3. Measurement of capacity and comparison of two capacities.
4. Measurement of ionisation current.
5. Measurement of dielectric constant.
6. Comparison of potentials.

Important cases are discussed here separately as follows:

3. Measurement of Capacity

If one pair of the quadrants is earthed and the other is connected to a potential V, then the deflection of needle given by equation (7.12) *i.e.*,

$$\theta = K (V_a - V_b)$$

here, $V_b = 0$ and $V_a = V$

hence, $\theta = K V$ ...(7.15)

This equation is used for the measurement of the capacity of a quadrant electrometer and for the comparison of two capacities.

(a) Capacity of a Quadrant Electrometer

The circuit arrangement is shown in Fig. 7.6. Let C be the capacity of the electrometer, one pair of quadrants of the electrometer is charged to a constant V_1 with a standard cell S by depressing the key K' and the resulting deflection θ_1 is noted. Then according to equation (7.15), we have,

$$\theta_1 = K V_1$$

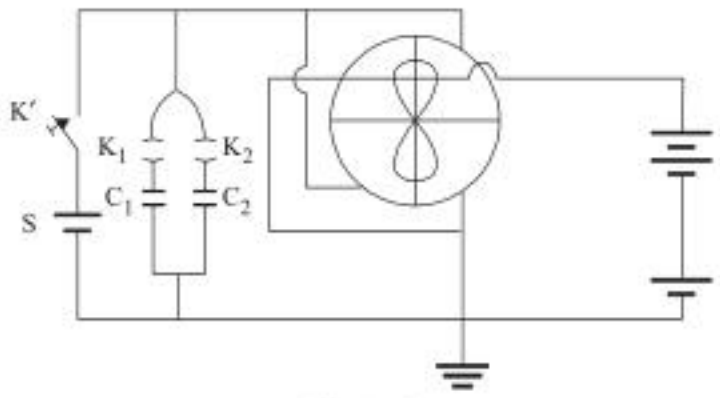


Fig. 7.6

Now K' is released and the standard condenser C_1 is put in circuit by closing the key K_1. The condenser C_1 shares the charge of the electrometer and the common potential becomes V_2 which gives the deflection θ_2 given by,

$$\theta_2 = KV_2$$

then the total charge on the electrometer Q is given by,

$$Q = CV_1 = (C + C_1)V_2$$

or $$\frac{C + C_1}{C} = \frac{V_1}{V_2} = \frac{\theta_1}{\theta_2}, \quad \left(\text{Since } \frac{KV_1}{KV_2} = \frac{\theta_1}{\theta_2}\right)$$

or $$\frac{C_1}{C} = \frac{\theta_1 - \theta_2}{\theta_2}$$

or $$C = C_1 \frac{\theta_2}{\theta_1 - \theta_2} \qquad \ldots(7.16)$$

Thus C can be calculated. A guard-ring capacitor is preferred as a standard condenser C_1 because its capacity can be calculated from its geometrical dimensions, as,

$$C_1 = \frac{\varepsilon_0 A}{d}$$

(b) Comparison of two Capacities

In this case, first the condenser C_1 put in the circuit by closing the key K_1 and then the key K' is depreseed so that both the capacity C_1 and the electrometer are charged to a common potential V. Let θ be the corresponding deflection in electrometer. Further, the key K_1 is opened and the condenser C_2 in put in the circuit by closing the key K_2, C_2 shares some of the charge acquired by C_1 and electrometer as shown in Fig. 7.6. Consequently the potential falls to V' and the corresponding deflection becomes b', then the charge,

$$Q = (C_1 + C)V = (C_1 + C_2 + C)V'$$

or $$\frac{C_1 + C_2 + C}{C_1 + C} = \frac{V}{V'}$$

But $V = K\theta$ and $V' = K\theta'$.

Hence, $$\frac{C_1 + C_2 + C}{C_1 + C} = \frac{\theta}{\theta'}$$

or $$\frac{C_2}{C_1 + C} = \frac{\theta - \theta'}{\theta} \qquad \ldots(7.17)$$

Since C, the capacity of the electromeer, is known from equation (7.16), the ratio C_2/C_1 becomes known, however, if C is very small in comparison to C_1 and C_2, we can neglect C in equation (7.17), then,

$$\frac{C_2}{C_1} = \frac{\theta - \theta'}{\theta'} \qquad \ldots(7.18)$$

Determination of Dielectric Consant

As dielectric constant K is defined as,

$$K = \frac{\text{Capacity of condenser with the substance as dielectric}}{\text{Capacity of the condenser with air as dielectric}}$$

Hence, the methods of determination of capacity and dielectric constant are same. Any method used for comparison of two capacities can be used for determining dielectric constant.

4. Measurement of Feeble (Ionisation) Currents

A quadrant electrometer can be used to measure "very feeble ionisation current." The ionisation current is produced by exposing a chamber containing gas, by some ionising agent *e.g.*, X-rays, ultraviolet or by radiations from radioactive materials.

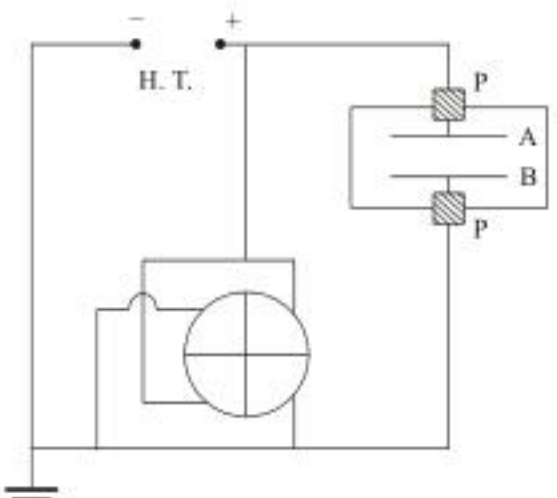


Fig. 7.7

An ionisation chamber, consisting of a metallic box containing two metal plates A and B, insluated and separated from each other. The plate A is charged to a high positive potential and to one pair of quadrants. The other plate B is connected to one pair of quadrants of the electrometer, which is earthed. The needle of the electrometer is given a high positive potential as shown in Fig. 7.7.

When the gas in the chamber is ionised, the needle of the electrometer is deflected because due to ionisation the negative ions are produced and they move toward the electrode A. *The movement of the ions constitutes ionisation current. The ratio of change of deflection is proportional to the magnitude of the ionisation current.*

Let C be the capacity of the electrometer together with the electrode A to which it is connected, and if V be the potential at any instant t due to the charge Q, the ionisation current,

$$I = \text{rate of flow of charge}$$

$$= \frac{dQ}{dt}$$

$$= \frac{C\,dV}{dt}$$

And from equation (7.15), $\theta = KV$, where K is constant.

Hence, $$I = C\,\frac{V_1 - V_2}{t} \qquad \ldots(7.19)$$

or $$I = \frac{1}{K}\,C\,\frac{d\theta}{dt}$$

$$= K'\,\frac{C\,d\theta}{dt} \qquad \ldots(7.20)$$

where $K' = \frac{1}{K}$ another constant.

$\frac{d\theta}{dt}$ is the rate of change of deflection of the needle, C is determined by experiment described earlier and consant $K' = \frac{1}{K}$, is determined from the calibration graph. Hence I can be calculated. Currents of the order of 10^{-15} amp. can be measured with a quadrant electrometer.

5. Measurement of Dielectric Constant

(*i*) Hopkinson's Null Method for Solids

We can determine the dielectric constant by this method if the solid is available in the form of slab. The circuit arrangement is shown in Fig. 7.8, C_2 is a guard ring condenser in which the lower plate is movable. C_1 is a variable cylindrical condenser consisting of co-axial cylindrical conductors, the inner cylinder can slide inside the outer one, hence the capacity of this condenser can be increased or decreased by increasing or decreasing the area of overlap. The battery B is earthed at its mid-point so that its terminal points are at equal and opposite potentials with respect to the earth-point. The needle of the electrometer is charged by battery E to a high positive potential. When the two keys K_1 and K_2 are depressed, the condensers C_1 and C_2 are charged such that the upper plate of C_1 is at a positive potential and that of C_2 to an equal negative potential. Hence, if two condensers are of equal capacity, they will acquire equal and opposite charges. Now, if the keys K_1 and K_2 are raised such that they are in contact with C_1 and C_2 and the connection with battery is broken, the charges will neutralise one another and there will be no deflection in the electrometer. This is obtained by adjusting the capacity of the variable capacitor C_1.

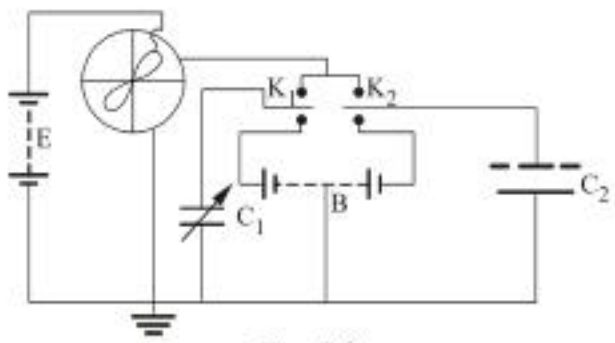


Fig. 7.8

Let the distance between the plates A and B of the capacitor C_2 be *d*. The given dielectric slab of thickness *t* is placed on the plate B. This increases the capacity of C_2. The previous adjustment of C_1 is kept undisturbed and the plate B is moved downward until there is again no deflection in the electrometer when K_1 and K_2 are raised such that C_1 and C_2 are connected to electrometer. Let *x* be the distance by which the plate B is moved.

$$\text{Capacity of } C_2 \text{ without dielectric} = \frac{\varepsilon_0 A}{d}$$

where A is the effective area of the guard ring capacitor. Capacity of C_2 after introducing the dielectric slab, when the electrometer showns no deflection,

$$= \frac{\varepsilon_0 A}{\left[d + x - t\left(1 - \frac{1}{K}\right)\right]}$$

Therefore, $$\frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 A}{\left[d + x - t\left(1 - \frac{1}{K}\right)\right]}$$

or $$d = d + x - t\left(1 - \frac{1}{K}\right)$$

or $$K = \frac{t}{t - x} \qquad \ldots(7.21)$$

Hence *K* can be calculated.

(ii) Dielectric Constant of a liquid by Hopkinson's Method

Procedure is the same as that for solid except that the guarding condenser C_2 in Fig. 7.8 is replaced by a double-walled cylindrical condenser in which an insulated cylinder C could hang as shown in Fig. 7.9. This condenser is connected in place of C_2 in circuit of Fig. 7.8 and C_1 is suitably adjusted, so that there is no deflection of the needle of the electrometer. Then the capacity of condenser C is equal to the capacity of the condenser C_1.

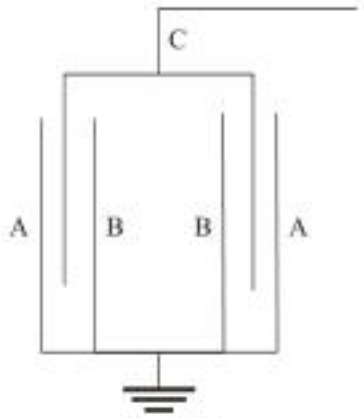


Fig. 7.9

If a and b are the inner and outer radii of the cylindrical condenser C_1 and l_1 is the length of overlap of inner and outer cylinders, then the capacity of the condenser C_1 is equal to the capacity of C_1 without dielectric,

$$C_{\text{without dielectric}} = \frac{2\pi\varepsilon_0 l_1}{\log_e \frac{b}{a}}$$

Now the given liquid is filled in C between A and B and the position of C_1 is again adjusted for zero deflection. If l_2 is the length of overlap, then capacity of the condenser C with dielectric,

$$C_{\text{with dielectric}} = \frac{2\pi\varepsilon_0 l_2}{\log_e \frac{b}{a}}$$

the ratio,

$$\frac{C_{\text{with dielectric}}}{C_{\text{without dielectric}}} = \frac{l_2}{l_1} = K \qquad \ldots(7.22)$$

Hence K, *i.e.*, dielectric constant, or the relative permittivity of the liquid is known.

(iii) Boltzmann's method of determining the dielectric constant of gases

The circuit used by Boltzmann is shown in Fig. 7.10. Two plates P and Q forming a parallel plate condenser are enclosed in a metal chamber C which is earthed, the chamber C can be exhausted or filled with gas. The connection from the plates P and Q are insulated from the chamber C. The plate P is connected to the positive of H.T. battery of n cells each of voltage V through key K_1. The plate Q is connected to one pair of the quadrants AA and the other quadrants are earthed. The key K_2 helps to connect plate Q and quardants AA to the earth.

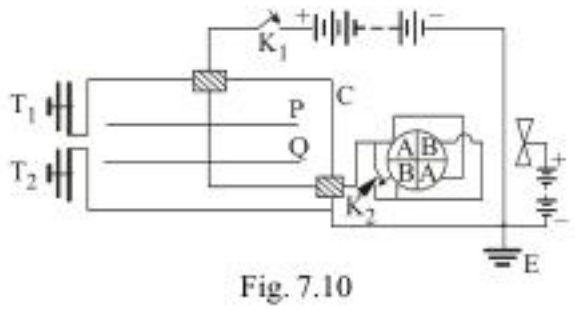


Fig. 7.10

The chamber C is first evacuated then the key K_1 is closed which charges the plate P to a potential nV. If the K_2 is simultaneously closed as K_1, there will be no deflection in the electrometer because both the pairs of quadrants are earthed. Next, keys K_1 and K_2 are opened and the chamber is filled with given gas. The capacity of the condenser formed by P and Q will increase K times and consequently, the potential of P falls to nV/K.

The key K_1 is again closed and the plate P is charged to a potential nV. It receives additional charge from the source due to its increased capacity. This additional charge induces an equal and opposite charge on Q. Thus Q is charged by the same amount as that of P. As Q is connected to electrometer, hence electrometer needle is deflected. Let this deflection be θ, then according to eq. (7.15), $\theta \propto V$.

Hence the change in potential $= n\,\mathrm{V} - \dfrac{n\,\mathrm{V}}{\mathrm{K}}$

$$= n\,\mathrm{V}\left(1 - \frac{1}{\mathrm{K}}\right)$$

Therefore, $\theta \propto n\,\mathrm{V}\left(1 - \dfrac{1}{\mathrm{K}}\right)$

or $$\theta = \mathrm{K}'\, n\,\mathrm{V}\left(1 - \frac{1}{\mathrm{K}}\right) \qquad \ldots(7.23)$$

where K' is a constant.

Now if one more cell is added to H.T. battery, then the potential of P will increase to $n\mathrm{V} + \mathrm{V} = (n + 1)\,\mathrm{V}$ and the potential of Q will be also increased by an equal amount. This increase of potential changes the deflection of the needle, say θ', then,

$$\theta' \propto \mathrm{V}$$

or $$\theta' = \mathrm{K}'\mathrm{V} \qquad \ldots(7.24)$$

Hence, $\dfrac{\theta}{\theta'} = n\left(1 - \dfrac{1}{\mathrm{K}}\right)$

or $$1 - \frac{1}{\mathrm{K}} = \frac{\theta}{n\theta'}$$

or $$\frac{1}{\mathrm{K}} = 1 - \frac{\theta}{n\theta'} = \frac{n\theta' - \theta}{n\theta'}$$

or $$\mathrm{K} = \frac{n\theta'}{n\theta' - \theta}$$

Thus knowing n, θ and θ', K can be calculated.

6. Comparison of Potentials

Potentials can be compared either by connecting the electrometer heterostatically or idiostatically and using eq. (7.12) and (7.13) respectively.

7.6 ELECTOSTATIC VOLTMETER

The electrostatic voltmeter was first designed by Lord Kelvin and is a modified form of quadrant electrometer. It consists of only one pair of opposite quadrants AA placed vertically. A light aluminium needle N is pivoted at the centre so that it can rotate about a horizontal axis in a vertical plane symmetrically between the quadrants The needle N is insulated from the quadrants. The upper part of the needle carries a pointer which moves over the scale S of the instrument and the lower part carries a controlling weight which can be adjusted for changing the range of the instrument as shown in Fig. 7.11.

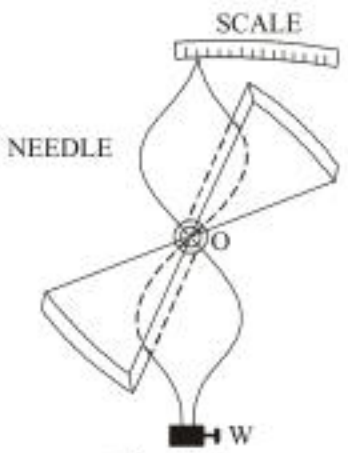


Fig. 7.11

The potential difference to be measured is applied between the needle and the quadrants. The needle N rotates due to this potential difference and the deflection of the needle is proportional to the square of the applied

potential difference. The scale is directly calibrated to read volts by actual comparison with the standard cell or attracted disc electrometer. Thus the device is handy and is capable of measuring potential of several thousand volts.

7.7 ELECTROSTATICS MACHINES

The devices used to produce electrical charges on a large scale are called electrostatics machines or generators. In fact, *these machines do not generate the charge but they only separate the charges*. In general, there are two types of machines, namely,

(1) Frictional machines, (2) Induction machines

Now a days the frictional machines are not used.

The commonly used induction machines are:

(1) Electrophorus devised by Volta in 1775.
(2) Wimshurst machines,
(3) Van de Graaff generator.

Here, we shall discuss only the Van de Graff generator, for other *induction machines see some school books.*

7.8 VAN DE GRAAFF GENERATOR

This was designed by Van de Graaff in 1931. It can produce high voltage of the order of 50 mega volt. The principle of the generator is as follows.

It uses the *discharge action of points and charge collecting action of hollow spherical conductors.* It consists of a large hollow sphere S mounted on a large pillar P of highly insulating material. B is an endless belt of some insulating material, *i.e.* rubber or silk as shown in Fig. 7.12. Belt runs on motor driven pulleys P_1 and P_2 and enters the sphere through the slit S_1 and leaves the sphere through S_2. A conductor, whose one end is a point A, is charged to a high potential (about 10,000 volts) is placed near the belt. This charge leaks out to the belt which becomes positively charged. The belt moves up and passes to another close point H near P_2 connected to the metallic sphere S.

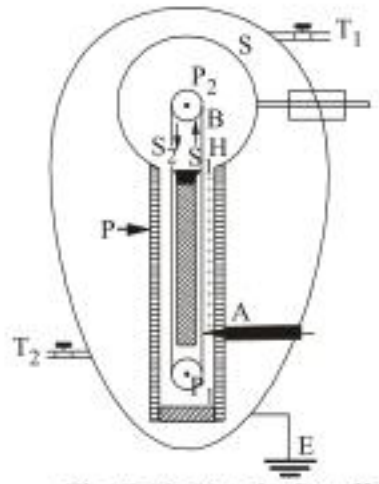


Fig. 7.12 Van de Graaff generator

This induces a negative charge on the point H and a positive charge on the sphere. The negative charge on the point H is soon neutralised by the positive charge on the belt. The sphere thus gets positively charged. As the belt is revolving continuously, the charge on the sphere is accumulating continuously. *Consequently the potential of the sphere rises considerably depending on its size and the smoothness of its surface. The bigger the sphere and the smoother its surface, the greater is the potential it can retain without leakage.*

To prevent the discharge taking place from S, the generator is completely enclosed in an earthed steel tank which is filled with air under pressure.

Example 7.4 *An electrometer is charged by connecting it momentarily to a battery and a deflection of 100 divisions is oberved. Then a condenser of 50 μF capacitance is connected in parallel to the electrometer and deflection falls to 75 divisions. Calculate the capacity of the electrometer.*

Solution. Let C be the capacity of the electrometer and C_1 is the capacity of the condenser. In the first case,

$$Q = CV$$

and in second case,

$$Q = (C + C_1)\, V_1$$

or $$\frac{V}{V_1} = \frac{C + C_1}{C}$$

But from equation (7.15), *i.e.* $\theta = KV$

$$\frac{V}{V_1} = \frac{\theta}{\theta_1} = \frac{100}{75}$$

Therefore, $$\frac{C + C_1}{C} = \frac{4}{3}$$

or $$\frac{C_1}{C} = \frac{1}{3}$$

or $$C = 3C_1 = 3 \times 50\ \mu F$$

$$= 150\ \mu F$$

Example 7.5 *Two leyden jars are exactly similar in size and shape but one has glass as dielectric and the other ebonite. The glass jar is charged but when the charge is shared between the two jars the potential falls to 0.6 of its original value. If dielectric constant of ebonite is 2, find that of glass.*

Solution. Using Eq. (7.18),

$$\frac{C_1}{C_2} = \frac{\theta''}{\theta - \theta'} \qquad (\text{but } \theta = K\,V)$$

hence, $$\frac{C_1}{C_2} = \frac{V'}{V - V'}$$

Let V be the original potential, then,

$$V' = \frac{6}{10} V$$

Therefore,

$$\frac{C_1}{C_2} = \frac{\frac{6V}{10}}{V - \frac{6V}{10}} = \frac{3}{2}$$

Since the capacitors are exactly similar, their capacitance without dielectric will be same. Hence,

$$C_1 = K_1 C_0$$

where K_1 is dielectric constant of glass

and $$C_2 = K_2 C_0$$

where K_2 is dielectric constant of ebonite.

Then,

$$\frac{C_1}{C_2} = \frac{K_1}{K_2} = \frac{3}{2}$$

or $$K_1 = \frac{3}{2} K_2$$

$$K_{glass} = 3 \qquad (\text{as } K_2 = 2)$$

Example 7.6 *When the air inside an electroscope of 20 cm capacitance is ionised by a beam of X-rays, it is found that the potential f the gold leaf changes from 154 to 100 volts in one minute. Find the current (1 farad = 9×10^{11} cm).*

Solution. $C = 20 \text{ cm} = \frac{20}{9 \times 10^{11}}$ farad

and $$I = \frac{CdV}{dt} = \frac{20}{9 \times 10^{11}} \times \frac{54}{60} = 2 \times 10^{-11} \text{ amp.}$$

REVIEW QUESTIONS

7.1 Give the principle and theory of the attracted disc electrometer.

7.2 (a) Why is it called absolute electrometer?

(b) What advantage has it over other forms of electrometer?

(c) How would you find dielectric constant of a substance with attracted disc electrometer?

(d) What are the advantages and disadvantages of an electrometer over a voltmeter?

7.3 Give the theory of quadrant electrometer. Explain clearly the heterostatic and idiostatic arrangements and discuss the relative advantages.

7.4 Describe the quadrant electrometer and explain how it may be used to measure

(1) alternating potential

(2) direct potential

(3) ionisation current

7.5 Give the theory of quadrant electrometer. How will you use it to compare two capacities?

7.6 What is Van de Graaff generator? Discuss its principle and working.

PROBLEMS

7-1 Calculate the force of attraction between lower and upper discs of an attracted disc electrometer when a potential difference of 1000 volts is applied between them. Given that they are 0.5 cm apart and area 10 sq. cm.

7-2 Two circular metal plates each of radius 2 cm are separated by a distance of 2 mm in air. Calculate the potential difference in volts between them when the force of attraction between them is equal to one gm weight.

7-3 Two brass plates are arranged horizontally 2 cm apart and the lower plate is earthed. The plates are charged to a potential difference of 400 volts. A drop of oil carrying a charge 4.8×10^{-10} esu is at rest between the plates. Calculate the mass of drop.

7-4 A small current flows through a resistance of 10^{10} ohms, the ends of which are connected to opposite quadrants of an electrometer. The deflection is 120 scale divisions. When a denial cell (emf 1.04 volts) is connected across these quadrants, the deflection is 54 scale divisions. Calculate the current.

ANSWERS

1. 177×10^6 Nt **2.** 1062.6 volts **3.** 3.27×10^{-11} kg

4. 2.4×10^{-19} amp

SELF-TEST OBJECTIVE QUESTIONS

7.1 Electrometer is an accurate device because
(*a*) high current is required for its working
(*b*) low current is required for its working
(*c*) the current completely avoided
(*d*) none of these

7.2 Absolute measurement of potential difference is done by
(a) Gold leaf electroscope (b) Quadrant electrometer
(c) Kelvin's attracted disc electrometer (d) voltmeter

7.3 Kelvin's absolute electrometer is less sensitive than other electrometers. This statement is
(*a*) False (*b*) True

7.4 The dielectric constant can be measured by electrometers. This is
(*a*) True (*b*) False

7.5 Quadrant electrometer is more accurate when used.
(a) iodiostatically (*b*) heterostatically

7.6 For measuring the alternating potential difference the quadrant electrometer must be used
(a) iodiostatically (*b*) heterostatically

7.7 For measuring small potential difference the quadrant electrometer is used
(a) iodiostatically (*b*) heterostatically

7.8 For measuring large potential difference the quadrant electrometer is used
(a) iodiostatically (*b*) heterostatically

7.9 Measurement of feeble ionisation current is possible by
(*a*) Ammeter (*b*) Quadrant electrometer

7.10 A quadrant electrometer can be used to measure
(*a*) capacitance (*b*) inductance
(*c*) potential difference (*d*) none of the above

ANSWERS

1. (c) **2.** (c) **3.** (b) **4.** (a) **5.** (b) **6.** (a)
7. (b) **8.** (a) **9.** (b) **10.** (c)

CHAPTER 8

STEADY CURRENT

8.1 METAL ELECTRODE IN AN ELECTROLYTE

When an electrode of pure zinc is immersed in an electrolyte, as in Fig. 8.1, the chemical forces cause zinc ions to enter the liquid. Since zinc ions are positively charged, the electrode acquires a negative charge. *This process continues until the electric field due to the static charge ion the electrode becomes great enough to conteract the chemical forces.* Thus, when the zinc ions are in electrostatics equilibrium, we have,

$$\bar{E}_c + \bar{E}_{ch} = 0 \qquad \ldots(8.1)$$

where $\bar{E}_c$ is the electric field intensity due to the static charge on the electrode and $\bar{E}_{ch}$ in the chemical force per unit charge.

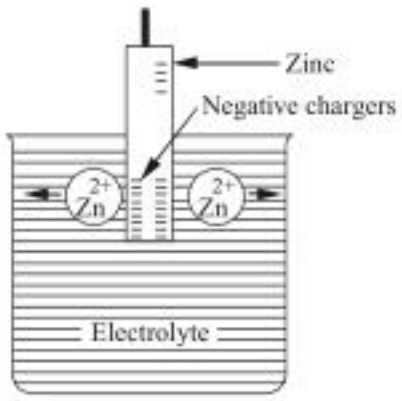


Fig. 8.1

8.2 BATTERY ON OPEN CIRCUIT

An electric cell is shown in Fig. 8.2. Positively charged copper ions tend to leave copper electrode just as positively charged zinc ions tent to leave the zinc electrode. The chemical forces may be considered as sum of two electric fields, $\bar{E}_{ch}$ acting on the surface of zinc and $\bar{E}''_{ch}$ on the copper surface. $\bar{E}'_{ch}$ is greater in magnitude then $\bar{E}''_{ch}$ and the direction of $\bar{E}'_{ch}$ is opposite to that of $\bar{E}''_{ch}$. Thus consider $\bar{E}_{ch}$, the chemical force per unit charge to be a vector function of position in the electrolyte, *i.e.* resultant of these two, we have,

$$\bar{E}_{ch} = \bar{E}'_{ch} + \bar{E}''_{ch} \qquad \ldots(8.2)$$

The number of zinc ions entering the electrolyte is greater than copper ions. As a consequence thezinc electrode acquires a greater negative free charge than the copper electrode. Hence, *outside of the electrolyte, there is a field $\bar{E}_c$ is directed from copper to zinc electrode.* In the electrolyte, a layer of positive ions surrounds each electrode. The electrode surface and the surrounding ions constitue a double layer of charge, the electrostaic field intensity $\bar{E}_c$ is directed toward the electrode shown in Fig. 8.2. When equilibrium is established in the double layer, we must have,

$$\bar{E}_c + \bar{E}_{ch} = 0 \qquad \ldots(8.3)$$

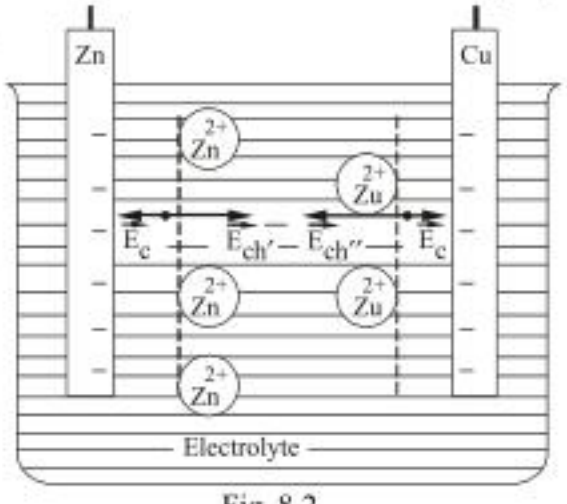


Fig. 8.2

This equation must also be valid for points in the main body of the electrolyte where the ions are in electrostatic equilibrium. Otherwise there would be a continuous migration of ions. Since $\vec{E}_{ch}$ is zero at points relatively far from the electrode $\vec{E}_c$ must be zero in the main body of electrolyte.

8.3 DEFINITION OF EMF

The *electromotive force (EMF) represents the work done by the non-electrostatic (chemical) force during the transfer of a unit charge from the negative to the positive terminal.* The emf ξ of an electric cell is given by,

$$\xi = \int_m^p \vec{E}_{ch} \cdot d\vec{L} \qquad \ldots(8.4)$$

Since the electric field $\vec{E}_{ch}$ exists only in the electrolyte, equation (8.4) may be written, here *m* denostes minus terminal and *p* positive,

$$\xi = \oint \vec{E}_{ch} \cdot d\vec{L} \qquad \ldots(8.5)$$

where the closed path extends from *m* to *p* through the electrolyte and then back to *p* through the space above the electrolyte. *No contribution to the final result is obtained by intergrating from p to m because E_{ch} is zero in the space above the electrolyte.* According to equation (8.5) $\vec{E}_{ch}$ *is "non-conservative", because the line integral of* $\vec{E}_{ch}$ *around a closed path is not zero.*

8.4 DEFINITION OF POTENTIAL DIFFERECNE

Since there are now two electric fields, $\vec{E}_c$ and $\vec{E}_{ch}$, we must re-examine the definition of potential difference. The field $\vec{E}_c$ is due to the time dependent (static) charge distribution in the cell and is, therefore, *"Conservative" and the potential difference is independent of path between the two points and can be computed only from the conservative electric field.* So the potential difference between the two plates is given by,

$$V_p - V_m = -\int_m^p \vec{E}_c \cdot d\vec{L} \qquad \ldots(8.6)$$

Since the cell is on open circuit, we have from Eq. (8.3),

$$\vec{E}_c + \vec{E}_{ch} = 0$$

Therefore, substituting for $\vec{E}_c$, we have,

$$V_p - V_m = +\int_m^p \vec{E}_c \cdot d\vec{L} \qquad \ldots(8.7)$$

Now according to equal on (8.4) the right hand side of equation (8.7) is emf. Therefore,

$$V_p - V_m = \int_m^p \vec{E}_{ch} \cdot d\vec{L} = \xi \qquad \ldots(8.8)$$

Thus, the potential difference across a cell on open circuit is equal to the emf of the cell.

Since $\oint \vec{E}_c \cdot d\vec{L} = 0$, we can rewrite equation (8.5) as follows,

$$\xi = \oint \left(\vec{E}_c + \vec{E}_{ch}\right) \cdot d\vec{L} \qquad \ldots(8.9)$$

The emf of an electric cell is equal to the "line integral" of the resultant electric field around a closed path that extends from the minus terminal to the plus terminal through electrolyte and then returns to the minus terminal through the space above the electrolyte.

8.5 BATTERY IN CLOSE CIRCUIT

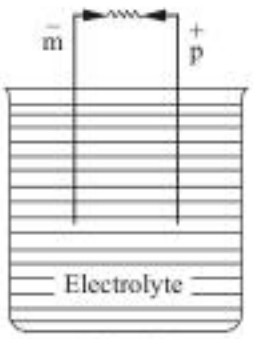


Fig. 8.3

When an electric cell is connected to a resistance as in Fig. 8.3, the negative charge flows through the resistance fromthe electrode at lower potential (m) to the electrode at higher potential (p). The effect, of course, is the same as that of positive charge flowing from p to m.

Here, for *simplicity and algebraic consistency,* we assume that only positive charge is moving through a wire. When the negative charge on m is reduced, additional positive ions from m enter the solution. This means that in the vicinity of m, $\left|\vec{E}_{ch}\right| > \left|\vec{E}_c\right|$. When the negative charge on p is increased, practically no positive ions leave p to enter the solution. This means that in the vicinity of p,

$$\left|\vec{E}_c\right| > \left|\vec{E}_{ch}\right|$$

The positive ions diffuse towards p and negative ions towards m. The current in the electrolyte consists of both of these streams of charges. The electric forces are such that positive charge is driven through the electrolyte from m to p. The charge returns to m through the external circuit because of the field $\vec{E}_c$. In a well-constructed cell, this process may continue for a long time.

Now the movement of charge in the electrolyte is related to the electric fields, $\vec{E}_c$ and $\vec{E}_{ch}$. We introduce the symbol $\vec{j}$ called current density which actually related to motion of charge in a conductor is proportional to the field or $\vec{j}$ is function of E,

$$\vec{J} = f\left(\vec{E}\right), J = 0 = f(0) \qquad \ldots(8.10)$$

Equation applies to ordinary conductors and excludes "super-conductors" for which $f(0)$ is a constant, for many conductors f is linear.

$$\vec{J} = \sigma\vec{E} \qquad \ldots(8.11)$$

In this case,

$$\vec{J} \propto \left(\vec{E}_c + \vec{E}_{ch}\right)$$

$$= \sigma\left(\vec{E}_c + \vec{E}_{ch}\right) \qquad \ldots(8.11a)$$

where the constant σ is called "conductivity". Note that when the cell is disconnected from the resistor, $\vec{J} = 0$, hence, $\vec{E}_c + \vec{E}_{ch} = 0$. This is same as open circuit case discussed previously.

8.6 CURRENT AND CURRENT DENSITY

If a net charge q passes through any cross-section of the conductor in time t, the current i is given by,

$$i = \frac{q}{t} \qquad \ldots(8.12)$$

Unit. The appropriate RMKS unit is ampere.

$$1 \text{ Ampere} = \frac{1 \text{ Coul}}{1 \text{ Sec.}}$$

The current given by equation (8.12) is the constant for the time. If the rate of flow of charge with time is not constant, then i varies with time and is given by,

$$i = \frac{dq}{dt} \qquad \ldots(8.13)$$

Here we shall consider only constant currents.

Current Density

Current i is a "characteristic" of a particular conductor. Current is a macroscopic quantity, like mass. Voluem, or length. A *related* microscopic quantity is the *current density* $\vec{j}$. It is a vector and is *characteristic of a point inside a conductor rather than conductor as a whole*. If the current is distributed uniformly across a conductor of cross-sectional area A, the magnitude of current density for all points on the cross-section is,

$$J = \frac{i}{A} \qquad \ldots(8.14)$$

If the current is *not uniformly distributed* then equation (8.14) gives the *average current density*. However, it is often of interest to consider the current density at a point. *This is defined as the current* Δi *through a small area* Δs *divided by* Δs, *with the limit of this ratio taken as* Δs *appraches zero*. Hence, the current density at a point is given by,

$$\vec{J} = \lim_{\Delta s \to 0} \frac{\Delta i}{\Delta s} \qquad \ldots(8.14a)$$

Unit. Its unit in RMKS is ampere per square metre, as evident from equation (8.14).

The vector $\vec{j}$ at any point is oriented in the direction that a positive charge carrier would move at that point. An electron at that point would move in the direction $-\vec{J}$.

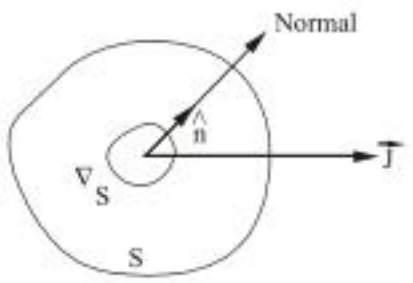


Fig. 8.4

The relationship given in equation (8.14) can be *generalised* for any surface, for a particular surface in a conductor, i is the flux of the vector $\vec{j}$ over that surface or in other words, the *charge flow is represented by a vector* $\vec{j}$ *which gives the amount of change passing per unit area per unit time through a surface element at right angles to the flow.*

If we consider the small area Δs shown in Fig. 8.4 at a given place in the material through which the charges are flowing, then amount of charge flowing across that area in a unit time is,

$$\Delta q = \vec{J} \cdot \hat{n}\, \Delta s, \qquad \ldots(8.15)$$

[For details see eq. (3.39)] where $\hat{n}$ is a unit vector normal to Δs. Hence, the total charge passing per unit time through any surface,

$$q = \int_s \vec{J}.d\vec{S} \qquad \ldots(8.16)$$

From the definition of the current $i = \frac{q}{t}$, for unit time *i.e.*, $t = 1$, therefore,

$$i = q = \int_s \vec{J} \cdot d\vec{S} \qquad \ldots(8.17)$$

8.7 DRIFT SPEED (v_d) OF CHARGE CARRIERS IN A CONDUCTOR

The drift speed V_d of charge carriers in a conductor can be computed from the current density $\vec{j}$. Suppose we have a charge distribution shown in Fig. 8.5, the conduction electrons (or charge) are moving to right with the constant drift speed v_d. As this charge distribution passes over the surface elements Δs, the charge Δq flowing across this surface element in a time Δt is equal to the charge contained in a parallelopiped with Δs as its base and height $v_d\, \Delta t$. If ρ is the volume density of the charge, then charge contained in the parallelopiped will be given by fig. 8.5.

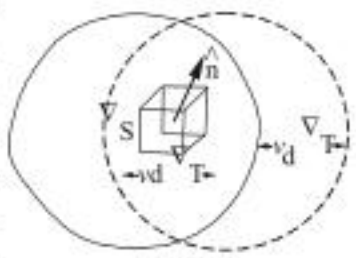


Fig. 8.5

Charge passing through Δs in unit time = volume of parallelopiped × ρ

= projection of Δs perpendicular to v_d × height of parallelopiped × ρ

$$\Delta q = \Delta s \cos\theta \, v_d \, \Delta t_\rho$$

or $$\Delta q = \rho \vec{v} d \,.\, \hat{n} \, d\mathrm{S} \, \Delta t$$

$$\text{Charge per unit time} = \rho \vec{\mathrm{v}} d \,.\hat{n} d\mathrm{S} = \rho \vec{v} d \,.d \vec{S} \qquad \ldots(8.18)$$

Comparing equations (8.18) and (8.16), we get

$$\vec{\mathrm{J}} = \rho \vec{\mathrm{v}} d \qquad \ldots(8.19)$$

Let us condsider the simple charge distribution of electrons each of charge e and N are the number of electrons per unit volume, then,

$$\rho = \mathrm{N}e, \text{ hence, } \vec{\mathrm{J}} = \mathrm{N}e\vec{\mathrm{v}} d$$

or $$\vec{\mathrm{v}}_d = \vec{\mathrm{J}} / \mathrm{N}e \qquad \ldots(8.20)$$

Example 8.1 *Calculate the "drift speed" of electrons shown in Fig. 8.6 and its value when $\vec{J}$ and $\hat{n}$ are parallel.*

Solution. We shall use eq. (8.17)

$$i = \int_s \vec{\mathrm{J}} \,.\, \vec{d}\mathrm{S}$$

Here, the value of integral is independent of the shape of the surface.

and, $$i = \vec{\mathrm{J}} \, \vec{\mathrm{A}}$$

where, A is the area of cross-section of cylinder.

If area A is normal to the surface,

$$I = \mathrm{J}.\,\mathrm{A}, \qquad \text{but } \mathrm{J} = \mathrm{N}e\,\mathrm{v}_d$$

Hence,

$$i = \mathrm{N}e v_d \,\mathrm{A}$$

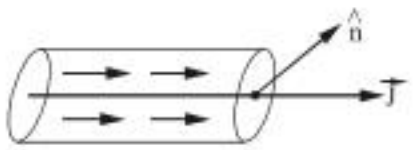


Fig. 8.6

or $$v_d = \frac{i}{\mathrm{NA}\,e}$$

8.8 CONSERVATION OF CHARGE I.E., CONTINUITY EQUATION

As we known that the electric charge is "*indestructible*". it is never *lost or created that* is the charge is conserved. The current i out of a closed surface represent the rate at which the charge leaves the volume v enclosed by s. If there is a "*net current*" out of a closed surface *i.e.* the net charge has crossed the surface. **The amount of charge inside must decrease correspondingly according to conservation rule,**

$$i = \int_s \vec{\mathrm{J}} \,.\hat{n} \, d\mathrm{S} = -\frac{d}{dt}(Q_{in}) \qquad \ldots(8.21)$$

where Q_{in} is the charge inside the surface. But

$$Q_{in} = \int_v \rho d \, \mathrm{v}$$

therefore, $i = \int_S \vec{J} . \hat{n} d\vec{S} = -\frac{d}{dt} \int_v \rho dv$

using divergence between, we get

$$\int_S \vec{J} . \hat{n} d\vec{S} = \int_v \vec{\nabla} . \vec{J} dv$$

hence $$\int_v \vec{\nabla} . \vec{J} . dv = -\frac{\delta}{\delta t} \int_v \rho dv$$

i.e., $$\vec{\nabla} . \vec{J} = -\frac{\partial \rho}{\delta t} \quad \ldots(8.22)$$

This is an important equation regarding "*charge flow*", *called* **continuity equation** *and describes the fact that the charge is conserved.* The decrease in value of charge inside a small volume must correspond to a flow of charge.

A steady state is one in which all time derivatives are zero so we must have $\frac{\delta \rho}{\delta t} = 0$ and

$$\vec{\nabla} . \vec{J} = 0 \text{ (Steady state)} \quad \ldots(8.23)$$

or $$\int_s \vec{J} . d\vec{S} = 0 \quad \ldots(8.24)$$

Now, if the medium obeys Ohm's law, we have,

$$\vec{J} = \sigma \vec{E}$$

then we get,

$$\vec{\nabla} . \vec{J} = \sigma \vec{\nabla} . \vec{E} = 0$$

So for a steady state we *conclude that E has zero divergence, hence according to Gauss' law, ρ vanishes and Laplace equation is valid.* Since

$$\vec{E} = -\vec{\nabla} . V,$$

we have *Laplace equation*,

$$-\vec{\nabla}^2 V = \vec{\nabla} . \vec{E} = \frac{\rho}{\varepsilon_0} = 0$$

8.9 OHM'S LAW

Ohm's law states that the potential difference between the ends of a conductor varies directly as the current flowing in it, provided the temperature does not change and the physical state of the conductor remains the same.

$$\text{Potential difference} \propto \text{Current}$$

or $$\frac{\text{Potential difference}}{\text{Current}} = \text{Constant}$$

$$= \text{Resistance of the conductor.}$$

Resistance

By definition, the resistance R_{ab} between the terminals *a* and *b* on a conductor is given by,

$$R_{ab} = \frac{V_a - V_b}{i} \quad \ldots(8.25)$$

where the current *i* is directed from *a* to *b*. This relation is called Ohm's law.

Unit:

Resistance R_{ab} is expressed in ohms

1 ohm = 1 Volt/ampere.

The ohm *is defined as the resistance of a conductor in which a potential difference of 1 volt is developed when a current of 1 ampere flows through it.*

Actually, *resistance has value beyond that of a mere definition*. For many conductors, the resistance depends only on the temperature and does not depend on the potential difference or the current. *In other words, the resistance is a property of the conductor and is not related to the circuit in which the conductor is connected.*

The equation $R = V/i$ can be used to calculate the current in a conductor if we know the resistance and the potential difference across the conductor.

8.10 RESISTANCE OF CYLINDRICAL CONDUCTOR AND SPECIFIC RESISTANCE

As an example let us calculate the resistance of the cylindrical conductor of radius r and length l as shown in Fig. 8.7. The potential difference,

$$V_a - V_b = -\int_b^a \vec{E}.d\vec{l}$$

Now using $\vec{J} = \sigma\vec{E}$ and reversing the limit,

$$V_a - V_b = \int_a^b \frac{\vec{J}.d\vec{l}}{\sigma}$$

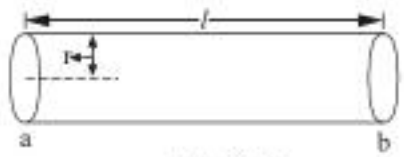


Fig. 8.7

From definition, the magnitude of $J = \frac{i}{A} = \frac{i}{\pi r^2}$.

Hence,

$$V_a - V_b = \int_a^b \frac{i\,d\vec{l}}{\sigma\pi r^2}, \quad \text{(Since } i \text{ and } dl \text{ are parallel)}$$

$$= \frac{il}{\sigma\pi r^2}$$

Therefore,

$$R = \frac{V_a - V_b}{i} = \frac{1}{\sigma\pi r^2} \qquad \ldots(8.26)$$

The resistance of a conductor of uniform cross-section is directly proportional to the length and inversely proportional to the conductivity and the cross-sectional area.

Specific Resistance

In relation (8.26), let us take a wire of unit length and unit cross-section, then,

$$R = \frac{1}{\sigma} = \rho_s \qquad \ldots(8.27)$$

The quantity ρ_s is called "*specific resistance*" or "*resistivity*". Thus, the *specific resistance is the resistance offered by a conductor of unit length and unit cross-section.* Its unit is ohm-metre. "*Conductance*," the "*reciprocal of resistivity*" is electrical conductivity or simply conductance. Its unit is mho.

8.11 OHM'S LAW AT A POINT

Consider a block of conducting material as shown in Fig. 8.8. Let a small imaginary rectangular shell of length l and cross-section a be constructed around a point P in the interior of the block with a normal to $\vec{j}$ as indicated in Fig. 8.8.

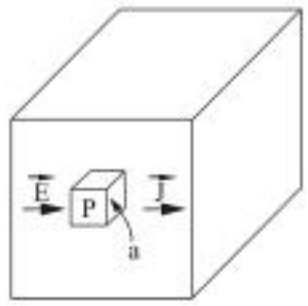


Fig. 8.8

Then, on applying Ohm's law to this shell, we have,

$$V = iR$$

where V = potential difference between ends of the shell. But $V = El$, where E is the electric field and $J = i/a$ or $i = Ja$. Therefore,

$$El = V = iR = JaR$$

or $$\vec{J} = \frac{l}{aR} E$$

But according to equation (8.26), *i.e.*,

$$R = \frac{l}{\sigma a}$$

since here, $\pi r^2 = a$,

or $$\sigma = \frac{l}{aR}$$

Therefore, $\vec{J} = \sigma \vec{E}$ ...(8.28)

By making the shell enclosing P as small as we wish, this relation may be made to apply at the point P and we may write as Eqn. (8.28).

8.12 RESISTANCE ANALOGOUS TO RECIPROCAL OF CAPACITANCE

In fact, this is the relationship between "conductivity" and "permittivity". In many respects, the calculation of the resistance between two electrodes is the same as that of capacitance. The similarity may be seen by comparing the definition of R and C, namely,

$$R = \frac{V_a - V_b}{i}, \quad C = \frac{q}{V_a - V_b}.$$

The current i is given by

$$i = \oint_s \vec{J} . d\vec{S}$$

Similarly, the charge q may be written by Gauss' law

$$q = \oint_s \vec{D} . d\vec{S}$$

If the medium has a conductivity σ and permittivity ε, then

$$\vec{J} = \sigma \vec{E} \text{ and } \vec{D} = \varepsilon \vec{E}$$

Therefore, $i = \oint_s \sigma \vec{E} . d\vec{S}$ and $q = \oint_s \varepsilon \vec{E} . d\vec{S}$

Thus, $$R = \frac{V_a - V_b}{\oint_s \sigma \vec{E} . d\vec{S}} \text{ and } C = \frac{\varepsilon \oint_s \vec{E} . d\vec{S}}{V_a - V_b}$$

Consequently, $$RC = \frac{\varepsilon}{\sigma}$$

$$R = \frac{\epsilon}{\sigma C} \qquad \ldots(8.29)$$

If the capacitance between two conductors separated by a medium of permittivity ε is known, then the resistance between the same two conductors separated by a medium of conductivity σ may be calculated by means of Eq. (8.29).

Example 8.2 *As an example, let us calculate the resistance between two parallel plates. The capacitance C between two parallel plates of area A separated by distance l is,* $C = \frac{\varepsilon A}{l}$.

Solution. By Eq. (8.29), the resistance,

$$R = \frac{\varepsilon}{\sigma\left(\frac{\varepsilon A}{l}\right)} = \frac{l}{\sigma A} = \frac{\rho_s l}{A} \qquad \ldots(8.30)$$

Same result as expressed by (8.26), and we can again define a specific resistance.

8.13 WIEDMANN AND FRANZ LAW

As we know that good conductors of heat are also the good conductors for electric current. The Wiedmann and Franz *established "empirically" the relation between the thermal and electrical conductivities*. If σ_h is the thermal conductivity then the ratio σ_h/σ at higher temperature is proportional to the absolute temperature,

$$\frac{\sigma_h}{\sigma} = LT \qquad \ldots(8.31)$$

where *L is the constant of proportionality, is constant for all metals and is known as "Lorentz number".*

8.14 THE RELAXATION TIME

The approach of current in an extended medium to a steady state can be studied by using,

$$\vec{\nabla} . \vec{J} + \frac{\delta\rho}{\delta t} = 0 \text{ and } \vec{J} = \sigma \vec{E}$$

together with $\vec{\nabla} . \vec{E} = \frac{\rho}{\varepsilon_0}$, where ρ is charge density and we assume that the medium is conducting but not also polarizable. Let us substitute $\vec{J} = \sigma\vec{E}$ in continuity equation, we get,

$$\vec{\nabla} . \sigma\vec{E} + \frac{\delta\rho}{\delta t} = 0$$

or $$\sigma\vec{\nabla} . \vec{E} + \frac{\delta\rho}{\delta t} = 0$$

or $$\frac{\sigma\rho}{\varepsilon_0} + \frac{\delta\rho}{\delta t} = 0, \text{ since } \nabla . \vec{E} = \frac{\rho}{\varepsilon_0}$$

or $$\frac{\delta\rho}{\rho} = -\frac{\sigma dt}{\varepsilon_0}$$

Integrating the above equation,

$$\int \frac{d\rho}{\rho} = \log_e \rho = -\int \frac{\sigma}{\varepsilon_0} dt$$

$$= -\frac{\sigma t}{\varepsilon_0} + \text{Const } (a)$$

or $$\log_e \rho = -\frac{\sigma t}{\varepsilon_0} + a$$

Taking antilog, we get,

$$\rho = \exp\left[-\frac{\sigma t}{\varepsilon_0} + a\right]$$

or $$= e^a e^{-\sigma t/\varepsilon_0}$$

e^a can be evaluated by setting $t = 0$, then, $\rho = \rho_0$, the fixed charge *i.e.*,

$$e^a = \rho_0$$

hence, $$\rho = \rho_0 e^{\frac{-\sigma t}{\varepsilon_0}} \qquad \ldots(8.32)$$

We see that the time required for an initial charge density to fall to 1/e of its value is ε_0/σ. It is called "relaxation time".

8.15 ELECTRIC POWER AND WORK

An electric generator is shown in Fig. 8.9 connected to an electric heater. Let us calculate the amount of power supplied to the resistance R and the amount of heat developed in the resistor in a time *t*. The potential difference,

$$V_a - V_b = -\int_b^a \vec{E} . d\vec{l} = \int_a^b \vec{E} . d\vec{l}$$

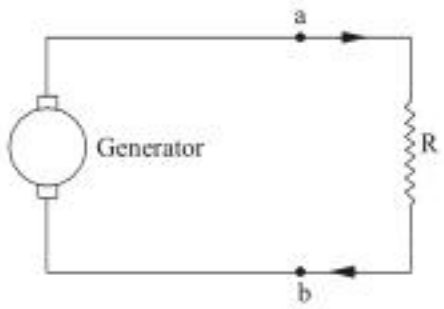


Fig. 8.9

The work done in moving ε charge dq from a to b is,

$$dw = (V_a - V_b)\, dq \qquad \ldots(8.33)$$

(Joules) = (Volts) (Coulombs)

By definition, power P is the rate of doing work. Thus,

$$P = \frac{dw}{dt} = (V_a - V_b)\frac{dq}{dt}$$

or $$P = (V_a - V_b)\, i$$

(Watts) = (Volts) (Amperes)

Since, by Ohm's law,

$$V_a - V_b = iR$$

hence, $P = i^2R$ joules/sec or watt

or, $$P = \frac{(V_a - V_b)^2}{R} \text{ joules/sec. or watt} \qquad \ldots(8.34)$$

This is the amount of energy absorbed per second in the resistance R. Total heat developed in R in time T,

$$w = \int_0^T P\, dt = \int_0^T i^2\, R\, dt \qquad \ldots(8.35)$$

where w is expressed in joules, equation (8.33) indicates that joule = watt. sec.

EMF and Internal Resistance

If a source of emf is connected in series with resistance R as in Fig. 8.10. If r is the internal resistance of B, the work done by the generator is the work done by the field E causing the charge to move from a to b. Consequently the rate at which the generator does work on the battery is given by,

$$V_b - V_a = -\xi - ir$$

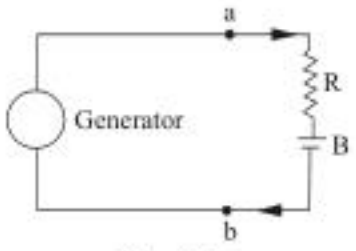


Fig. 8.10

where, $-\xi$ is the drop in voltage (due to emf) in passing from the plus terminal to the minus voltage nal and $-ir$ is the drop in voltage in passing through the internal resistance r. The generator maintains the potential difference $V_a - V_b$. The work done by the generator is the work done by the field E in moving charge from a to b. Consequently, the rate at which the generator does work on the battery is,

$$(V_a - V_b)\, i = \xi i + i^2 r$$

That the power absorbed in the battery due to emf is ξi and the power absorbed in the battery due to the evolution of heat is $i^2 r$.

Finally, let us take the discharge of battery through the resistance as shown in Fig. 8.11.

$$\xi = i\,\mathrm{R} + ir$$

multiply both sides of this equation by i, we get,

$$\xi i = i^2\,\mathrm{R} + i^2 r$$

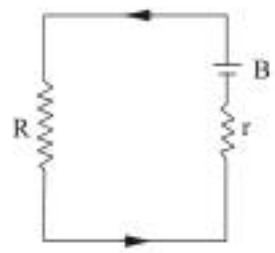


Fig. 8.11

Here ξi is the rate at which work is being done by chemical forces within the battery. This power is equal to the rate at which the heat is developed in the external resistance (i^2R) plus the rate at which heat is developed in the internal resistance (i^2R).

8.16 MAXIMUM POWER THEOREM

Let the cell of emf ξ and internal resistance r be connected in series with an external resistance R. Then the current i in the circuit,

$$i = \frac{\xi}{\mathrm{R}+r}$$

Power consumed $\quad \mathrm{P} = i^2\,\mathrm{R}$

$$= \frac{\xi^2\,\mathrm{R}}{(\mathrm{R}+r)^2}$$

$$= \xi^2\,\mathrm{R}\,(\mathrm{R}+r)^{-2}$$

Differentiating w.r.t. R,

$$\frac{d\mathrm{P}}{d\mathrm{R}} = \xi^2\left[(\mathrm{R}+r)^{-2} - 2\,\mathrm{R}\,(\mathrm{R}+r)^{-3}\right]$$

$$= \xi^2(\mathrm{R}+r)^{-2}\,[\mathrm{R}+r-2\mathrm{R}]$$

$$= \xi^2(\mathrm{R}+r)^{-3}\,(r-\mathrm{R})$$

$$\frac{d\mathrm{P}}{d\mathrm{R}} = \frac{\xi^2(r-\mathrm{R})}{(\mathrm{R}+r)^3}$$

Now, $\frac{dP}{dR} = 0$

when R = *r* or when R = ∞. If R = *r, the power consumed is maximum and if* R = ∞ *power consumed is zero.* This is known as "maximum power theorem" and *states that the power consumed is maximum in the internal circuit if the external resistance is equal to the resistance of the source of emf.* Theorem is applicable to cells, thermocouples, dynamos, thermionic valves etc.

Example 8.3 *A silver wire 1 mm in diameter carries a charge of 90 coulmbs in 1 hour and 15 minutes. Silver contains* 5.8×10^{22} *free electrons/cm*^{3}. *Calculate, (i) the current in the wire in amperes, (ii) the drift velocity of the elctrons in wire in metres/sec.*

Solution. (1) 1 hour and 15 minutes = 4500 secs.

$$\text{Current } i = \frac{q}{t}$$

$$i = \frac{90}{4500} = 0.02 \text{ ampere}$$

(2) Current density $\vec{j} = \frac{\text{Current}}{\text{Area}}$

$$= \frac{0.02 \text{ amp}}{\pi \times (0.0005)^2 \text{ m}^2}$$

$$= \frac{0.02}{3.14 \times 5 \times 5 \times 10^{-8}} \text{ amp/m}^2$$

$$= 2.55 \times 10^4 \text{ amp/m}^2$$

The drift velocity,

$$v_d = \frac{J}{ne} = \frac{2.55 \times 10^4}{1.6 \times 10^{-19} \times 5.8 \times 10^{28}} \text{ m/sec}$$

(Since 1 cm^3 = 10^{-6} m^3)

hence, $n = 5.8 \times 10^{28}$ elect./m^3

$$v_d = 2.69 \times 10^{-6} \text{ metre/sec.}$$

Example 8.4 *In a neon gas discharge tube* 2.9×10^{16} Ne^+ *ions move to the right through a cross-sectinof the tube each second, while* 1.2×10^{18} *electrons move to the left in the same time. Find the magnitude and the direction of the current.*

Solution. The charge on each neon ion,

$$q_1 = e = 1.6 \times 10^{-19} \text{ coul.}$$

Let us take the positive direction from left to right and negative from rigth to left.

The current due to Ne^+ ions $= \frac{n_1 q_1}{t_1}$

$$= \frac{2.9 \times 10^{18} \times 1.6 \times 10^{-19}}{1} \text{ coul/sec.}$$

$$= 0.446 \text{ amp.}$$

The charge on each electron, $q_2 = -1.6 \times 10^{-19}$ coul.

The current due to electrons $= -\dfrac{n_2 q_2}{t_2} = -\dfrac{(1.2\times10^{18})(-1.6\times10^{-19})}{1}$

$$= 1.2\times1.6 \text{ coul/sec} = 1.92 \text{ amp}.$$

Therefore, the net current through the discharge tube = 0.446 + 1.92 = 2.366 amp.

Thus, the magnitude of the current is 2.366 amp. and its direction is from left to right.

Example 8.5 *An aluminium wire whose diameter is 0.245 cm is welded end to end to a copper wire with diameter 0.1626 cm. The composite wire carries a steady current of 10 amp.*

(a) What is the current density in each wire?

(b) Find the drift speed of the electrons is copper wire. Assume one free electron per atom in copper.

Given Avagodro Number = 6 × 10^{23} atoms per gm-mole.

Molecular weight of copper = 64

Density of copper = 9 gm/cc.

Solution. (a) The current density is given by

$$\vec{\mathrm{J}} = \frac{i}{\mathrm{A}}$$

The area of aluminium wire, $\mathrm{A}_1 = \pi r_1^{\,2}$

$$= 3.14\left[\frac{0.245\times10^{-2}}{2}\right]^2 = 4.713\times10^{-6}\ \mathrm{m}^2$$

The area of copper wire, $\mathrm{A}_2 = \pi r_2^{\,2}$

$$= 3.14\left[\frac{0.1626\times10^{-2}}{2}\right]^2 = 2.075\times10^{-6}\ \mathrm{m}^2$$

Therefore, the current density in aluminium wire,

$$\mathrm{J}_{Al} = \frac{i}{\mathrm{A}_1} = \frac{10}{4.713\times10^{-6}} = 2.12\times10^{6}\ \text{amp/m}^2$$

Similarly, the current density in copper wire,

$$\mathrm{J}_{cu} = \frac{i}{\mathrm{A}_2} = \frac{10}{2.075\times10^{-6}} = 4.818\times10^{6}\ \text{amp/m}^2$$

(b) The number of atoms per unit volume = ρ N/M where ρ is the density of copper, N is Avogadro number and M is atomic weight of copper.

As given, there is one free electron per atom in copper, the number of free electrons per unit volume,

$$n = \frac{\rho \mathrm{N}}{\mathrm{M}}$$

$$= \frac{9\times(6\times10^{23})}{6} \text{ electrons/cm}^3$$

$$= 8.437\times10^{23} \text{ electrons/m}^3$$

But, $J = ne v_d$

Therefore,

$$v_d = \frac{J_{cm}}{n_e} = \frac{4.818 \times 10^6}{8.437 \times 10^{28} \times 1.6 \times 10^{-19}}$$

$$= 2.834 \times 10^{-4} \text{ m/sec}.$$

Example 8.6 *The potential difference across a wire of 10^{-3} cm^2 cross-sectional area and 50 cm length is 2 volt, when a current of 0.25 amp. exists in the wire. Calculate,*

(a) The field strength in the wire,

(b) The current density, and

(c) The conductivity of the metal

Solution. (a) The field strength in the wire is given by,

$$E = \frac{V}{d} = \frac{2 \text{ volt}}{0.5 \text{ metre}} = 4 \text{ volt/metre}.$$

(b) The current density,

$$J = \frac{i}{A} = \frac{0.25 \text{ amp}}{10^{-3} \text{ cm}^2}$$

$$= \frac{0.25 \text{ amp}}{10^{-3} \times 10^{-4} \text{ m}^2}$$

$$= 2.5 \times 10^6 \text{ amp/m}^2.$$

(c) The electrical conductivity of metal σ is given by

$$J = \sigma E$$

or $$\sigma = \frac{J}{E} = \frac{(2.5 \times 10^6 \text{ amp/m}^2)}{(4 \text{ volts/metre})}$$

$$= 6.25 \times 10^5 \text{ mho/metre}.$$

Example 8.7 *Calculate the insulation resistance of a cable having a cylindrical conductor of radius r_1 and the outer radius of insulation r_2. The specific resistance of the insulation is ρ.*

Solution. Let r and $r + dr$ be the radii of thin concentric layer of insulation, then the surface area $= 2\pi rl$, where l is the length of the cable and resistance of the layer $= \frac{\rho dr}{2\pi rl}$, according to eq. (8.30).

Therefore, total resistance of the cable of length l

$$R = \int_{r_1}^{r_2} \frac{\rho dr}{2\pi rl} = \frac{\rho}{2\pi l} \log_e \frac{r_2}{r_1}$$

Example 8.8 *Find the resistance of 1 cc of copper when drawn into a wire of diameter 0.32 mm specific resistance of copper = 1.59 × 10^{-6} ohms. cm.*

Solution. Radius r of the wire = 0.016 cm

$$\text{Length of the wire} = \frac{\text{Volume}}{\pi r^2} = \frac{1}{3.14 \times 0.016^2}$$

and according to equation (8.30),

$$R = \frac{\rho l}{a} = \frac{1.59\times10^{-6}}{3.14\times0.016^2\times3.14\times0.016^2} = 2.46 \text{ ohm.}$$

8.17 KIRCHHOFF'S LAWS

Ohm's law is unable to give values of currents in various parts of a complicated circuit. Kirchhoff in 1842 gave two general laws for calculating the currents and resistances in networks at junctions. *These laws are obtained from the law of "conservation of energy".*

1. Kirchhoff's First Law *i.e.* Current Law

It applies to circuit nodes (or junctin) and states *that in any network the algebraic sum of the currents at any junction in a circuit is zero, i.e.,*

$$\Sigma i = 0 \qquad \ldots(8.37)$$

the currents included in the above sum differ in sign according to whether they flow to or from the junction.

Proof. We can prove this law as follows. For steady current, the equation of continuity is given in equation (8.24), i.e.,

$$\int_s \vec{J} \cdot d\vec{S} = 0$$

and for a closed surface,

$$\oint \vec{J} \cdot d\vec{S} = 0.$$

This relation can be applied to any volume. For example, the volume may be entirely inside of a conducting medium, or it may be only partially filled with conductors. The conductors may form a network inside the volume, or they may meet at a point.

As an illustration of this latter case, the surface S in Fig. 8.12 encloses a volume that contains five conductors meeting at a junction point P. *Taking the current flowing away from the junction as negative and the current flowing towards the junction as positive,* according to relation $\oint_s \vec{J}\, d\vec{S} = 0$. we get,

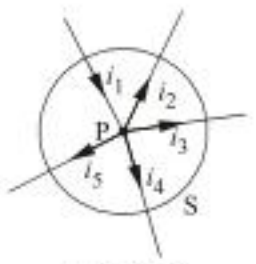


Fig. 8.12

$$i_1 - i_2 - i_3 + i_4 - i_5 = 0$$

It follows,

$$i_1 + i_4 = i_2 + i_3 + i_5$$

In accordance with the above equation, Kirchhoff's current law can be stated in different way, *that the total current entering any junction (or node) of a circuit equals the total current leaving that junction.*

"Physically," Kirchhoff's current law *implies that there can be no "accumulation" of electric charge at any junction of a circuit and is nothing more than a restatement of the principle of conservation of charge.*

2. Kirchhoff's Voltage (Second) Law or Loop Rule

It applies to close circuits (meshes) and states that in any closed circuit, *algebraic sum of the products of the current and resistance of the each part of the circuit is equal to the total emfs acting in the circuit, i.e.,*

$$\Sigma ir = \Sigma\xi \qquad \ldots(8.38)$$

and if ξ is not acting i.e. if there is no source of emf, then,

$$\Sigma ir = 0$$

Proof. To prove Kirchhoff's voltage aw, let us consider the simpe electric circuit show in Fig. 8.13. The current is i at al points in the circuit, we have from Ohm's Law at a point,

$$\mathrm{J} = \sigma \vec{\mathrm{E}}$$

where $\vec{\mathrm{E}}$ is total field at the point.

In general the total field $\vec{\mathrm{E}}$ may not only do due to static charges but also to other causes such as the chemical action in a battery. We may write,

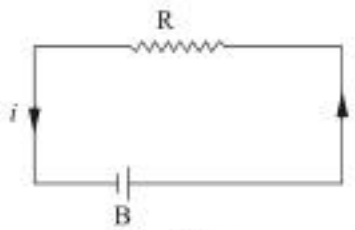


Fig. 8.13

$$\vec{\mathrm{E}} = \vec{\mathrm{E}}_c + \vec{\mathrm{E}}_{ch}$$

where $\vec{\mathrm{E}}_c$ is field due to static charge on the electrodes and $\vec{\mathrm{E}}_{ch}$ is the field due to chemical forces.

If a is the cross-sectional area of the conductor, then,

$$\mathrm{J} = \frac{i}{a} \text{ and } \sigma = \frac{l}{a\mathrm{R}}$$

(See equation (8.26), where l is the length of the conductor).

Hence,

$$\mathrm{E} = \frac{\vec{\mathrm{J}}}{\sigma} = \frac{i}{a} = \frac{a\mathrm{R}}{l} = \frac{i\mathrm{R}}{l} = \vec{\mathrm{E}} = \vec{\mathrm{E}}_c + \vec{\mathrm{E}}_{ch}$$

Integrating around the complete circuit, we get,

$$\oint \vec{\mathrm{E}}_c . d\vec{l} + \oint \vec{\mathrm{E}}_{ch} . d\vec{l} = i \oint \frac{\mathrm{R}}{l} dl$$

$\oint \vec{\mathrm{E}}_c . d\vec{l} = 0$, because $\vec{\mathrm{E}}_c$ is a conservative field and $\oint \vec{\mathrm{E}}_c . d\vec{l}$ is emf ξ according to equation (8.8), and $\oint \frac{\mathrm{R}}{l} dl = \mathrm{R}_\mathrm{T}$. Hence,

$$\xi = i\mathrm{R}_\mathrm{T}$$

where R_T is the total resistance and is equal to R if internal resistance is zero. We can generalize for number of closed circuits *i.e.*

$$\Sigma\xi = \Sigma ir$$

If ξ were absent, then, $\Sigma ir = 0$.

While applying the second law, the *positive value of the current is taken when we are traversing in the direction of the current and the emf is taken positive when the traverse from the negative to the positive electrode through the electrolyte.* In circuit ABDA in Fig. 8.14.

$$i_1 r_1 + i_2 r_2 + i_3 r_3 + ir = \xi \text{ and in circuit ABCA}$$

$$i_2 r_2 - i_5 r_5 - i_4 r_4 = 0$$

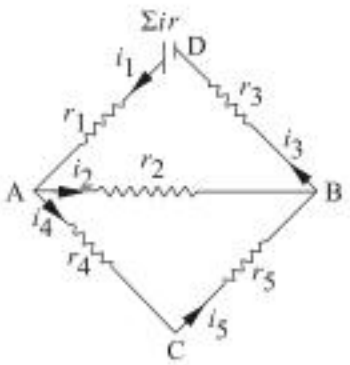


Fig. 8.14

8.18 APPLICATION OF KIRCHHOFF'S LAW TO WHEATSTONE'S BRIDGE

Fig. 8.15 represents the Wheatstone's bridge circuit where P, Q, R and S are connected to form a closed circuit. A cell is connected between the points A and C and a galvanometer is connected between the points B and D. The currents through the various branches are indicated in the figure. i_g is the current through galvanometer and G is its resistance. Applying the Kirchhoff's first law, we get, at the junction B,

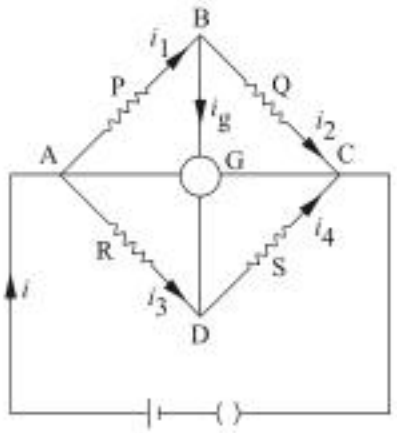


Fig. 8.15

$$i_1 - i_2 - i_g = 0 \qquad \ldots(1)$$

at the junction D,

$$i_3 + i_g - i_4 = 0 \qquad \ldots(2)$$

Now, applying Kirchhoff's second law to the meshes ABDA and ABCDA, we get,

$$i_1 \mathrm{P} + i_g \mathrm{G} - i_2 \mathrm{R} = 0 \qquad \ldots(3)$$

$$i_1 \mathrm{P} + i_2 \mathrm{Q} - i_4 \mathrm{S} - i_3 \mathrm{R} = 0 \qquad \ldots(4)$$

When the galvanometer shows, "zero deflection", the point B and D are at the same potential and $i_g = 0$. Hence, equations (1), (2) and (3) reduce to

$$i_1 = i_2 \qquad \ldots(5)$$

$$i_3 = i_4 \qquad \ldots(6)$$

$$i_1 \mathrm{P} = i_3 \mathrm{R} \qquad \ldots(7)$$

Putting the values of (5) and (6) in equaiton (4), we get,

$$i_1 \mathrm{P} + i_1 \mathrm{Q} - i_3 \mathrm{S} - i_3 \mathrm{R} = 0$$

or $$i_1 (\mathrm{P} + \mathrm{Q}) = i_3 (\mathrm{R} + \mathrm{S}) \qquad \ldots(8)$$

Now divide equation (8) by (7), we get

$$\frac{i_1(\mathrm{P}+\mathrm{Q})}{i_1 \mathrm{P}} = \frac{i_3(\mathrm{R}+\mathrm{S})}{i_3 \mathrm{R}}$$

or $$\frac{\mathrm{P}+\mathrm{Q}}{\mathrm{P}} = \frac{\mathrm{R}+\mathrm{S}}{\mathrm{R}}$$

or $$\frac{\mathrm{Q}}{\mathrm{P}} = \frac{\mathrm{S}}{\mathrm{R}}$$

or $$\frac{\mathrm{P}}{\mathrm{Q}} = \frac{\mathrm{R}}{\mathrm{S}}. \qquad \ldots(8.39)$$

Thus at balance point, *i.e., where the galvanometer shows zero deflection, we have,* $\frac{P}{Q} = \frac{R}{S}$. *We can find the value of an unknown resistance S in terms of known resistance P, Q and R.*

Galvanometer current i_g when the Wheatstone bridge is unbalanced. When the bridge is unbalanced, the galvanometer current i_g can be determined by Kirchhoff's law.

At the point A Kirchhoff's first law gives,

$$i - i_1 - i_3 = 0$$

or $i_3 = i - i_1$ ...(1)

At the point B, we get,

$$i_1 - i_2 - i_g = 0$$

or $i_2 = i_1 - i_g$ ...(2)

and at the point C, we have,

$$i_2 + i_4 - i = 0$$

or $i = i_2 + i_4 = i_1 - i_g + i_4$

or $i_4 = i - i_1 + i_g$ ...(3)

Applying Kirchhoff's second law to the mesh ABDA, we get,

$$i_1 \mathrm{P} + i_g G - i_3 \mathrm{R} = 0$$

Substituting i_3 from (1), we get,

$$i_1 \mathrm{P} + i_g \mathrm{G} - (i - i_1)\mathrm{R} = 0$$

or $i_1 (\mathrm{P} + \mathrm{R}) + i_g \mathrm{G} - i \mathrm{R} = 0$ ...(4)

From the mesh BCDB, we get,

$$i_2 \mathrm{Q} - i_4 \mathrm{S} - i_g \mathrm{G} = 0$$

Substituting i_2 and i_4 from (2) and (3) in the above equation, we get,

$$(i_1 - i_g)\mathrm{Q} - (i - i_1 + i_g)\mathrm{S} - i_g \mathrm{G} = 0$$

or $i_1 (\mathrm{Q} + \mathrm{S}) - i_g (\mathrm{Q} + \mathrm{S} + \mathrm{G}) - i\mathrm{S} = 0$ ...(5)

Multiplying equation (4) by (Q + S) and (5) by (P + R), then subtracting resulting (5) from resulting (4), we get,

$$i_g [\mathrm{G}(\mathrm{Q} + \mathrm{S}) + (\mathrm{P} + \mathrm{S})(\mathrm{Q} + \mathrm{S} + \mathrm{G})] - i [\mathrm{R}(\mathrm{Q} + \mathrm{S}) - \mathrm{S}(\mathrm{P} + \mathrm{R})] = 0$$

or $i_g [\mathrm{G}(\mathrm{Q} + \mathrm{S}) + (\mathrm{P} + \mathrm{R})(\mathrm{Q} + \mathrm{S} + \mathrm{G})] - i\ (\mathrm{QR} - \mathrm{SP}) = 0$

or $$i_g = i \frac{\mathrm{QR} - \mathrm{SP}}{\mathrm{G}(\mathrm{P} + \mathrm{Q} + \mathrm{R} + \mathrm{S}) + (\mathrm{Q} + \mathrm{S})(\mathrm{P} + \mathrm{R})}$$ ...(8.40)

From the above equation, we can also find the Wheatstone's bridge condition of balance by putting $i_g = 0$. Then,

$$\mathrm{QR} - \mathrm{SP} = 0$$

or $$\frac{\mathrm{P}}{\mathrm{Q}} = \frac{\mathrm{R}}{\mathrm{S}}$$

which is same as equation (8.39).

Example 8.9 *A battery of 6 volts emf and 0.5 ohm interval resistance is joined in parallel with another of 10 volts emf and 1 ohm internal resistance. The combination is used to send a current through an external resistance. The combination is used to send a current through an external resistance of 12 ohm. Calculate, by application of Kirchhoff's law, the current through each battery.*

Solution. Let the current through the batteries B_1 and B_2 be I_1 and I_2 respectively [Fig. 8.16]. Then applying Kirchhoff's first law to the point C, the current through the external resistance is $(I_1 + I_2)$.

Applying Kirchhoff's second law to the mesh ABFEA

$$I_1 \times 0.5 + (I_1 + I_2)\,12 = 6$$

or $$12.5\,I_1 + 12\,I_2 = 6$$

$$25\,I_1 + 24\,I_2 = 12 \qquad \ldots(1)$$

Similarly applying Kirchhoff's second law to the mesh CDFEC,

$$I_2 \times 1 + (I_1 + I_2)\,12 = 10$$

or $$12\,I_1 + 13\,I_2 = 10 \qquad \ldots(2)$$

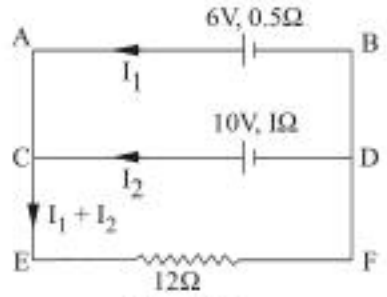


Fig. 8.16

Solving (1) and (2), we get

$$I_1 = -2.27 \text{ amp}$$

$$I_2 = 2.865 \text{ amp}.$$

The negative sign in I_1 indicates that the current I_1 in the battery B is opposite to the direction shown in Fig. 8.16.

Example 8.10 *12 wires, each of resistance r ohms, are connected in the form of a skeleton cube. Find the equivalent resistance of the cube when the current enters at one corner and leaves at the diagonally opposite corner.*

Solution. Let a current of 16 enter the point A and leave the point G at the opposite end of the diagonal.

The distribution of the current is shown in Fig. 8.17 as well as equivalent circuit and let R be the equivalent resistance.

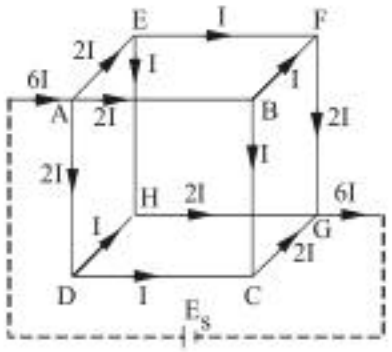


Fig. 8.17

Applying Kirchhoff's law to any path between AG, say ABFG, we get,

$$2Ir + Ir + 2Ir = V = 6IR$$

or $$5Ir = 6IR$$

or $$R = \frac{5}{6}r.$$

Example 8.11 *12 wires, each of resistance r ohms, are connected in the form of a skeleton cube. Find the equivalent resistance of the cube when a cell is joined across any one of the wires forming the cube.*

Solution. Applying Kirchhoff's law to the mesh DCGHD and ABCDA as shown in Fig. 8.18,

$$zr-(y-z)r-2(y-z)r-(y-z)r=0$$

and $xr-yr-zr-yr=0$

or $5z-4y=0$

or $z=\frac{4}{5}y$

and $x-2y-z=0$

or $x-2y-\frac{4}{5}y=0$

i.e., $y=\frac{5}{14}x.$

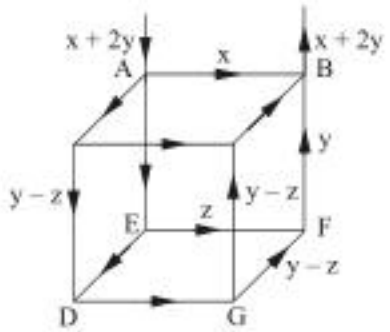


Fig. 8.18

Let the equivalent resistance be R.

Then,

P.D. across AB = $(x+2y)$ R = xr.

Therefore,

$$R=\frac{x}{x+2y}r=\frac{x}{x+\frac{10}{14}x}r=\frac{7}{12}r$$

Example 8.12 *What is the equivalent resistance between the terminal points x and y in the network shown in Fig. 8.19? Assume that the resistance of each resistor is 10 ohms..*

Solution. Assume the current distribution as shown in Fig. 8.19. Applying Kirchhoff's first law at C,

i.e., $(i-i_1)+(i_1-i_2)-(i-i_2)=0$

or $i-i_2=(i-i_1+i_1-i_2)$

This is the current along CF.

Now again applying the Kirchhoff's second law to the closed loops, ADCBA and CDEFC. we get,

$$i_1R+(i_1-i_2)R-(i-i_1)2R=0$$

or $4i_1-i_2=2i$ ...(1)

and $2i_2R-(i-i_2)R-(i_1-i_2)R=0$

or $-i_1+4i_2=i$ ...(2)

Solving (1) and (2), we get,

$$i_1=\frac{3}{5}i \text{ and } i_2=\frac{2}{5}i \quad \ldots(3)$$

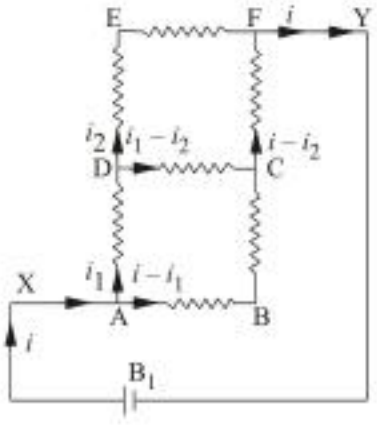


Fig. 8.19

Again applying second law to mesh XABCFYB$_1$X

$$(i-i_1)2R+(i-i_2)R=\xi$$

Substituting i_1 and i_2 from (3), we get

$$\left(i-\frac{3}{5}i\right)2R+\left(i-\frac{2}{5}i\right)R=\xi$$

or $$\frac{7}{5} iR = \xi$$

If r is the equivalent resistance between the terminal points x and y, then

$$\xi = ir$$

Hence, $$r = \frac{7}{5} R.$$

As given $R = 10$, hence

$$r = \frac{7}{5} \times 10 = 14 \text{ ohms}.$$

Example 8.13 *A source of emf ξ volts and internal resistance r ohms, supplies current to a heating coil. Calculate the resistance R of the heading coil in terms of r so that,*

(1) heating takes place most rapidly,

(2) 3/4 of the total energy delivered by the source goes into the heating coil.

Solution. The current through the heating coil

$$i = \frac{\xi}{R + r} \qquad \ldots(1)$$

The rate of energy delivered by the source to the heating coil,

$$W = i^2 R = \left(\frac{\xi}{R+r}\right) R = \frac{\xi^2 R}{(R+r)} \qquad \ldots(2)$$

(1) As we have discussed in equation (8.16), for maximum power, the condition is that $R = r$.

(2) The rate of total energy delivered by the source,

$$= \xi i = \frac{\xi^2}{R + r}$$

As given, $$\frac{\xi^2 R}{(R + r)^2} = \frac{3}{4} \frac{\xi^2}{R + r}$$

hence, after simplification, we get, $R = 3r$.

8.19 SENSITIVITY OF WHEATSTONE'S BRIDGE

The condition for the balance of Wheatstone's bridge is P/Q = R/S. This corresponds to zero current in the galvanometer. Now, if the birdge is slightly out of balance, there will be a current in the galvanometer, but the current is too feeble, hence cannot be detected easily so that the bridge may not be very sensitive. Thus, the sensitivities of the bridge depends on the sensitivity of the galvanometer.

In Wheatstone bridge, if we interchange the galvanometer and battery, the condition for the null point will not change, however, the two arrangements will not have the same sensitivity. The following discussion about the sensitiveness is due to H.A. Callendar.

The unknown resistance r is connected in the arm CD and i be the current in it. Let the resistance in the arm AB, BC, DA and the galvanometer be mnr_1, mr_1, nr_1 and G respectively. Let the current in the galvanometer be i_g and in the arm AB i_1, then the current in the arm AD will be $i + i_g$ and in the arm BC will be $i + i_g$ as shown in Fig. 8.20. The bridge balance condition is according to equation (8.39),

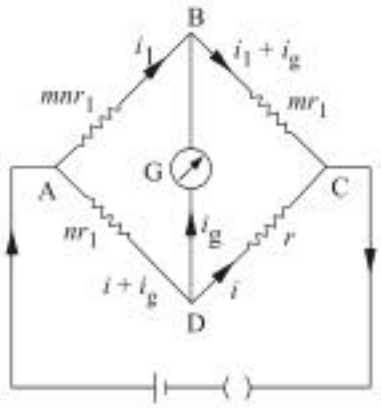


Fig. 8.20

$$\frac{mnr_1}{mr_1} = \frac{nr_1}{r}$$

or $r_1 = r$

Thus the bridge conditional is satisfied if $r_1 = r$ or in other words, if $r = r_1$, the bridge is balanced. Hence, *the difference $r - 1$ is the measure of want of balance i.e., the greater the difference, the bridge is far from the balance.* However, a bridge will be most sensitive, if the deflection in the galvanometer is large, even when $r - r_2$ is extremely small. Applying Kirchhoff's second law to the circuits ABDA and DCBD,

$$i_1 mnr_1 - i_g \, G - (i + i_g) nr_1 = 0$$

and $ir - (i_1 + i_g)\, mr_1 - i_g \, G = 0.$

On rewriting the above equations,

$$i_1 mnr_1 - i_g (G + nr_1) - inr_1 = 0 \qquad \ldots(1)$$

and $$-i_1 mr_1 - i_g (G + mr_1) + ir = 0 \qquad \ldots(2)$$

Multiplying equation (2) by n and adding the resulting equation to (1), we get

$$in\,(r - r_1) - i_g [n\,(G + mr_1) + (G + nr_1)] = 0$$

or $$\frac{i_g}{i} = \frac{n\,(r - r_1)}{(1+n)\,G + n\,(1+m)\,r_1}$$

or $$\frac{i_g}{i} = \frac{r - r_1}{\left(1 + \frac{1}{n}\right) G + (1+m)\,r_1} \qquad \ldots(8.41)$$

The ratio i_g/i depends on the want of balance $r - r_1$, the galvanometer resistance G and the ratio m and n. This ratio is independent of the resistance of the battery.

For greater sensitiveness for a given want of balance, $r - r_1$, the value of i_g should be large, therefore,

(*i*) i should be large,
(*ii*) G shoule be small
(*iii*) n should be large
(*iv*) m should be small.

For increasing the sensitivity the current i should be large. But the current i cannot be increased indefinitely as this will heat up the resistance and consequently alter their resistance. Further, the sensitivity increases with decrease in the resistance of the galvanometer, but this decrease in the galvanometer resistance is achieved by decreasing the number of turns in the coil of the galganometer and which *decrease the sensitivity of the galvanometer itself.*

Therefore, we have to increase sensitivity by increasing n and decreasing m. In the limiting case, if $n = \infty$ and $m = 0$,

$$\frac{i_g}{i} = \frac{r - r_1}{G + r_1} \qquad \ldots(3)$$

The condition gives the "*ideal sensitivity*" which is *impracticable* because (1) if $n = \infty$, the resistance of the arms AB and AD becomes infinite and (2) if $m = 0$; the points A and C are short circuited. However, making n very large and m very small has no great advantage, because even if $n = m = 1$, the sensitivity from equation (8.41) is

$$\frac{i_g}{i} = \frac{r - r_1}{2(G + r_1)} \qquad \ldots(4)$$

This value is still half of the value in the ideal limiting case given in equation (3).

Hence, the necessary condition is that $n > 1$ and $m < 1$ *i.e.* n should not be vary small or m too large because in either case the sensitivity will be very much reduced. Thus *the Callendar rule is that the sentivities of the bridge will be higher if the resistance in series with the unknown resistance is greater than the resistance connected in parallel to it.*

It can be easily shown that for greater sensitivity, the galvanometer should have the same order of the resistar as the unknown resistance.

Another rule for greater sensitiveness is the "greater of the two resistances, the galvanometer or the battery should be connected between the junctions of the two highest and two lowest resistances".

8.20 MAXWELL'S CYCLIC CURRENTS

Equations obtained by the application of Kirchhoff's law are sometimes very tedious to solve in a complicated network. Maxwell suggested a simple device, called cyclic currents method, which simplifies the solution of problems. According to this method, a certain magnitude of currents is assumed to flow in each mesh, all the cyclic currents being in the same direction *i.e.* either clockwise or anti-clockwise. The actual current flowing in any branch will then be the difference between the cyclic currents of the meshes, if separate.

8.21(A) APPLICATIONS OF MAXWELL'S CYCLIC CURRENT METHOD

1. Wheatstone's Bridge

Let x, y and z be the currents flowing in the clockwise direction through the meshes ABDA, BCDB and ADCA as shown in Fig. 8.21. Applying Kirchhoff's second law, respectively we get,

$$xP + (x - y)\,R_g + (x - z)\,R = 0 \qquad \ldots(1)$$

$$yQ + (y - z)\,S + (y - x)\,R_g = 0 \qquad \ldots(2)$$

$$(z - x)\,R + (z - y)\,S + zr = 0 \qquad \ldots(3)$$

Collecting the coefficient of x, y and z in equation (1) and (2), we have,

$$x(P + R + R_g) - yR_g - zR = 0$$

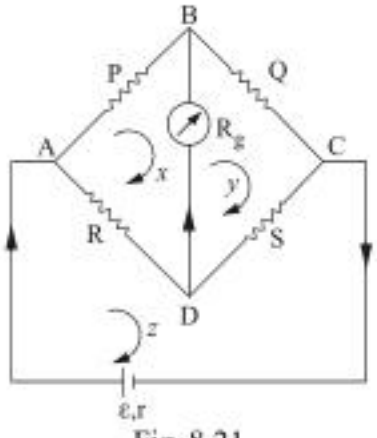


Fig. 8.21

and $\quad -xR_g + y\,(Q + S + R_g) - zS = 0$

Multiplying the first of these equations by S and second by R and subtracting, we get,

$$x\left[S\,(P + R + R_g) + RR_g\right] - y\left[SR_g + R\,(Q + S + R_g)\right] = 0$$

The current in the galvanometer = $x - y$. At the balance point this is zero, i.e. $x - y = 0$

or $\quad x = y$

Hence, we get,

$$S\,(P + R + R_g) + RR_g = SR_g + R\,(Q + S + R_g)$$

or $\quad SP = RQ$

hence, $\quad \dfrac{P}{Q} = \dfrac{R}{S}$

which is the required condition for balanced Wheatstone's bridge given in equation (8.39).

2. Kelvin's Method for Determining the Resistance of a Galvanometer

In this method the galvanometer is put in the unknown arm of the post officebox which is a Wheatstone bridge. The points B and D are connected by a tapping key K. Let the clockwise cyclic currents through the meshes, (1) ABDA, (2) BCDB and (3) ABCEA be x, y and z respectively as shown in Fig. 8.22.

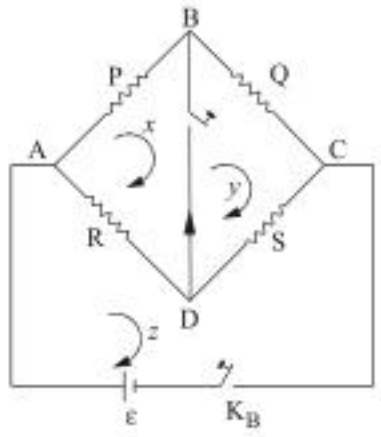


Fig. 8.22

Let P, Q, R, S, G and B be the resistance of the arms AB, BC, AD, DC, BD and AC respectively. Applying Kirchhoff's second law to the meshes ABDA and BCDB, we get,

$$xP + (x - y)\,G + (x - z)\,R = 0 \qquad \ldots(1)$$

$$yQ + (y - z)\,S + (y - x)\,G = 0 \qquad \ldots(2)$$

The values of resistance P, Q and R are adjusted such that there is no change in the deflection of the galvanometer when the key K_B is opened or closed.

This means that current in this arm must be zero *i.e.*, $x - y = 0$ or $x = y$,

Substituting in equation (1) and (2), we get,

$$yP + (x - z)\,R = 0$$

$$yQ + (y - z)\,S = 0$$

or $\quad x\,(P + R) = zR$

and $\quad y\,(Q + S) = zS$

or ratio, $\quad \dfrac{x\,(P + R)}{y\,(Q + S)} = \dfrac{R}{S}$

or $\quad \dfrac{P + R}{Q + S} = \dfrac{R}{S}$ since $x = y$

$$PS + RS = RQ + RS$$

or $\quad S = R\,\dfrac{Q}{P}$ $\qquad \ldots(8.42)$

Thus from the known values P, Q and R, S can be calculated.

Adjustement

(1) When the battery key K_B is pressed the galvanometer will always show a deflection as it is in one of the arms of the bridge. Generally, the deflection is out of scale, it may be reduced and brough within scale by connecting a high resistance box in the battery circuit.

(2) When the tapping key K_B is pressed the galvanometer deflection will change as some of the current will also flow in BD arm. P/Q is a fixed ratio for one observation, hence R is adjusted such that the galvanometer deflection remains the same before and after pressing key K_B. *In fact this method is a constant deflection method and not to be cnfused with null method.* This method was first employed by Kelvin hence it is called Kelvin's method.

8.21 (b) Mance's method of measuring the internal resistance of a Cell

Wheatstone's bridge circuit has been employed by Mance's to determine the internal resistance of the cell as shown in Fig. 8.23. The cell is connected in the unknown arm of the bridge and usual arm of the cell is connected by key K_B. Let P, Q, R, r and G be the resistance of the arms AB, BC, DA, CD, and BD respectively.

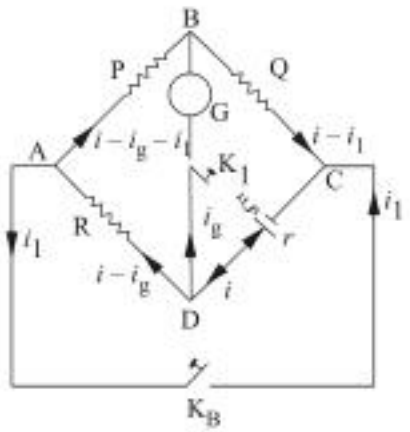


Fig. 8.23

Applying Kirchhoff's second law to the meshes ABDA and BCDB, we get

$$(i-i_g-i_1)\,P-i_g\,G+(i-i_g)\,R=0 \qquad \ldots(1)$$

$$ir+i_g\,G+(i-i_1)\,Q=\xi \qquad \ldots(2)$$

Collecting the coefficients of i, i_g and i_1, we get,

$$i\,(P+R)-i_g\,(P+G+R)-i_1\,P=0 \qquad \ldots(3)$$

$$i\,(r+Q)+i_g\,G-i_1\,Q=\xi \qquad \ldots(4)$$

Multiplying (3) by $(r+Q)$ and (5) by $(P+R)$, we get

$$i\,(P+R)\,(r+Q)-i_g\,(P+G+R)(r+Q)-i_1\,P(r+Q)=0 \qquad \ldots(5)$$

$$i\,(r+Q)\,(P+R)+i_g\,G\,(P+R)-i_1\,Q(P+R)=\xi\,(P+R) \qquad \ldots(6)$$

Subtracting (5) from (6),

$$i_g\left[G\,(P+R)+(P+G+R)(r+Q)\right]-i_1\left\{Q(P+R)-P(r+Q)\right\}=\xi\,(P+R) \qquad \ldots(7)$$

Adjustment

The values of P, Q and R are adjusted as in the case of Kelvin's method (the procedure is exactly the same) such that the galvanometer shows a "constant deflection" when the key K_1 is pressed or opened. That is the current in the galvanometer is constant. The equation (7) shows that i_g will remain constant whatever be the value of current i_1 if the coefficient of i_1 is zero.

$$Q\,(P+R)-P\,(r+Q)=0$$

or $$Q\,(P+R)=P\,(r+Q)$$

$$PQ+QR=Pr+PQ$$

$$r=\frac{Q}{P}R \qquad \ldots(8.43)$$

Hence r the internal resistance of the cell can be calculated by knowing P, Q and R.

8.22 KELVIN'S DOUBLE BRIDGE FOR LOW RESISTANCE

The Wheatstone's bridge is not suitable for comparing two very low resistance such as metal rode because the junction resistances and resistances of connecting wires are larger comparted to the low resistance to be measured. For accurate determination of low resistance, Kelvin devised double bridge as shown in Fig. 8.24

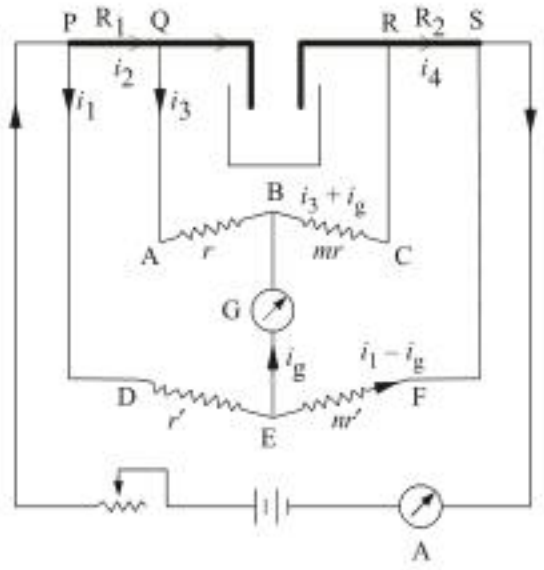


Fig. 8.24

PQ and RS are two low resistance of the same order of magnitude such as the *thick conducting rods of uniform cross-section*. One of which may be the known variable standard say PQ resistance and the other (RS) is unknown resistance. The rods PQ and RS are connected in series through a mercury cup. The bridge is completed by two resistance coils AC and DF whose middle points are at B and E which are connected to the rods PQ and RS as shown in Fig. 8.24.

Let the resistance coil ABC be divided at B in the ratio $r : mr$ and the coil DEF in the ratio $r' : nr'$. Let the currents in various conductors be those indicated in the figures. If i' be the current between Q and R then at the junction Q the Kirchhoff's current law gives,

$$i_2 - i_3 - i' = 0$$

or $$i' = i_2 - i_3 \qquad \ldots(1)$$

and at the junction R, we get,

$$i' + (i_3 + i_g) - i_4 = 0$$

or $$i' = i_4 - (i_3 + i_g) \qquad \ldots(2)$$

Equating (1) and (2),

$$i_2 - i_3 = i_4 - (i_3 + i_g)$$

or $$i_g = i_4 - i_3 \qquad \ldots(3)$$

Applying Kirchhoff's law to the meshes PQABEDP and RSFEBCR, we get,

$$i_2 \, R_1 + i_3 \, r - i_g \, G - i_1 \, r' = 0 \qquad \ldots(4)$$

$$i_4 \, R_2 - (i_1 - i_g) \, nr' + i_g \, G + (i_3 + i_g) \, mr = 0 \qquad \ldots(5)$$

The point P of the variable resistance which is in the form of knife edge so that it can slide along the rod is so adjusted that there is no deflection in the galvanometer *i.e.* when $i_g = 0$, then equation (4) and (5) becomes,

$$i_2 R_1 + i_3 r - i_1 r' = 0 \quad \ldots(6)$$

$$i_4 R_2 - i_1 nr' + i_3 mr = 0 \quad \ldots(7)$$

But from equation (3), when $i_g = 0$

$$i_4 = i_2$$

therefore,

$$i_2 R_1 + i_3 r - i_1 r' = 0$$

$$i_2 R_2 - i_1 nr' + i_3 mr = 0$$

or $$i_2 R_1 = i_1 r' - i_3 r$$

$$i_2 R_2 = i_1 nr' - i_3 mr$$

or $$\frac{R_2}{R_1} = \frac{i_1 nr' - i_3 mr}{i_1 r' - i_3 r} \quad \ldots(8.44)$$

If the points B and E divide the two coils ABC and DEF in the same ratio, *i.e.* when $m = n$, then,

$$\frac{R_1}{R_1} = m = n \quad \ldots(8.45)$$

Usually, we make,

$$m = n = 1$$

then, $R_2 = R_1$.

Thus, by suitably adjusting the variable point P of the known resistance R_1 for zero deflection in the galvanometer, we get directly the unknown resistance R_2. This bridge is suitable to measure resistances from 0.1 to 0.003 ohm.

8.23 POTENTIOMETER

A potentiometer is a device used to measure potential difference. The device is simply to compare an unknown emf with a known one. It consists of a uniform wire AB of resistance R. A steady current is passed through the wire AB by a constant source of emf ξ through the rheostat Rh. ξ_1 is emf of the cell which is to be measured and is connected in the circuit as shown in Fig. 8.25. When the current is steady and constant, the potential gradient is constant.

Let resistance per unit length of the wire AB be ρ and the steady current passing through the wire be I. Then the potential gradient that is the fall of potential per unit length of the wire is $I\rho$. Connect ξ_1 by K and move the sliding contact till the galvanometer gives no deflection. If l_1 is the length of the potentiometer wire for balancing ξ_1, then

$$\xi_1 = I_\rho l_1$$

Similarly connected ξ_2 by K and repeat the process, now if l_2 is the length of the potentiometer wire for balancing ξ_2, then,

$$\xi_2 = I_\rho l_2 .$$

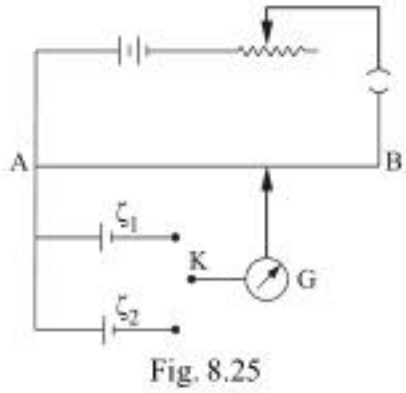


Fig. 8.25

Hence ratio,

$$\frac{\xi_1}{\xi_2} = \frac{I_\rho l_2}{I_\rho l_2} = \frac{l_1}{l_2} \qquad \ldots(8.46)$$

If one of the cells say ξ_2 is known (a standard cell) the emf of the other can be known. The equation (8.46) shows that for a steady current passing through the potentiometer wire, the potential difference across any length is proportional to the length of the wire.

Sensitiveness of a Patentiometer

In potentiometer, *the accuracy increases as the length of the wire is increased. But a potentiometer of greater length becomes unwieldy.* This difficulty is overcome by using compact forms of potentiometer.

8.24 TYPES OF POTENTIOMETERS

Many different types of compact forms have been designed. The principle of working of all types of potentiometer is the same.

The commonly used potentiometers are: (*i*) Crompton, (*ii*) Student and (*iii*) Rayleigh potentiometer.

Crompton Potentiometer

Since it is difficult to get a very long, uniform homogeneous wire, hence it is divided equal into fifteen segments. One of the wires is stretched on a wooden board and rest of the fourteen segments are replaced by the coils connected in series with the stretched wire, the resistance of each of coil is equal to the wire as shown in Fig. 8.26.

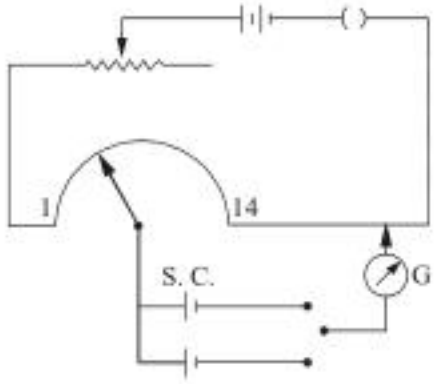


Fig. 8.26

Student Potentiometer

This type of potentiometer was designed by Dr. N.K. Sethi. In this potentiometer, a platinum wire of length 50 cm and of uniform cross-section is used. Its resistance is 1 ohm. There are ten coils each of one ohm resistance put in series with the wire and on the scale each cm. is indicated as two.

Standardisation of potentiometer of direct reading potentiometer

A standard cell (cadminum cell) is connected as shown in Fig. 8.26 the resistance is adjusted to 1018.3 ohm and the variable resistance (rheostat) so adjusted as to give no deflection with the standard cadmium cell in circuit. The current will be given by,

$$\frac{1.0183}{1018.3} = \times 10^{-3} \text{ amp}$$

where 1.0183 is the emf of the cadmium cell in volt. Now if the resistance with AB = 1 ohm as in the case of student potentiometer, then the potential difference per mm of the scale,

$$= 10^{-3} \times 0.001 \text{ volt/mm}$$

= 1 micro volt/mm = 1 milli. volt/cm

When the wire is calibrated to read one millivolt per cm the potentiometer will read the current in milliamperes per cm. *Thus, we can read current and voltage directly on the scale and such device is also called "direct reading" potentiometer.*

8.25 APPLICATIONS OR POTENTIOMETER

Some important uses of potentiometer are discussed below:

1. Calibration of an ammeter and voltmeter

The principle of standardisation is employed for the calibration of an ammeter and voltmeter. Connections are made as shown in Fig. 8.27.

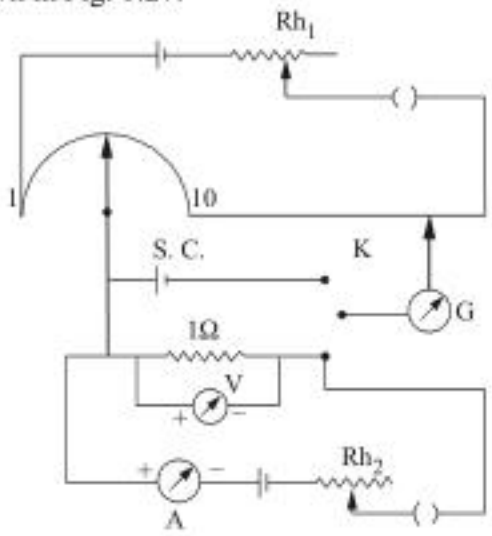


Fig. 8.27

First the potentiometer wire is calibrated in milli-volt by obtaining the balance point with the standard cell of emf 1.0183 volts at 1018.3 cm. The balance point is obtained by varying Rh_1. Next with the help of Rh_2, the current in the ammeter A is adjusted and corresponding to this value of current the balance point on the potentiometer is noted. Then the current is calculated *i.e.* directly the reading is current in amp, when multiplied by 10^{-3}. Similarly the voltmeter gives the voltage. The experimeter is repeated for various readings of the ammeter and voltmeter. A calibration graph between the observed and calculated values is drawn.

2. Measurement of very small thermo emf

The emf of the order of 30 or 40 mill-volts *e.g.* thermo emf can be measured with potentiometer. First the potentiometer wire is calibrated in milli-volts with standard cell as discussed above. The connections are made as shown in Fig. 8.28.

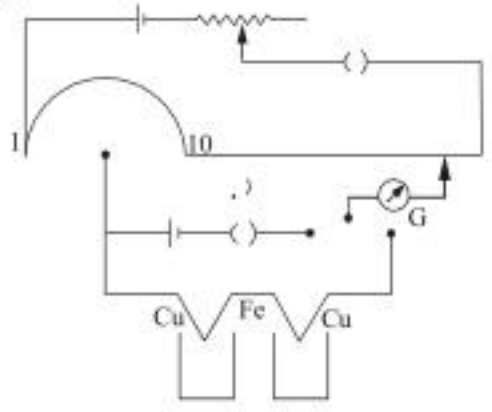


Fig. 8.28

Let us take copper-iron couple to produce the thermo emf. As the potentiometer wire is calibrated *i.e.* the *p.d.* per mm of the scale = $10^{-3} \times 0.001$ volt = 1 microvolt/mm = 1milli-volt/cm. We can read directly the termo-emf generated at various temperatures.

3. Measurement of large potential differecne

Though the potentiometer is beast suited for measuring the potential difference of the order of less than 2 volts, we can measure large potential difference by dividing it into fractions by volt box. A volt box consistas of a very large resistance, about 10,000 ohms. The resistance is divided in fractions by intermediate tappings. The large potential to be measured is applied to this resistance as shown in Fig. 8.29.

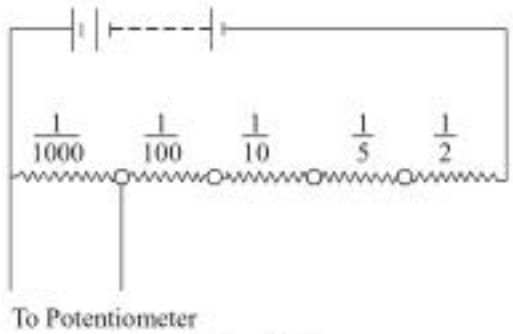


Fig. 8.29

Take any fraction, say 1/100 and balance it against the potentiometer wire and the p.d. for this fractin is known, then multiply it by 100 to obtain the requisite high potential.

4. Determination of Internal Resistance of a Cell

The connections are shown in Fig. 8.30.

First the key K is open and the emf, ξ of the cell is balanced, say l_1 is the balancing length of the potentiometer wire. Now close the key K and introduce the resistance in the box R and the potential drop V = IR is balanced against a length l_2 of the potentiometer wire. If r is the internal resistance of the cell, then,

$$\frac{\xi}{V} = \frac{I(R+r)}{IR} = \frac{R+r}{r} = \frac{l_1}{l_2}$$

or $$r = \frac{l_1 - l_2}{l_2} - R \qquad \ldots(8.47)$$

Hence r can be calculated.

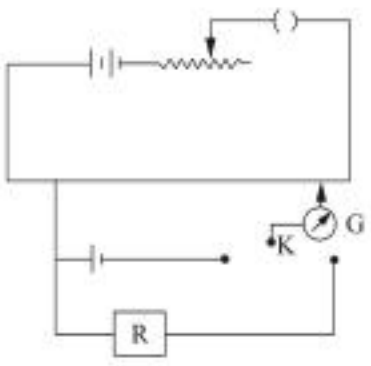


Fig. 8.30

5. Comparison of two low resistance

Two low resistance of the same order can be compared using a potentiometer. The conections are made as shown in Fig. 8.31. As the same current is flowing through R_1 and R_2, then potential drops are IR_1 and IR_2 respectively. If the balancing lengths on the potentiometer wire are l_1 and l_2 respectively, then,

$$\frac{IR_1}{IR_2} = \frac{R_1}{R_2} = \frac{l_1}{l_2} \qquad \ldots(8.48)$$

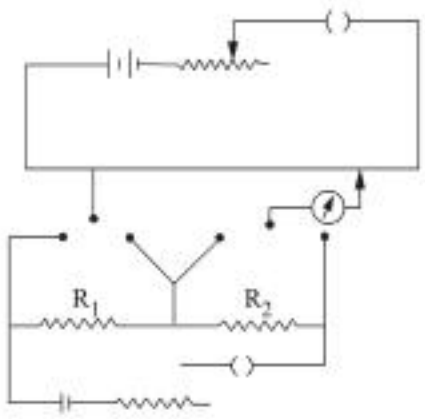


Fig. 8.31

6. Determination of a given low resistance

If in the aboev experiment one of the resistance (R_2) is standard resistance then the other (R_1) can be calcualted by relation

$$R_1 = R_2 \frac{l_1}{l_2} \qquad \ldots(8.49)$$

REVIEW QUESTIONS

8.1 Define emf and potential difference. What is the difference between an emf and potential difference?

8.2 What do you mean by electric current and current density? How are they related to each other? Establish the relation between current density and the velocity of charge carrier.

8.3 What is the conservation of charge rule? Show that

$$\nabla . \vec{J} + \frac{\delta p}{\delta t} = 0$$

8.4 Defind (a) resistance, (b) resistivity, (c) conductivity. Establish the relation $\vec{J} = \sigma \vec{E}$.

8.5 What is the relation between conductivity and permittivity? Show that R = e/sc.

8.6 State and prove Kirchhoff's laws of distribution of currents in an electrical network and apply these laws to deduce the condition of
(a) Balance of Wheatstone's bridge.
(b) Maximum sensitivity of Wheatstone Bridge.

8.7 Explain the Maxwell's cyclic current method. Apply it to obtain the Wheatstone's bridge condition.

8.8 Give Kelvin's method for measuring the galvanometer resistance.

8.9 Describe how you will measure the interval resistance of a cell using Wheatstone's bridge.

8.10 Why is Wheatstone's bridge unsuitable for measurement of very low resistance? Describe Kelvin's double Bridge method for determining a low resistance.

8.11 Give the principle of a potentiometer.

8.12 Describe the constructin of a Crompton potentiometer. How is it used for measuring
(1) Large potential difference
(2) Very small thermo emf
(3) Low resistance.

PROBLEMS

8.1 A current of 1 amp is passing through a copper wire 10 meter in length and 0.08 mm in diameter. If the resistivity of copper is 1.65×10^{-8} ohm-meter, calculate

(1) the resistance of copper wire; and

(2) the potential difference between the ends of the wire.

8.2 In copper there are 10^{23} free electrons per cc. all of which contribute to a current of 1 amp in a wire of copper of 0.01 cm^2 cross-section area.

(1) What is the average drift speed of electrons in the copper?

(2) What is the electric field in the wire?

Given specific resistance of copper = 1.6×10^{-8} ohm. meter.

8.3 In a cloud of electrons having 6×10^{29} electrons per $meter^3$, 40 percent of the electrons have a drift velocity 1 mm per second in x-direction, 40 per cent have drift velocity of 2 mm per second in y-direction and remaining 20 per cent have have drift velocity of 3 mm per second in z-direction. Calculate the current density $\bar{j}$.

8.4 (a) What is the electric field strength in a copper conductor of cross-sectional area 1 cm^2 when a current of 200 amp. flows through it? The resistivity of copper is 1.7×10^{-8} ohm-meter.

(b) Also calculate the potential difference between point 2 metres apart along the length of the above conductor.

8.5 An aluminium wire of diameter 0.10 cm is welded end to end to a copper wire of diameter 0.064 cm. The composite wire carries a steady current of 10 amp. What is the current density in each wire? What is the drift velocity of electrons in copper wire? Given that the density of copper is 9.0 gms/cc and Avagadro's number is 6×10^{23} atoms/mol.

8.6 Referring to Fig. 8.32, calculate the currents in the two resistances.

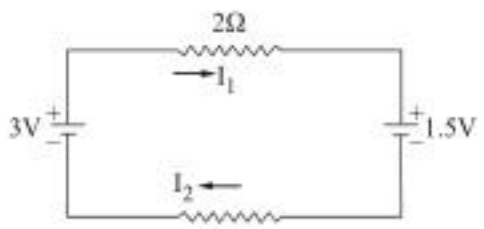


Fig. 8.32

8.7 In the network shown in Fig. 8.33, obtain the total resistance between the terminals A and B.

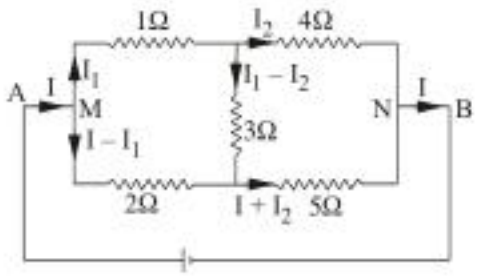


Fig. 8.33

8.8 A network is shown in Fig. 8.34. Calculate the value of resistance X in terms of r and R. So that the total resistance between A and B may also be equal to X.

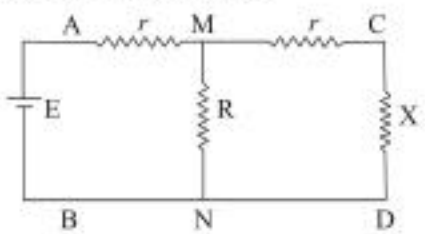


Fig. 8.34

8.9 Determine the loops current I_1 and I_2 and then calculate the voltage drop across 3 ohm resistor in the network of Fig. 8.35.

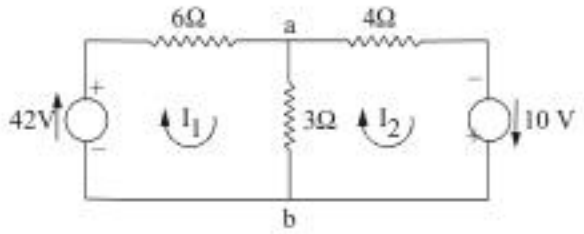


Fig. 8.35

8.10 Referring to Fig. 8.36, calculate the current I_1 and I_2.

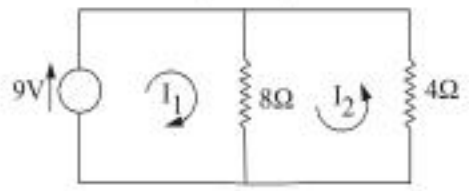


Fig. 8.36

8.11 (a) Let *n* resistances R_1, R_2...R_n be connected in series. Show that the equivalent resistance R is given by,

$$R = \sum_{i=1}^{n} R_i$$

(b) Let n resistances R_1, R_2...R_n be connected in parallel. Show that the equivalent resistance R is given by ,

$$\frac{1}{R} = \sum_{i=1}^{n} \frac{1}{R_i}$$

ANSWERS

1. 32.85 ohm, 32.85 volts **2.** 6.25×10^{-4} m/sec; 1.6×10^{-2} volt/metre

3. $\vec{J} = -1.92 \times 10^7 (2\hat{i} + 4\hat{j} + 3\hat{k})$ ap/metre^2

4. (a) 3.4×10^{-2} volt/metre (b) 6.8×10^{-2} volts

5. 127.27×10^5 amps/met^2, 310.73×10^5 amp/metre^2, 23.03×10^{-4} metre/sec.

6. $I_1 = I_2 = 0.3$ amp. **7.** 2.90 ohms **8.** $X = (r^2 + 2r\,R)^{1/2}$

9. $I_1 = 6$ amp; $I_2 = 4$ amp; 6 volts **10.** $I_1 = -1$ amp and $I_2 = 2$ amp

SELF-TEST OBJECTIVE QUESTIONS

8.1 In cells

(*a*) Chemical energy changes into mechnical

(*b*) Chemical energy changes into heat

(*c*) Chemical energy changes into electricity

(*d*) none of these

8.2 E.M.F. of a cell is always greater than P.D.

(a) True (b) False

8.3 If there is a regular movement of .....within a material, we say that an electric current is flowing
(*a*) atoms (*b*) ions
(*c*) electrons (*d*) protons

8.4 $\bar{J} = \sigma\bar{E}$ is
(*a*) A Maxwell's equation (*b*) Continuity equation
(*c*) Ohm's law (*d*) Ampere's law

8.5 Ohm's law does not apply to
(*a*) a.c. circuits (*b*) conductors (*c*) semi-conductors
(*d*) Conductors when there is a change in temperature

8.6 The ampere represents a specific amount of electrons passing given point in a circuit in a specific.....
(*a*) Direction (*b*) quantity
(*c*) time (*d*) quantity and time

8.7 The sensitivty of a Wheatstone bridge is maximum when
(*a*) P/Q = R/X (*b*) R = X, P = Q
(*c*) P = Q = R = X (*d*) none of these

8.8 If you were to double the length of a wire, the resistance would
(*a*) half (*b*) remain the same
(*c*) double (*d*) depend on the voltage

8.9 Electrical conductivity is the reciprocal of
(*a*) conductance (*b*) remanence
(*c*) resistivity (*d*) permittivity

8.10 The unit of conductance is
(*a*) Henry (*b*) Ohm
(*c*) Mho (*d*) Farad

8.11 The reciprocal of the resistance is termed
(*a*) admittance (*b*) Suscepectance
(*c*) Conductance (*d*) Conductivity

8.12 A potentiometer works on the principle that an unknown.....is measured by balancing it, wholly or in part against a known.....
(*a*) e.m.f. (*b*) current
(*c*) resistance (*d*) reactance

8.13 While using a Wheatstone bridge
(*a*) both the keys should be pressed simultaneously
(*b*) battery key should be pressed first
(*c*) galvanometer key should be pressed first
(*d*) none of these

8.14 The low resistance can be most accurately measured by
(*a*) The Wheatstone bridge method (*b*) Kelvin's bridge method
(*c*) Potentiometer (*d*) None of the above

8.15 For steady state continuity equation is

(*a*) $\bar{\nabla} \cdot \bar{J} = 0$ (*b*) $\nabla \bar{J} = -\frac{d\rho}{dt}$

(*c*) $\nabla \bar{J} = 0$ (*d*) $\nabla \cdot \bar{J} = \frac{d\rho}{dt}$

8.16 Kirchhoff's laws are applicable to
(*a*) D.C. only (*b*) A.C. only
(*c*) A.C. and D.C. both (*d*) none of the above

8.17 Closed circuit technique are based on
(*a*) Thevenin theorem (*b*) Norton theorem
(*c*) Kirchhoff's current law (*d*) Kirchhoff's voltage law

ANSWERS

1. (c)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (c)
7. (c)	**8.** (c)	**9.** (c)	**10.** (c)	**11.** (c)	**12.** (a)
13. (b)	**14.** (b)	**15.** (a)	**16.** (c)	**17.** (d)	

CHAPTER 9

MAGNETOSTATICS

9.1 THE MAGNETIC FIELD *i.e.* MAGNETIC EFFECT OF THE CURRENT

Let us now consider the magenetic effect of the current. Before the discovery of this effect the electricity and magnetism were treated as separate branches. But in 1820 Orested discovered that a current in a wire can also produce magnetic effect. It was demonstrated that when a compass needle is brought near a wire carrying current, then the compass needle shows deflection.

We define the space around a magnet or a current carrying conductor as the site of a magnetic field, just as we define an electric field. The basic magnetic field vector $\overline{B}$ is called magnetic induction. It is represented by lines of induction. $\overline{B}$ is related to its ines of induction as follows:

1. The tangent to a line of induction at any point gives the direction of $\overline{B}$ at that point.
2. The lines of induction are drawn so that the number of lines per unit cross-sectional area is proportional to the magnitude of the magnetic field vector $\overline{B}$ where the liens are close together, B is strong and where they are far apart, B is weak.

9.2 THE DEFINITION OF $\overline{B}$

We have defined electric field $\overline{E}$ in terms of force experienced by a test charge q_0, which depend on the situation of the charge q_0. The fact that the force on an electric charge depends not only on its situation, but also on the velocity of the charge is considered for defining $\overline{B}$. A similar approach would be to define the field $\overline{B}$ of a moving charge in terms of force experienced by it.

Let us bring a positive test charge with velocity v at a point P situated in magnetic field. The test charge experiences a deflecting force as shown in Fig. 9.1

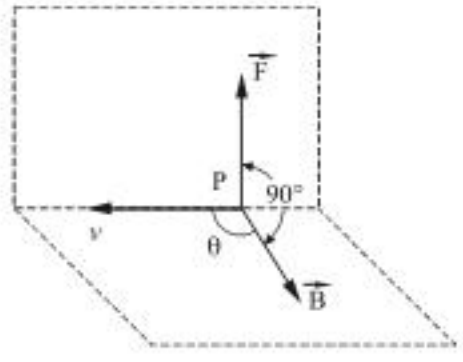


Fig. 9.1

The magnetic induction or the magnetic field vector $\overline{B}$ is defined in terms of this force and other measurable quantities. This magnetic force has a strange directional character. At any particular point in space, both the direction fo the force and its magnitude depend on the direction of the motion of the

particle. At every instant the force is always at right angles to the velocity vector, also at any particular point, the force is always at right angles to a fixed direction in space, and finally the magnitude of the force is proportional to the component of the velocity at right angles to this unique direction. It is possible to describe all these behaviour by defining the magnetic field vector $\vec{B}$ as follows.

If we vary the direction of $\vec{v}$ through point P, keeping the magnitude of $\vec{v}$ unchanged, though $\vec{F}$ will always remain at right angles to $\vec{v}$, its magnitude F changes and for particular orientation the force becomes zero. We defined this direction as the direction of $\vec{B}$.

Now, if the test charge is again oriented such that it moves at right angles to this direction of $\vec{B}$ for which force is zero, then the magnitude of force is maximum, say this force is denoted by $F\perp$ and we define the magnitude of $\vec{B}$ from the measured magnitude of this maximum force $F\perp$, *i.e.*,

$$B = \frac{F\perp}{q_0 v} \qquad \ldots(9.1)$$

or in general,

$$\vec{F} \times q_0 \vec{v} \times \vec{B} \qquad \ldots(9.2)$$

This equation is known as the Lorentz force law. Because for relation (9.1), $\vec{B}$ and $\vec{v}$ are right angles, the magnitude of deflection force,

$$F = q_0 \, v \, B \sin\theta \qquad \ldots(9.3)$$

$\vec{F}$ is right angles to the plane formed by $\vec{v}$ and $\vec{B}$, will always be right angles to $\vec{v}$ and also $\vec{B}$ and thus always a side ways force.

Unit

The RMKS unit of $\vec{B}$ from equation (9.2) is

$$= \frac{\text{Newtons}}{\text{Coul-metre/sec}} = \frac{\text{Newtons}}{\text{ampere.metre}}$$

Since, coul/sec. = ampere.

or as will be shown later, the weber per square metre.

Work done in moving a test charge in a static magnetic field

Since force is always right angles to the direction of motion, for an element of the path $d\vec{L}$, the work done is $\vec{F}.d\vec{L}$. This work done is zero because $\vec{F}$ and $d\vec{L}$ are right angles. *Thus, a static magnetic field cannot change the kinetic energy of a moving charge.* It can only deflect it sideways, whereas when a charge moves in static electric field, its kinetic energy changes and is given by

$\left(\frac{1}{2}mv^2 = eV\right)$, where V is potential.

9.3 THE LORENTZ FORCE

If a charged particle moves through a region where both an electric and a magnetic field are present, then the resultant force is given by,

$$\vec{F} = q_0\vec{E} + q_0\vec{v}\times\vec{B}$$

$$= q_0\left[\vec{E} + \vec{v}\times\vec{B}\right] \qquad \ldots(9.4)$$

This resultant force is called Lorentz force. This equation defines the field $\vec{E}$ and $\vec{B}$ in free space, where the force on a moving test charge can be measured.

9.4 FORCE ON A CURRENT CARRYING WIRE

The equation (9.2) which gives the force on moving charge in a magnetic field, can be used for calculating the force exerted by a magnetic field on a wire carrying current. Basically the current *i* in a metal wire is carried by the free or conduction electrons.

If there are N charge per unit volume so number of charges in volume $dV = N dV$, therefore the force $d\vec{F}$ due to these changes is given by

$$d\vec{F} = N\,dV\,(q\vec{v} \times \vec{B})$$

where $(q\vec{v} \times \vec{B})$ is force on the charge q, but $Nqv = J$, the current density, [see eq. (8.20)]

Hence,

$$d\vec{F} = (\vec{J} \times \vec{B})\,dV$$

Let us apply it to the case of a cylindrical wire of cross-section A as shown in Fig. 9.2. Then we get force dF due to length dl.

$$d\vec{F} = \left(\vec{J} \times \vec{B}\right) A\,dl, \qquad \text{[since } dV = A\,dl\text{]}$$

$$= \vec{J}\,A\,dl \times \vec{B}$$

But, $$J = \frac{i}{A}$$

hence, $$d\vec{F} = i\,d\vec{l} \times \vec{B} \qquad \ldots(9.5)$$

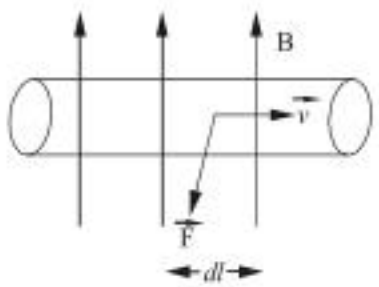


Fig. 9.2

Therefore, for the total wire of length l, the resultant force is obtained by integrating the above relation, *i.e.*

$$\vec{F} = \int d\vec{F} = i\vec{l} \times \vec{B} \qquad \ldots(9.6)$$

idl is called "*current element*". From Eq. (9.5), the force $d\vec{F}$ is perpendicular to $\vec{B}$ and $id\vec{l}$. The relation (9.6) shows that force depends on the total current and not on the amount of charge carried by each particle or even its sign.

The direction of $\vec{B}$ is determined by mean of the *right-hand rule for linear currents*. If the wire is grasped in the palm of the right hand such that thumb pointing in the direction of the current as in Fig. 9.3 then the fingers encircling the wire point in the direction of the magnetic field or lines of Magnetic flux. This is applicable for linear conductors. *The right-hand rule for circular currents* states that if the direction of the circular current coincides with the direction of the closed fingers, then the thumb will point in the direction of the magnetic field or the lines of the magnetic flux.

Fig. 9.3

9.5 THE FORCE PER CURRENT ELEMENT OF MAGNETIC FLUX DENSITY $\vec{B}$

The magnetic induction (or magnetic field vector) $\vec{B}$ is also called the magnetic flux density $\vec{B}$. The quantity idl = current × length = current moment. Unit of the current moment is a ampere-metre. In electro-statics the electric field intensity $\vec{E}$ is defined as the force per unit charge. In magnetostaics, the magnetic flux density $\vec{B}$ is defined as the force per current moment. The magnitude B from eq. (9.6) is,

$$B = \frac{F}{il} = \frac{\text{Force}}{\text{Current moment}} \qquad \ldots(9.7)$$

The unit for B is the Newton/ampere-metre, or as will be shown later, the weber per metre^2. In electrostatics we define $\vec{E}$ as follows

$$\vec{E} = \frac{\vec{F}}{q} = \frac{\text{Force}}{\text{Charge}} \qquad \ldots(9.8)$$

In order to establish an analogy with electrostatics, the product il is called pole strength denoted by qm, then equation (9.7) reduces to

$$\vec{B} = \frac{\vec{F}}{qm} \qquad \ldots(9.9)$$

we define $\vec{B}$ as the force per pole.

Becase of the analogy between (9.8) and (9.9) $\vec{B}$ is appropriately called the magnetic field intensity. $\vec{B}$ is also called as the *magnetic flux density* or *magnetic induction*. The equation (9.6) is used to define $\vec{B}$ and is derived from the basic definition of $\vec{B}$ given by equation (9.2). But the definition given here is more clear and a good analogy with $\vec{E}$ is established as shown in Fig. (9.9).

9.6 MAGNETIC FLUX ϕ_B

The magnetic flux denoted by ϕ_B can be defined in exact analogy with electric flux ϕ_E or in general flux of any vector field discussed earlier. The net outward flow or flux is the average out-drawn normal component of the field $\vec{B}$ times the area of the surface.

$$\phi_B = (\text{average normal component of } \vec{B}) \times \text{area}$$

The magnetic flux through area dS as shown in Fig. 9.4

$$d\phi_B = B\, dS \cos\theta$$

$$= \vec{B} \,.\, \hat{n}\, dS$$

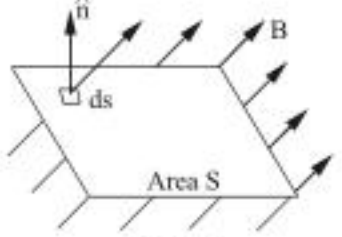


Fig. 9.4

where θ is the angle between $\vec{B}$ and normal to the area dS and $\hat{n}$ is unit vector along this normal. In general for any surface we have

$$\phi_B = \int_S \vec{B} \,.\, d\vec{S} \qquad \ldots(9.10)$$

where integral is taken over the entire surface.

Unit. The rationalised MKS unit for magnetic flux is obtained from equation (9.10),

$$\text{Unit of } \phi_B = \frac{\text{Newton}}{\text{Amp.metre}} \text{ metre}^2$$

Since unit of B is Newton/ampere-metre and Area = metre^2, therefore, unit of

$$\phi_B = \frac{\text{Newton metre}}{\text{Ampere}}$$

$$= \text{Weber}$$

Thus, in place of Newton-metre/ampere, a new unit weber is introduced. Also,

$$\text{Weber} = \frac{\text{Newton.metre}}{\text{Coul/sec.}} = \text{Volt sec}$$

$$\text{since Volt} = \frac{\text{Newton.metre}}{\text{Coulomb}}$$

For the magnetic field $\overline{B}$, we have, earlier,

$$\frac{\text{Newton}}{\text{amp-metre}} = \frac{\text{Newton.metre}}{\text{ampere.metre}^2}$$

But, $$1 \text{ Weber} = \frac{1 \text{ Newton.metre}}{1 \text{ ampere}}$$

hence, unit for $\overline{B} = \frac{\text{Weber}}{\text{metre}^2}$

We can obtain directly, the same result from Eq. (9.10). The $\overline{B}$ is called flux density and defined as the number of magnetic flux lines passing per unit area through a surface normal to the lines. Hence, the total flux is the measure of number of lines passing over the area through a surface normal to the lines.

9.7 MAGNETIC FLUX OVER A CLOSED SURFACE (OR GAUSS' LAW IN MAGNETOSTATICS)

The flux tubes (or field lines) of a static electric field originate and end on electric charges. On the other hand, tubes of magnetic flux (or lines of magnetic field $\overline{B}$,) are continuous, that is they have no sources or sinks. This is fundamental difference between static electric and magnetic fields. To describe this continuous nature of lines of $\overline{B}$ or flux tubes, it is said that magnetic field $\overline{B}$ is solenoidal (as we have discussed in chapter of vectors that a field is solenoidal when the liens are closed curves and do not have sources or sinks). Since magnetic field $\overline{B}$ is continuous, the magnetic flux entering any region is equal to the flux leaving it. Hence, when the result must be zero, or

$$\oint_S \overline{B} . d\vec{S} = 0 .$$

This is the Gauss's law applied to the magnetic field. Since flux entering any volume equals to flux leaving, the net flux over the volume is zero. Mathematically, it means at any point,

$$\text{div } \overline{B} = 0$$

$$\nabla . \overline{B} = 0$$

Example 9.1 *A uniform magnetic field of magnitude 1.5 Weber/metre² points horizontally from south to north. A proton of energy 5.0 MeV moves vertically downward through this field. Calculate the force on it.*

Solution. Kinetic energy of the proton is

$$K = 5.0 \text{ MeV} = 5 \times 10^6 \text{ eV}$$

$$= 5 \times 10^6 \text{ eV} \times 1.6 \times 10^{-19} \text{ joule}$$

$$= 8.0 \times 10^{-13} \text{ joule}.$$

But $K = \frac{1}{2} mv^2$, where m is the mass and v is the velocity, or

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2 \times 8.0 \times 10^{-12}}{1.7 \times 10^{-27}}} = 3.1 \times 10^7 \text{ metres/sec}.$$

Now, according to equation (9.2),

$$F = qv\,B \sin\theta.$$

As given $\vec{B}$ is horizontal and proton moves vertically downward, hence angle between $\vec{v}$ and $\vec{B}$ is 90°,

therefore, $\sin\theta = \sin 90 = 1$

$$F = qvB = (1.6\times10^{-19}\ \text{Coul})\times 3.1\times10^{7}\ \text{metre/sec.})\times 1.5\ \text{weber/metre}^2$$

But, $$\text{Weber} = \frac{\text{Newton.metre}}{\text{Ampere}}$$

hence, $F = 7.4\times10^{-12}$ Newtons.

Since $\vec{F}$ is perpendicular to $\vec{v}$ and $\vec{B}$ from relation,

$$\vec{F} = q\vec{v}\times\vec{B}$$

the direction of this force is east.

Example 9.2 *A 10 cm long wire carrying a current of 10 amp. is held at an angle 30° with the direction of a uniform magnetic field of strength 1 weber/metre². Calculate the force acting on the wire.*

Solution. According to equation (9.6), the force is

$$\vec{F} = i\vec{l}\times\vec{B}$$

$$F = il\text{B}\sin\theta.$$

Putting the given values,

$$F = 10\times10\times10^{-2}\times1\sin 30$$

$$= 0.5\ \text{Newton}.$$

Example 9.3 *A semi-circular wire of radius R metres carries a current I amp and is placed in a uniform field of B where/metre² acting perpendicular to the plane of the semi-circle. Calculate the force acting on the wire.*

Solution. In Fig. 9.5 AB represents the semi-circular wire with centre at O, the magnetic field is perpendicular to the plane of the paper in which the circle lies, *i.e.* acting towards the reader as shown, by ⊕.

Consider an element of length dl of the wire. The $d\vec{F}$ force on this element is obtained by equation (9.5)

$$d\vec{F} = \text{I}d\vec{l}\times\vec{B}$$

the magnitude of this force,

$$dF = \text{I}\,dl\ \text{B}\sin 90$$

$$= \text{I}\,(\text{R}d\theta)\ \text{B}, \qquad (dl = \text{R}d\theta)$$

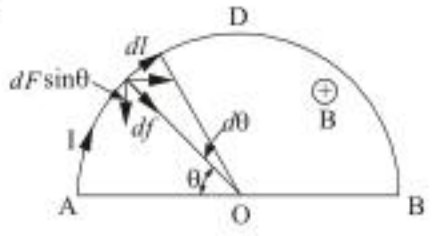


Fig. 9.5

and whose direction is radially toward O, the centre of the arc. By symmetry, the horizontal components *i.e.* the components along AB of all such elements cancel each other with the corresponding are element on the other side of O, while the downward components *i.e.* components perpendicular to AB (*i.e.* along DO) will add up. Thus, the total force on the semi-circle of wire about O points *i.e.* along DO is,

$$F = \int_0^{\pi} dF \sin\theta$$

$$= \int_0^{\pi} IBR d\,\theta \sin\theta = 2IBR.$$

9.8 TORQUE ON A CURRENT COIL IN UNIFORM MAGNETIC FIELD

A coil carrying a current in magnetic field experiences a torque. This is the basic principle underlying the operation of direct current measuring instruments, such as ammeters, voltmeters and galvanometers. We shall calculate the torque exerted on the one turn rectangular coil DEFG suspended in a uniform field $\overline{B}$ as shown in Fig. 9.6. The direction of $\overline{B}$ being parallel to the plane of the coil, the magnitude of force on the side DE is according to equation (9.6),

$F_{DE} = ih\,B.$ (angle between B and ih is 90°)

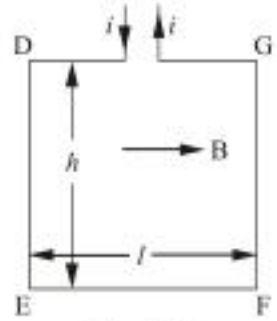


Fig. 9.6

The direction of this force is normal to the page towards the reader. Similarly, the magnitude of force on FG is,

$$F_{FG} = ih\,B$$

and the direction of F_{FG} is normal to the page away from the reader because the direction of current is just opposite in DE and FG arms. The *force* F_{DE} and F_{FG} *are equal in magnitude and opposite in direction and, therefore, constitute a couple.* The torque τ exerted by this couple on the coil,

$$\tau = ihl\,B,$$

since Torque = one of the force × perpendicular distance.

There is no force on the side FF since a current element EF makes an angle of zero degrees with B or in other words EF and $\overline{B}$ are parallel, hence the product $(i\vec{l} \times \overline{B})$ is zero. Similarly, the force on the sdie GD is zero since the current here makes an angle of 180 degrees with $\overline{B}$. Thus, the total torque on the coil is,

$$\tau = ihl\,B$$

or $$\tau = i\,AB$$

where A is the area of the coil = lh.

If the coil has N turns instead of one turn, the torque is given by,

$$\tau = Ni\,AB$$

Finally, if the normal to the plane of the coil makes an angle θ with $\overline{B}$, as, shown in Fig. 9.7, then

$$\tau = Ni\,AB \sin\theta \qquad \ldots(9.11)$$

whereas electric dipole when placed in an external electic field it experiences a torque given by,

$$\tau = \vec{P} \times \vec{E}$$

$$\tau = PE \sin\theta$$

where $\vec{P}$ is the electric dipole moment.

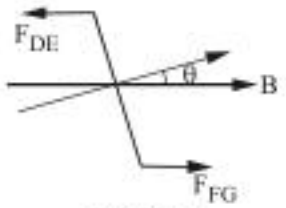


Fig. 9.7

Let us compare this equation with the expression for torque on a current loop, *i.e.*,

$$= (NiA)\, B \sin\theta$$

In each case the field ($\vec{E}$ or $\vec{B}$) appears. *Comparison suggests that (niA) in equation (9.11 can be taken as magnetic dipole moment m, analogous to* $\vec{P}$. Hence,

$$\vec{m} = NiA \qquad \ldots(9.12)$$

then, $$\tau = \vec{m} \times \vec{B} \qquad \ldots(9.13)$$

Thus, coil behaves as a magnetic dipole. The direction of magnetic dipole moment lies along the axis of loop by the following rule. Let the fingers of the right hand curl around the loop in the direction of the current the extended right hand thumb will then point in the direction of $\vec{m}$.

Unit. The unit of magnetic diple moment from equation (9.12) is ampere metre^2.

Potential energy of a current loop

Since a torque acts on a current loop when placed in an external field, it follows that work (positive or negative) must be done by an external agent to change the orientation of such dipole. This work donw is restored as potential energy. This energy may be taken to be zero for any arbitrary position, taking that potential energy be zero when $\vec{m}$ and $\vec{B}$ are at right angles that is, when $\theta = 90°$.

The magnetic potential energy is equal to the work done by the external agent in turing the loop from zero potential energy to a position θ. Thus,

$$U = \int_{90}^{\theta} \tau\, d\theta = \int_{90}^{\theta} NiAB \sin\theta\, d\theta$$

$$= mB \int_{90}^{\theta} \sin\theta\, d\theta$$

$$= -mB\cos\theta$$

or $$U = -\vec{m} \cdot \vec{B} \qquad \ldots(9.14)$$

which is in perfect corresponding with potential energy of electric dipole.

$$U = -\vec{P} \cdot \vec{E}$$

Example 9.4 *A circular coil of 100 turns has an effective radius 50 cm and carries a current 0.10 amp. Calculate the work done required to turn it in an external uniform magnetic field 1.5 weber/ metre^2 through 180°.*

Solution. Let us take the coil be turned from position $\theta = 0$ to $\theta = 180°$, the work required is the difference in energy between the two positions, the potential energy $\vec{U} = -\vec{m} \cdot \vec{B}$.

Hence, work required $= U_{\theta=180} - U_{\theta=0}$

$$= -mB\cos 180 - (-mB\cos 0)$$

$$= 2mB$$

But $m = NiA$

Hence, work required $= 2\,NiAB$

$$= 2Ni\,(\pi r^2)\,B$$

$$= 2\times100\times(0.10)\times\pi\times(5\times10^{-2})\,1.5$$

$$= 0.24 \text{ joule}$$

9.9 EQUIVALENCE OF CURRENT LOOP TO A SHORT MAGNETISED BAR (OR MAGNETIC DIPOLE)

Let us consider a bar market of pole strength Qm placed in magnetic field B making an angle θ with the axis of magnet as shown in Fig. 9.9.

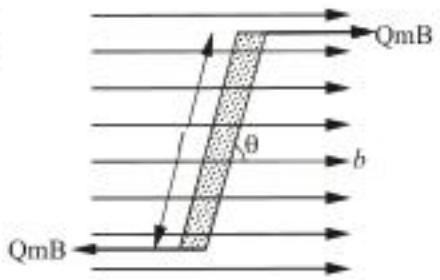


Fig. 9.8

$$\text{Torque} = \text{Force} \times \text{perpendicular}$$

$$= Qm\,Bl\sin\theta$$

$$= \vec{m} \times \vec{B}$$

where m = magnetic moment of bar, magnet given by

m = pole strength × distance between the poles.

Comparing this result with that of a torque due to a single coil carrying current, i.e. equation (9.13), they are identical. So, that we can sya that a current carrying coil is equivalent to a bar magnet, its magnetic dipole moment must be iA. The pole strength in magnet is equivalent to current moment in coil as given in equation (9.7).

However, the equivalence is valid for coil of any shape. Therefore, the equation (9.11) is valid even if the coil is not rectangular. Let us consider a coil as shown in Fig. 9.9. The plane coil may be divided into a large number of narrow rectangular circuits. The current in the coil (or loop) remains same because in the common boundary of any two rectangular circuits, there are two equal but oppositely directed currents. The torque due to the j-th rectangular circuit is,

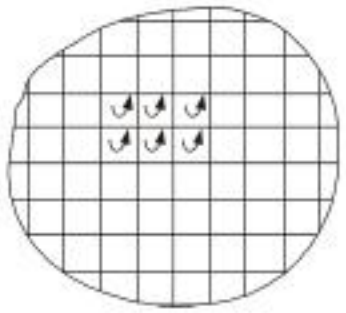


Fig. 9.9

$$\tau_j = iA_j\,B\sin\theta$$

Total torque τ is given by,

$$\tau = \sum_{J=1}^{J=n} \tau_j = iB\sin\theta \sum_{J=1}^{J=n} A_j$$

$$= iAB\sin\theta \qquad \ldots(9.15)$$

where, $A = \sum_{J=1}^{J=n} A_j$ is the total area of the coil. This is, of course, the same result as expressed in equation (9.11).

Thus, any current carrying coil is equivalent to a magnet (or magnetic dipole).

9.10 BALLISTIC GALVANOMETER

The ordinary galvanometers measure current. The B.G. measures charge. The B.G. are of two types, namely,

(*i*) The moving magnet of B.G.
(*ii*) The moving coil B.G.

The Moving Coil Ballistic Galvanometer

A ballistic galvanometer coinsists essentially of a rectangular coil suspended in a magnetic field as shown in Fig. 9.10. Let the current *i* flow through the coil. There is no resultant force along *bc* and *ad* which are parallel to $\vec{B}$. There is equal to opposite force on *ab* and *dc*. The torque is given by,

$$\tau = i\,AB$$

where A is area of coil *lb*.

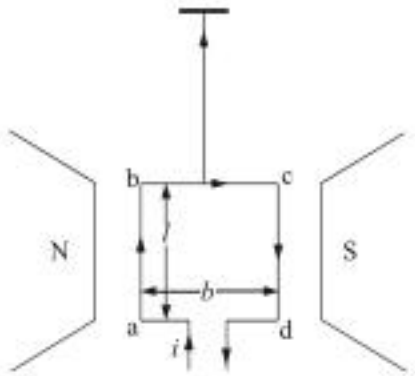


Fig. 9.10

If the current *i* acts for a short interval *dt*, then the moment of impulse is *i*AB*dt* and total moment of impulse,

$$\int_0^\tau i\,AB\,dt = q\,AB, \text{ since } q = \int_0^\tau i dt$$

But moment of impulse = mass × velocity × distance from the axis of rotation
= moment × distance from the axis of rotation
= angular momentum.

If I is moment of inertia of the coil about the suspension fibre and ω is angular velocity, then angular momentum = Iω

$$qAB = I\omega \qquad \ldots(1)$$

The kinetic energy of the system is $\frac{I\omega^2}{2}$ and is used up in twisting the suspension fibre. If *c* is the restoring force per unit twist, the couple on twist θ is $c\theta$ and work done for an additional twist $d\theta$ is $c\theta\,d\theta$ so that the whole work done in twisting the suspension fibre is,

$$\int_0^\theta c\theta\,d\theta = \frac{1}{2}c\theta^2$$

therefore,

$$\frac{1}{2}c\theta^2 = \frac{1}{2}I\omega^2$$

or $$\omega^2 = \frac{c}{I}\theta^2$$

$$= \frac{q^2A^2B^2}{I^2}$$ from equation (1)

or $$\frac{c}{I}\theta^2 = \frac{q^2 A^2 B^2}{I^2}$$

or $$q^2 = \frac{c^2}{B^2 A^2}\frac{I}{c}\theta^2$$

Now the period of vibration of a body of moment of inertia I, when c is the restoring couple per unit twist is given by,

$$T = 2\pi\sqrt{\frac{I}{c}}$$

or $$\frac{I}{c} = \frac{T^2}{4\pi^2}$$

therefore, $$q^2 = \frac{c^2}{B^2 A^2}\frac{T^2}{4\pi^2}\theta^2$$

or $$q = \frac{c}{BA}\frac{T}{2\pi}\theta \qquad \ldots(9.16)$$

or $$q = K\frac{T}{2\pi}\theta \qquad \ldots(9.17)$$

where $K = \frac{c}{BA}$.

This gives a relation between the charge flowing and the ballistic throw θ of the galvanometer, where K is the galvanometer constant. The torque or couple on suspension fibre is $c\theta$. Therefore,

$$iAB = c\theta$$

$$i = \frac{c}{BA}\theta \qquad \ldots(9.18)$$

Current Sensitivity

Quantitatively, the current sensitivity of a galvanometer is defined as the deflection, as read from the scale per unit current. So from equation (9.18),

$$\text{current sensitivity} = \frac{\text{deflection}}{\text{current}}$$

$$= \frac{\theta}{i} = \frac{BA}{c} \qquad \ldots(9.19)$$

As we known that mirror and scale arrangement is used for reading the deflection, if the current i is in amp., the scale deflection θ in millimetres and scale distance L in metres, the current sensitivity,

$$S_i = \frac{\theta}{Li}$$

If i = 1amp, then Si is a large number for a given value of L. Consequently for convenience we prefer microampere which gives microampere current sensitivity.

Micro-ampere sensitivity is the deflection in mm produced on a scale 1000 mm away from the galvanometer mirror, when a current of one micro ampere (10^{-6} amp.) is passed in the galvanometer.

Charge Sensitivity of Quantity Sensitivity

The charge sensitivity of the B.G. is defined as the deflection per unit charge. So from equation (9.17),

$$\text{Charge sensitivity} = \frac{\text{deflection}}{\text{charge}}$$

$$= \frac{\theta}{q} = \frac{2\pi}{T} \cdot \frac{BA}{c}$$

But from equation (9.19). $\frac{BC}{c}$ is current sensitivity.

$$\text{Hence, charge sensitivity} = \frac{2\pi}{T} \times \text{current sensitivity}. \qquad \ldots(9.20)$$

Hence the charge sensitivity of a B.G. is $\frac{2\pi}{T}$ times the current sensitivity.

Voltage Sensitivity

The voltage sensitivity under any given conditions is the deflection per unit voltage and is consequently, current sensitivity divided by the resistance, *i.e.*,

$$\text{Voltage sensitivity} = \frac{\text{Current sensitivity}}{\text{Resistance}}$$

The resistance is that of entire circuit and not that of the instrument alone. Voltage sensitivity is large number, hence for convenience, we prefer micro-volt sensitivity.

Micro-volt Sensitivity. Micro-volt sensitivity is the deflection in millimeter on a scale 1 metre away from the galvanometer mirror, when a potential difference of one micro-volt (10^{-6} volts) is maintained across the galvanometer terminals.

Figure of merit of galvanometer. Figure of merit of galvanometer is the current which will produce a deflection of one scale division, when we use lamp and scale arrangement, it is the current which will produce a deflection of 1 mm on a metre scale one metre away from the galvanometer mirror.

It may be noted here that the current sensitivity or charge sensitivity is sometimes defined as the current or charge per unit deflection and then it is reciprocal of what have been defined here by equation (9.19) and (9.20). We prefer to call it figure of merit of the galvanometer.

$$\text{Figure of merit} = \frac{q}{\theta} \text{ or } \frac{i}{\theta}.$$

9.11 CONDITION WHEN A MOVING COIL GALVANOMETER IS DEAD BEAT OR BALLISTIC

In deriving equation (9.17) we have assumed that whole of the kinetric energy of the coil is used up in twisting the fibre and neglected the retarding forces present namely, (1) due to friction of air, (2) viscosity of suspension fibre and (3) induced current in any neighbouring mass or metal. The first two of these factors constitute mechanical damping, while the third factor accounts for electro-magnetic damping. The retarding couple due to damping varies as the angular velocity = $d\theta/dt$ may be taken as $pd\theta/dt$, where p is coefficient of mechanical damping.

When the circuit is closed, we have in addition the induced current produced in the coil which varies as the rate of change of displacement and inversely as the total resistance of the circuit, hence the retarding couple is $\frac{m}{\mathrm{R}}\frac{d\theta}{dt}$, where the constant m involves the magnetic flux due to the magnet and the area of coil. Thus, the total retarding couple due to damping.

$$=\left(p+\frac{m}{\mathrm{R}}\right)\frac{d\theta}{dt}$$

Now the angular acceleration is $\frac{d^2\theta}{dt^2}$.

Hence, according to Newton's second law, equation of motion is,

$$\frac{\mathrm{I}d^2\theta}{dt^2}=-c\theta-\left(p+\frac{m}{\mathrm{R}}\right)\frac{d\theta}{dt}$$

or $$\frac{\mathrm{I}d^2\theta}{dt^2}+\left(\frac{m}{\mathrm{R}}+p\right)\frac{d\theta}{dt}+c\theta=0$$

or $$\frac{d^2\theta}{dt^2}+\frac{\mathrm{I}}{\mathrm{I}}\left(\frac{m}{\mathrm{R}}+p\right)\frac{d\theta}{dt}+\frac{c\theta}{\mathrm{I}}=0$$

or $$\frac{d^2\theta}{dt^2}+2b\frac{d\theta}{dt}+\mathrm{K}^2\theta=0 \qquad \ldots(1)$$

where, $$2b=\frac{1}{\mathrm{I}}\left(\frac{m}{\mathrm{R}}+p\right) \qquad \ldots(2)$$

and $$\mathrm{K}^2=\frac{c}{\mathrm{I}} \qquad \ldots(3)$$

Let the solution of this equation be $\theta=\mathrm{A}e^{\omega}$.

Then, $\frac{d\theta}{dt}=\mathrm{A}\alpha e^{\omega}$ and $\frac{d^2\theta}{dt^2}=\mathrm{A}\alpha^2 e^{\omega}$

Putting thse values in equation (1),

$$\alpha^2+2b\alpha+\mathrm{K}^2=0$$

or $$\alpha=-b\pm\sqrt{b^2-\mathrm{K}^2}$$

This gives, $\theta=\mathrm{A}_1 e^{(-b+\sqrt{b^2+\mathrm{K}^2})t}+\mathrm{A}_2 e^{(-b-\sqrt{b^2-\mathrm{K}^2})t}$ where, $\mathbf{A}_2$ and A_1 are constants.

The quantity $b^2-\mathrm{K}^2$ may be positive, zero or negative depending upon whether

$b>\mathrm{K}$, $b=\mathrm{K}$ or $b<k$.

Case 1: When $b>K$, we have two real values of α, then solution of equation (1) is,

$$\theta=\mathrm{A}_1 e^{\alpha_1 t}+\mathrm{A}_2 e^{\alpha_2 t}$$

The α_1 and α_2 are negative. The deflection θ will go on decreasing with increasing time. *The motion is non-oscillating or dead beat and the oscillation is said to be over-damped.*

Case 2: Critical damping: When $b = K$; the two values of α are equal to $-b$ and solution of (1) becomes

$$\theta = (A_1 + A_2)\, e^{-bt}$$

$$= A_3\, e^{-bt}$$

That the coil after detiection comes to rest in minimum possible time and the galanometer is said to be *critically damped. However, the motion of the coil is just non-oscillatory.*

Case 3: When $b < K$, the quantity $\sqrt{b^2 - K^2}$ is imaginary so the values of α are complex *i.e.*,

$$\alpha = -b \pm jg, \text{ where } g^2 = K^2 - b^2$$

Hence, solution of equation (1) becomes,

$$\theta = A_1\, e^{(-b+jg)t} + A_2\, e^{(-b-jg)t}$$

$$= e^{-bt}\left[A_1\, e^{jgt} + A_2\, e^{-jgt}\right]$$

$$= e^{-bt}\left[A_1\{\cos gt + j \sin gt\} + A_2\{\cos gt - j \sin gt\}\right]$$

$$= e^{-bt}\left[(A_1 + A_2)\cos gt + j\,(A_1 - A_2)\sin gt\right]$$

Putting $A_1 + A_2 = B \sin \beta$

and $j\,(A_1 - A_2) = B \sin \beta$

$$\theta = Be^{-bt} \sin (gt + \beta) = Be^{-bt} \sin\left\{\left(\sqrt{K^2 - b^2}\right)t + \beta\right\}$$

The motion is oscillatory or ballistic and period of oscillation,

$$T = \frac{2\pi}{g} = \frac{2\pi}{\sqrt{K^2 - b^2}} \qquad \ldots(9.21)$$

In order that the whole charge may pass through the ballistic galvanometer before it has moved from its zer position, it is essential that its periodic time be large. Since we have,

$$T = 2\pi\sqrt{\frac{I}{c}},$$

therefore, I should be large and c should be small. As we have $b < K$

$$\frac{1}{2I}\left(\frac{m}{R} + p\right) < \frac{c}{I} < \frac{2\pi}{T}$$

Hence to make b small,

I should be large,

R should be large,

m should be small *and so the coil should be wound round a non-conducting frame, e.g.* paper, bamboo or plastic. p should be small, and so the air resistance should be small and the fibre should be very fine.

We just see that the requirement of dead-beat galvanometer are just opposite to those for the ballistic $b > K$. Hence the opposite conditions should prevail. The coil is made of thick copper wire and has smaller number of turns so to have smaller resistance and is wound round a metallic or conducting frame.

9.12 EFFECT OF ELECTROMAGNETIC DAMPING RESISTANCE ON SENSITIVITY

The sensitivity of a B.G. with only electromagnetic damping depends on the resistance of the circuit through which the damping current flows. For an instantaneous discharge, the sensitivity is given by,

$$\text{Charge sensitivity} = \left(\frac{2\pi}{T}\right) \times \text{current sensitivity}$$

and $$\text{voltage sensitivity} = \frac{\text{Current sensitivity}}{\text{Resistance}}$$

The electromagnetic damping coefficient is $\left(\frac{m}{R}\right)$, say K when entire resistance of damping circuit is R and Ke its value when damping is critical corresponding to resitance Re.

$$K = \frac{m}{R}$$

$$K_e = \frac{m}{R_e}$$

$$\frac{K_e}{K} = \frac{R}{R_e} = n, \qquad R_e \text{ is the critical damping resistance}$$

and, $R = n R_e$.

And as we know that the condition necessary for ballistic galvanometer is that T should be large and R should be large. Hence increases the sensitivity.

Thus, we see that the resistance of the galvanometer and the period of vibration have great influence on the sensitivity of the galvanometer. Therefore, the sensitivity a galvanometer is defined as the deflection in mm on a scale 1 metres away when a current of one micro ampere is passed when the resistance of the galvanometer is one ohm and the period of vibration one second.

Mega-ohm Sensitivity

It is the number of mega-ohms to be put in series with the galvanometer coil so that a potential difference of one volt across 1 metre away from the galvanometer mirror.

9.13 DAMPING

The oscillatory motion in the B.G. is given by,

$$\theta = Be^{-bt} \sin\left(\sqrt{K^2 - b^2}t + \beta\right)$$

However, the effort is made in the B.G. to make b as small as possible so that the motion is simple harmonic, but in actual practice b can never be zero so the amplitude of oscillation decreases with time.

The amplitude of motion is Be^{-bt} and

for $t = 0$, we have the amplitude $\theta_0 = B$

for $t = \frac{T}{4} = \frac{\pi}{2\sqrt{K^2 - b^2}}$ amplitude $\theta_1 = Be^{-\frac{b\pi}{2\sqrt{K^2 - b^2}}}$

for $t = \dfrac{3T}{4} = \dfrac{3\pi}{2\sqrt{K^2 - b^2}}$ amplitude $\theta_2 = Be^{-\frac{3b\pi}{2\sqrt{K^2-b^2}}}$

for $t = \dfrac{5T}{4} = \dfrac{5\pi}{2\sqrt{K^2 - b^2}}$ amplitude $\theta_3 = Be^{-\frac{5b\pi}{2\sqrt{K^2-b^2}}}$

and $$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = \frac{\theta_3}{\theta_4} \ldots\ldots = e^{+\frac{b\pi}{\sqrt{K^2-b^2}} = e^{\lambda} = d}$$

We see that the successive amplitude $\theta_1, \theta_2, \theta_3$ etc. are continuously decreasing and the ratio of one value to the next is always a constant d. This constant ratio d is called decrement and $\log_e d = \lambda$ is called the logarithmic decrement λ. The successive decrease in amplitude is shown in Fig. 9.11.

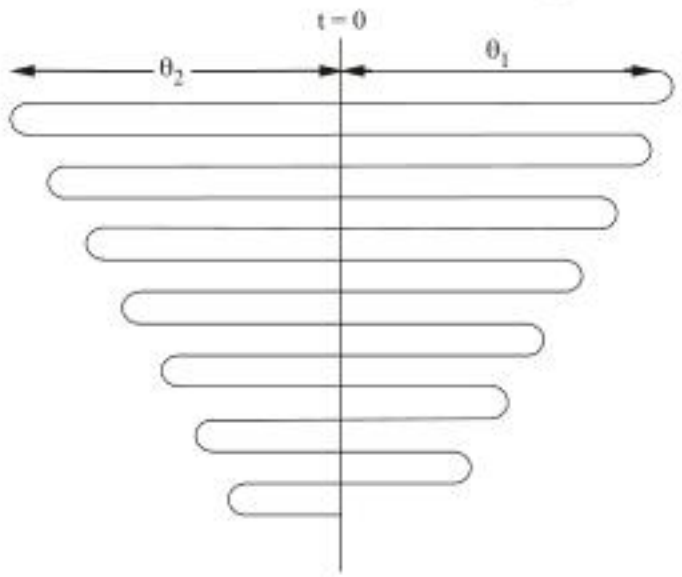


Fig. 9.11

And now, $\theta_1 = Be^{-\frac{b\pi}{2\sqrt{K^2-b^2}}}$

or $\theta_1 = Be^{-\lambda/2}$

or $\theta_1 = \theta_0 e^{-\lambda/2}$

Hence, $\dfrac{\theta_0}{\theta_1} = e^{\lambda/2}$

$$= \left[1 + \frac{\lambda}{2} + \frac{1}{1 \times 2}\left(\frac{\lambda}{2}\right)^2 + \cdots\right]$$

$$\frac{\theta_0}{\theta_1} = 1 + \frac{\lambda}{2} \text{ (approximately)}$$

or $\theta_0 = \theta_1\left(1 + \dfrac{\lambda}{2}\right)$.

Thus we can correct the effect of damping in B.G. although we cannot avoid it. Now the corrected charge,

$$q = \frac{c}{\mathrm{BA}} \frac{\mathrm{T}}{2\pi} \theta_1 \left(1 + \frac{\lambda}{2}\right) \qquad \ldots(9.22)$$

Experimental Determination of the Logarithmic Decrement

We have established a relation,

$$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = \frac{\theta_3}{\theta_4} = \cdots = \frac{\theta_9}{\theta_{10}} = \frac{\theta_{10}}{\theta_{11}} = d = e^{\lambda}$$

Multiplying all these factors, we get,

$$\frac{\theta_1}{\theta_{11}} = e^{10\lambda}$$

or $$\lambda = \frac{1}{10} \log_e \frac{\theta_1}{\theta_{11}} = \frac{2.3026}{10} \log_{10} \frac{\theta_1}{\theta_{11}}.$$

Thus observing θ_1 and θ_{11} *i.e.*, first throw and eleventh throw, one can calculate λ.

Calibration of the B.G.

In equation (9.22), the charge q canbe written as,

$$q = \mathrm{K}\theta_1 \left(1 + \frac{\lambda}{2}\right)$$

where K is ballistic constant given by,

$$\mathrm{K} = \frac{c}{\mathrm{AB}} \frac{\mathrm{T}}{2\pi}.$$

And the current is given by,

$$i = \frac{c}{\mathrm{AB}} \theta$$

or $$\frac{c}{\mathrm{AB}} = \frac{i}{\theta}.$$

$\frac{c}{\mathrm{AB}}$ is called current reduction factor and is determined by sending a known steady current i_0 through the galvanometer and observing the deflection θ. Then,

$$q = \frac{\mathrm{T}}{2\pi} i_0 \frac{\theta_1}{\theta} \left(1 + \frac{\lambda}{2}\right) \qquad \ldots(9.23)$$

The time period T is determined with the stop watch. Hence K can be calculated.

9.14 APPLICATIONS OF B.G.

The B.G. is used for various measurements. Some of very important are mentioned below:

1. Comparison of the emfs of two cells
2. Comparison of two capacitances

3. Determination of the following:
   (*a*) Angle of dip and components of earth magnetic field (Art. 12.3)
   (*b*) Self-inductance (Art. 11.15)
   (*c*) Mutual inductance (Art. 11.23)
   (*d*) Magnetic susceptibility and permeability (Art. 14.28)
4. Measurement of:
   (*a*) Strong magnetic field (Art. 12.4)
   (*b*) High resistance (Art. 11.26)

1. Comparison of emfs of two cells

The circuit used is shown in Fig. 9.12. A suitable condenser C is first charged with cell of emf ξ_1 and then discharge through the BG and observed the throw say it is θ_1. Similarly charge capacitance with ξ_2 and observed the throw say θ_2. Then,

$$q_1 = c\xi_1 \, K\theta_1 \left(1+\frac{\lambda}{2}\right)$$

and $$q_2 = c\xi_2 \, K\theta_2 \left(1+\frac{\lambda}{2}\right)$$

on dividing,

$$\frac{\xi_1}{\xi_2} = \frac{\theta_1}{\theta_2} \qquad \ldots(9.24)$$

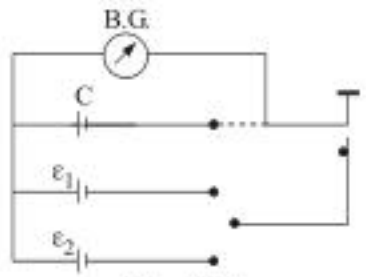


Fig. 9.12

The charge and discharge are easily effected with the help of charge discharge key K.

2. Comparison of two capacitances

The connections are shown in Fig. 9.13. Charge the two condenser with same battery and discharge through the BG and observed the deflection in both cases and say θ_1 and θ_2 successively. Then,

$$q_1 = c_1\xi = K\theta_1 \left(1+\frac{\lambda}{2}\right)$$

and $$q_2 = c_2\xi = K\theta_2 \left(1+\frac{\lambda}{2}\right)$$

On dividing, $$\frac{c_1}{c_2} = \frac{\theta_1}{\theta_2} \qquad \ldots(9.25)$$

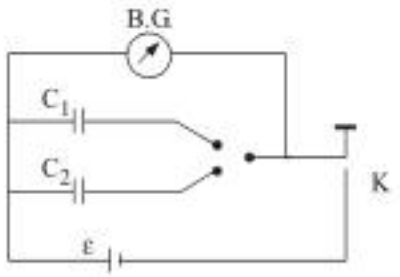


Fig. 9.13

Ballistic Constant

If we use a standard capacitance c and charge it to a known emf ξ and then discharge it through the B.G. when the throw θ is observed, then the B.G. constant K is easily calculated.

$$q = \xi c = K\theta\left(1+\frac{\lambda}{2}\right)$$

or $$K = \frac{\xi c}{\theta\left(1+\frac{\lambda}{2}\right)} \qquad \ldots(9.26)$$

Example 9.5 *When 0.1 coul. of charge is passed through a moving coil B.G, a deflection of 30 mm is observed on a scale one meter away. Find the current sensitivity of the galvanometer if the time period of the coil is 10 seconds.*

Solution. From equation (9.20)

$$\text{Current sensitivity} = \frac{T}{2\pi} \times \text{charge sensitivity}$$

$$= \frac{10\,\text{sec}}{2 \times 3.14} \times \frac{30\,\text{mm}}{0.1\,\text{Coul}}$$

$$= 477.5\ \text{mm/amp}$$

$$= 0.4775\ \text{m/amp}.$$

Example 9.6 *A condenser of capacity 0.2 μF is charged to 3 volts when discharged throgh a B.G is gives a deflection of 10 cm. Calculate the current sensitivity of galvanometer, if its periodic time be 10 sec.*

Solution. $\text{Charge sensitivity} = \frac{10}{0.2 \times 3} \frac{\text{cm}}{\text{micro coulomb}}$

$$= 16.6 \frac{\text{cm}}{\text{micro coulomb}}$$

$$\text{Current sensitivity} = \frac{2\pi}{T} . \text{Charge sensitivity}$$

$$= \frac{2 \times 3.14}{10\,\text{sec}} \times \frac{16.6\,\text{cm}}{\text{micro coul}}$$

$$= 10.4\ \text{cm/micro amp}.$$

Example 9.7 *A condenser charged to 2 volts is discharged through a ballistic galvanometer, when the carrected deflection is 9.6 cms and the current sensitivity is 4.54 × 10^2 mm/μA and the periodic time is 12 seconds. Calculate the capacity of the condenser.*

Solution. $\text{Charge sensitivity} = \frac{T}{2\pi} \times \text{current sensitivity}$

$$= \frac{12}{2 \times 3.14} \times 4.54 \times 10^2\ \text{mm/micro coul}$$

$$= \frac{12 \times 4.5 \times 10}{2 \times 3.14}\ \text{cm/micro coul}$$

But, $\text{charge sensitivity} = \frac{\text{deflection}}{\text{charge}}$

Or $\text{charge} = \frac{\text{deflection}}{\text{charge sesitivity}}$

$$= \frac{9.6\,\text{cm}}{\frac{12 \times 4.5 \times 10}{2 \times 3.14}}\ \text{cm/micro coul}$$

Or $$q = \frac{9.6 \times 2 \times 3.14}{12 \times 4.5 \times 10} \mu \text{ coul}$$

But $$= \frac{9.6 \times 2 \times 3.14 \times 10^{-6}}{12 \times 4.5 \times 10} \text{ coul}$$

Or $$q = c\,\text{V}$$

$$c = \frac{q}{\text{V}} = \frac{9.6 \times 2 \times 3.14 \times 10^{-6}}{12 \times 4.5 \times 10 \times 2} \text{ Coul/volt}$$

$$= 20.16 \times 10^{-6} \text{ Farad}$$

$$= 0.2016\ \mu\text{F}$$

Example 9.8 *The successive deflections to the right and left of the mean position in the case of ballistic galvanometer are found to be 25.0, 24.9 and 24.8 cms. Calculate the logarithmic decrement.*

Solution. As we know that $\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = e^{\lambda}$

$$e^{\lambda} = \frac{25}{24.9} = \frac{24.9}{24.8} = \frac{25 + 24.9}{24.9 + 24.8}$$

$$\lambda = 2.3 \log_{10} \frac{49.9}{49.7} = 0.0039$$

Example 9.9 *A condenser of 1,000 pf is charged to a potential difference of 1 volt and then discharged through a B.G. The find throw as noted a scale placed 1 meter away is 62.2 cms. If the time period of free vibrations be 10 seconds, and logarithmic decrement 0.02, calculate the ballistic constant of the galvanometer and its figure of merit.*

Solution. $$q = c\text{V}$$

$$= 1000 \times 10^{-12} \times 1 = 10^{-9} \text{ coul}$$

and also, $q = \text{K}\theta_1 \left(1 + \frac{\lambda}{2}\right)$,

where, $$\text{K} = \frac{\text{T}}{2\pi} \frac{c}{\text{NAB}}$$

or $$10^{-9} = \text{K} \times 62.2 \times 10^{-2} \times \left(1 + \frac{0.02}{2}\right)$$

or $$\text{K} = \frac{10^{-9}}{62.2 \times 10^{-2} \times \left(\frac{2 + 0.02}{2}\right)}$$

$$= \frac{10^{-9} \times 1000 \times 100}{622 \times 101} \frac{\text{Coul}}{\text{metre}}$$

Since, $$\text{K} = \frac{\text{T}}{2\pi}\left(\frac{c}{\text{NAB}}\right)$$

In fact, $\frac{c}{\text{NAB}}$ is the figure of merit

$$\frac{c}{\text{NAB}} = \text{K} \cdot \frac{2\pi}{\text{T}} = \frac{10^{-9} \times 1000 \times 100}{622 \times 101} \frac{2\pi}{10}$$

$$= 10^{-10} \text{ amp/radian}.$$

Let us consider a deflection of 1 mm on a scale, 100 cm away. Angular deflection $= \frac{10^{-3}}{1} = \frac{1}{1000}$ radian.

Current required to produce this deflection,

$$= 10^{-10} \text{ amp/radian} \times \frac{1}{1000} \text{ radian}$$

$$= 10^{-13} \text{ amp}.$$

9.15 DIRECT CURRENT AMMETER AND VOLTMETER

(or Conversion of galvanometer to Ammeter and Voltmeter)

A direct current ammeter or voltmeter operates on the principle that a torque is exerted on a coil carrying a current in a magnetic field. As in the ballistic galvanometer, a magnetic field is provided by curved pole pieces. Consequently the magnetic torque = N*i*AB,

where B = Magnetic field

A = Area of the coil

i = Current through the coil

N = Number of the coils.

Due to this torque the coil rotates. The rotationof the coil is opposed and the restoring torque acts due to spring by which coil is pivoted. When an ammeter or voltmeter is used in a circuit, there is a steady current in the meter coil. Equilibrium is established when the coil is at rest in a deflected position. Then,

$$\text{N}i\text{AB} = c\theta \qquad \ldots(9.27)$$

where c is constant restoring force per unit twist for a given coil. Clearly, the angle θ of deflection is proportional to the current in the coil as NAB are constant for a given ammeter, *i.e.*,

$$\theta \propto i$$

Construction of Ammeter

The internal circuit of a d.c. ammeter is shown in Fig. 9.14. Applying the Kirchhoff's laws to the closed loop in Fig. 9.14, we get,

$$i_g \text{R}_g - i_s \text{R}_s = 0 \quad \text{(from voltage law)} \qquad \ldots(9.28)$$

where the subscript g refers to the coil or galvanometer and s to the shunt.

From the Kirchhoff's current law, we get,

$$i - i_g - i_s = 0$$

or $$i = i_g + i_s \qquad \ldots(9.29)$$

where i is the current measured by the ammeter. After eliminating i_s from the above two equations, we have,

$$i = i_g \left(1 + \frac{\text{R}_g}{\text{R}_s}\right) \qquad \ldots(9.30)$$

Clearly, i is proportional to i_g a$_L$ d, therefore, it is possible to calibrate the ammeter to read i directly. The range of ammeter depends on the maximum value of the current (i_g) which the coil can carry safely and on the values of R_g and R_s.

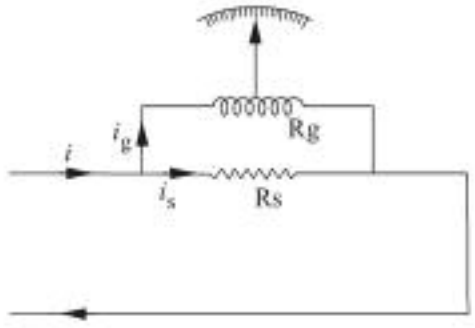


Fig. 9.14

Thus an ammeter is just a galvanometer with very low resistance, this is necessary so that ammeter may not appreciable alter the current in the circuit which is to be measured. Therefore, a shunt resistance is connected in parallel.

Construction of Voltmeter

The internal circuit of d.c. voltmeter is shown in Fig. 9.15

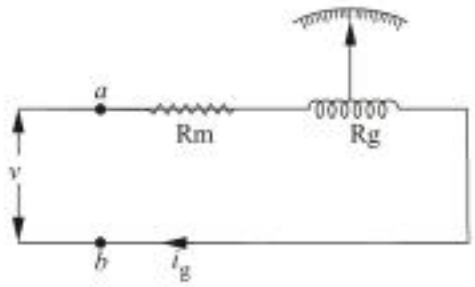


Fig. 9.15

The power multiplier resistance to be used may be computed as follows.

Let us consider V is the range of the voltmeter, then,

$$V = i_g (R_m + R_g)$$

where i_g is the maximum current which the coil can carry safely therefore, the proper multiplier resistance is given by,

$$R_m = \frac{V}{i_g} - R_g \qquad \ldots(9.31)$$

Example 9.10 *Calculate the resistance to be connected in series to convert an ammeter having a range of 0.02 ampere and a resistance (coil and shunt are parallel) of 0.5 ohm to a voltmeter having a range of 3 volts.*

Solution. The value of R_m may be calculated by Ohm's law, as shown in Fig. 9.16,

$$3 = 0.02 (R_m + 0.5)$$

$$R_m = 149.5 \text{ ohms}$$

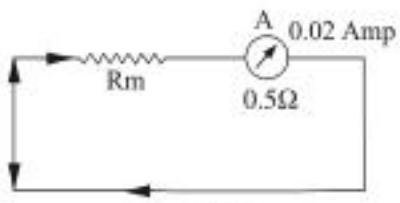


Fig. 9.16

Example 9.11 *A 50 ohm coil galvanometer can carry a current 10 MA. How can you convert it into*
(*a*) *voltmeter of range 500 volts*
(*b*) *ammeter of range 10 amp.*
Solution. (a) Using relation (9.31), *i.e.*

$$V = i_g (R_m + R_g) \text{ or } R_m = \frac{V}{i_g} - R_g$$

$$i_g = 10\,\text{MA} = 10 \times 10^{-3} \text{ amp}$$

$$R_g = 50 \text{ ohms}$$

$$V = 500 \text{ volts}$$

hence, $$R_m = \frac{500}{10 \times 10^{-3}} - 50 = 49{,}950 \text{ ohms}$$

$$R_m = 49950, \text{ is connected in series.}$$

(b) Using relation (9.30), *i.e.*

$$i = i_g \left(1 + \frac{R_g}{R_s}\right)$$

given, $$i = 10 \text{ amp}$$

$$i_g = 10 \times 10^{-3} \text{ amp.}$$

$$R_g = 50 \text{ ohms}$$

Hence, $$10 = 10 \times 10^{-3} \left(1 + \frac{50}{R_s}\right)$$

or $$\frac{10}{10 \times 10^{-3}} = \frac{R_s + 50}{R_s},$$

or $$1 + \frac{50}{R_s} = 1000$$

or $$\frac{50}{R_s} = 999$$

or $$R_s = \frac{50}{999} = 0.05005 \text{ ohm}$$

R_s is shunt resistance.
Example 9.12 *A pointer type galvanometer of 20 ohm resistance has a linear scale of 50 divisions. It gives a deflection of 30 divisions when connected in series with a resistance of 2980 ohms and an accumulator of 2 volts. How will you convert it into an ammeter of range 3 ampere?*
Solution. The current corresponding to 30 divisions by Ohm's law.

$$i_{30} = \frac{2}{2980 + 20} = \frac{2}{3000} \text{ amp.}$$

The maximum value of the current galvanometer can carry is corresponding to 50 divisions. Hence,

$$i_g = i_{50} = \frac{2}{3000} \times \frac{50}{30} \text{ amp.}$$

Now using the relation,

$$i = i_g \left(1 + \frac{R_g}{R_s}\right)$$

$$3 = \frac{2}{3000} \times \frac{50}{30} \left(1 + \frac{20}{R_s}\right)$$

or $$\frac{3 \times 3000 \times 30}{2 \times 50} = 1 + \frac{20}{R_s}$$

or $$\frac{20}{R_s} = \frac{3 \times 3000 \times 30}{2 \times 50} - 1$$

$$= 2700 - 1 = 2699$$

or $$R_s = \frac{20}{2699} = 0.00741 \text{ ohm (shunt)}$$

9.16 THE HALL EFFECT

The fact that the charged particle in magnetic field are deflected has changed our basic concept that in the metal the current is carried only by electrons. If a conductor is placed in a magnetic field such that the direction of flow of current is at right angle to the magnetic field, a voltage is developed across the conductor in the direction perpendicular to both current and field. The phenomena by which voltage developed is called Hall effect and voltage itself is Hall voltage.

Figure 9.17(a) shows a section of a metal bar carrying a current of current density j_x in the positive x-direction. The electrons are driven by a field $\bar{E}_x$ and drifting to the left. Since the current j_x is in the $+x$-direction, the current density j_x and the field $\bar{E}_x$ is related by,

$$\bar{j}_x = \sigma \bar{E}_x$$

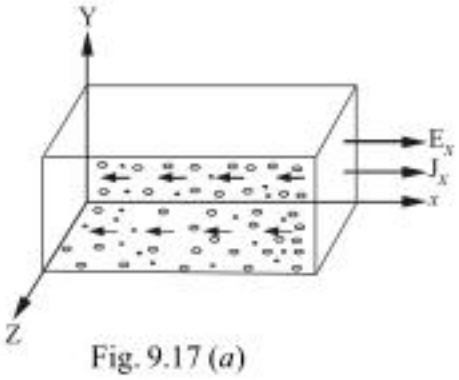


Fig. 9.17 (*a*)

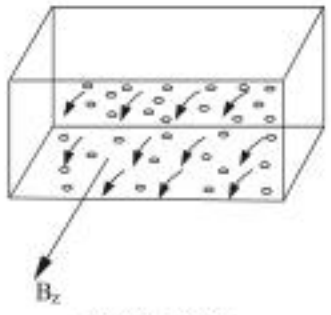


Fig. 9.17 (*b*)

Now an external field $\bar{B}_z$ in the z-direction is applied. The deflecting force due to $\bar{B}_z$ acts on electrons, consequently electrons are being deflected downward as shown in Fig. 9.17(b). But since they cannot escape at the bottom of the bar, they simply accumulate there until the surplus of negative charge at the bottom of the bar and the corresponding excess of positive charge at the top creates an

electric field $\vec{E}_y$, due to this field the upward force eE_y exactly balances the downward froce evx Bz (due to B_z, since v_x and B_z are perpendicular). In the steady state which is attained very quaickly the average motion is again horizontal. Due to this field (E_y) a downward force acts on the positive ions. This force accelerates the bar downward.

The Lorentz force on an electron is,

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

When the downward force balanced by upward force of developed transverse field, the resultant force is zero.

$$F_y = 0 = e(E_y - v_x B_z)$$

or $$E_y = v_x B_z$$

If d is width of the bar then the potential difference developed, across the bar,

$$V = E_y . d = v_x B_z d$$

or $$v_x = \frac{E_y}{B_z}$$

Now the current density,

$$j_x = Ne\, v_x$$

where N are number of charge carriers, *i.e.* electrons per cubic meter. Substituting v_x, we get

$$j_x = Ne . \frac{E_y}{B_z}$$

$$\frac{1}{Ne} = \frac{E_y}{j_x B_z} = R_H \qquad \ldots(9.32)$$

where R_H is called the Hall coefficient or constant and is negative for most metal where the current is carried by the free electrons.

The sign of Hall coefficients is positive for some emtals like beryllium, zinc and cadmium. The positive value of Hall coefficient predicts that the current is also carried by a positive charge. This act is explained on the basis of band theory as an example in semi-conductor the current is carried by both positive and negative. Eqn. (9.32) gives the number of current carriers per unit volumes.

Example 9.13 *A copper strip 2.0 cm wide and 1.0 mm thick is placed in a magnetic field of magnitude 1.5 webers/metre^2 as in Fig. 9.17 (b) in the z-direction. If a current of 100 amp is set up in the strip along the x-direction, calculate the Hall potential difference across the strip given the number of electrons per cubic metre = 9.4 × 10^{28} metre^3.*

Solution. From equation (9.32),

$$E_y = \frac{j_x B_z}{Ne}$$

But, $$E_y = \frac{V}{d}$$ where V is the potential difference

and $$j_x = \frac{i}{A} = \frac{i}{dh}$$

where h is thickness.

Substituting these values, we get

$$\frac{V}{d} = \frac{i}{dh} \cdot \frac{B_z}{Ne}$$

$$V = \frac{i B_z}{Neh} = \frac{200 \text{ amp} \times 1.5 \text{ weber/metre}^2}{8.4 \times 10^{28} / \text{m}^3 \times 1.6 \times 10^{-19} \text{ coul} \times 1.0 \times 10^3 \text{ m}}$$

$$= 2.2 \times 10^{-5} \text{ volt}$$

$$= 22 \text{ micro volt}.$$

9.17 THE FORCE ON A MAGNETIC DIPOLE IN A NON-UNIFORM EXTERNAL MAGNETIC FIELD

For simplicity, let us consider a small circular loop of radius r is placed in x-y plane. Further, consider an external fiedl $\vec{B}$ of some other current system, such a s solenoid, pointing along z-axis as shown in Fig. 9.18 (a). The field is non-uniform. Its magnitude decreases as we proceed in the z-direction which is indicated in the figure by the spacing of the field lines. Let us calculate the force acting on the magnetic dipole due to this external field.

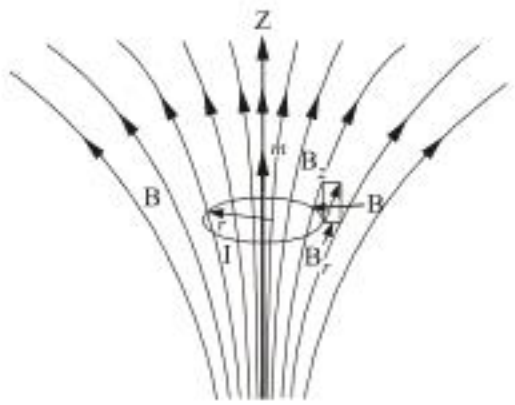


Fig. 9.18 (*a*) A current loop in a non uniform magnetic field.

From the spreading out of the field lines, it is evident that there is an outward (radial) component of $\vec{B}$ everywhere. Let this radial component be $\vec{B}_r$ acting in outward direction as shown in the figure. Due to this field component, each element of the loop will experience a force.

$$d\vec{F} = I\, d\vec{l} \times \vec{B}_r$$

Its magnitude,

$$dF = I dl B_r$$

because $\vec{B}_r$ is radial. The direction of this force is perpendicular to the plane of dl and B_r (or perpendicular to dl and $\vec{B}_r$) *i.e.*, z-direction which is downward if the direction of the current is as shown in diagram. Since we have considered a small loop, B_r can be taken to be constant for the entire loop. Therefore, the total downward force

$$F = \oint dF = I B_r \oint dl = I\, 2\pi r B_r$$

or $$\vec{F} = -2\pi r\, I B_r \hat{z} \qquad \ldots(9.33)$$

Let us consider a small cylinder of radius r and height dz situated close to the loop as shown in Fig. 9.18 (b). Let us calculate the flux, passing through this cylinder. Since B is not uniform, but changing, its space rate of change in +z direction is $\frac{\delta B_z}{\delta z}$.

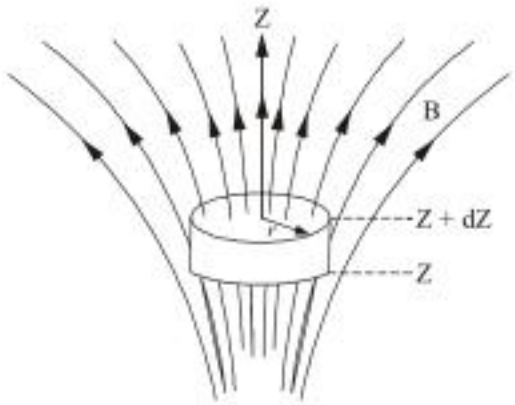


Fig. 9.18 (*b*)

As we know, the flux of magnetic field is $\mathrm{B}.d\vec{\mathrm{S}}$. Therefore,

The outward flux from the curved surface of the cylinder

$$= 2\pi r dz \mathrm{B}_r$$

The inward flux from the lower end-face of the cylinder is

$$= \pi r^2 \mathrm{B}_z$$

Since $\frac{\delta \mathrm{B}_z}{\delta z}$ is the rate of change of B_z, the change for the height dz of the cylinder $= \frac{\delta \mathrm{B}_z}{\delta z} dz$.

Therefore, the field on the top face of the cylinder is

$$= \mathrm{B}_z + \frac{\delta \mathrm{B}_z}{\delta z} dz$$

Thus the outward flux from the top face of the cyinder is

$$= \pi r^2 \left(\mathrm{B}_z + \frac{\delta \mathrm{B}_z}{\delta z} dz \right)$$

Therefore, the net outward flux,

$$= 2\pi r dz \, \mathrm{B}_r + \pi r^2 \left(\mathrm{B}_z + \frac{\delta \mathrm{B}_z}{\delta z} dz \right) - \pi r^2 \, \mathrm{B}_z$$

$$= 2\pi r dz \, \mathrm{B}_r + \pi r^2 \frac{\delta \mathrm{B}_z}{\delta z} dz$$

But we know that div $\vec{\mathrm{B}} = 0$, which means that the net flux of the magnetic field $\vec{\mathrm{B}}$ out of any given volumes must be zero, hence

$$2\pi r dz \mathrm{B}_r + \pi r^2 \frac{\delta \mathrm{B}_z}{\delta z} dz = 0$$

or $$\mathrm{B}_r = -\frac{r}{2} \frac{\delta \mathrm{B}_z}{\delta z}$$

The negative sign indicates the rate of decrease of B_r with increasing z. Substituting in equation (9.33), the force on the magnitic dipole is given by,

$$F = 2\pi rI \frac{r}{2} \frac{\delta B_z}{\delta z}$$

$$= I\pi r^2 \frac{\delta B_z}{\delta z}$$

But $\pi r^2 = A$ area of the loop

and $IA = m$, the magnetic dipole moment,

therefore, $F = m\frac{\delta B_z}{\delta z}$ ...(9.34)

The magnetic dipole moment points in the positive z-direction and in our case parallel to the direction of the external magnetic field.

We thus conclude that when the moment of the dipole is parallel to the external field, the force acts in direction of the incresing field strength.

Clearly, from relation (9.34) the force is independent of the shape of the loop, therefore, small loops of al shapes will experience the same force in the same magnetic field.

If the magnetic moment $\vec{m}$ is not in the direction of the field and making certain angle with $\vec{B}$ and the components of $\vec{B}$ varying in difference directions say, x, y and z, then,

$$\left.\begin{aligned} \vec{F}_x &= \vec{m}.\,\text{grad}\, B_x \\ \vec{F}_y &= \vec{m}.\,\text{grad}\, B_y \\ \vec{F}_z &= \vec{m}.\,\text{grad}\, B_z \end{aligned}\right\} \quad \ldots(9.35)$$

when $\frac{\delta B_z}{\delta_z} = 0$, then in equation (9.34) force is also zero.

REVIEW QUESTIONS

9-1. Explain how magnetic induction $\vec{B}$ may be defined without using the concept of magnetic poles.

9-2. Obtain an expression for the force on the wire placed in magnetic field $\vec{B}$. Explain current element. Why $\vec{B}$ is also called as the magnetic flux density?

9-3. What do you mean by flux of magnetic field? Calculate the magnetic flux over a closed surface. What is the physical significance of the result?

9-4. When a current carrying wire is placed in uniform magnetic field show that torque acting on it is $\tau = m \times \vec{B}$ where m is the magnetic dipole moment.

9-5. Explain magnetic dipole moment and discuss the equivalent of a current carrying coil to magnetic shell (dipole). Show that potential energy $= -\vec{m}.\vec{B}$.

9-6. Given the theorey and working of a ballistic galbvanometer.

9-7. Under what condition does a moving coil gavanometer behave as ballistic?

9-8. Explain electromagnetic damping, critical damping and logarithmic decrement. Show that the charge sensitivity (or quantity sensitivity) is equal to $T/2\pi$ times its current sensitivity.

9-9. What is Hall effect and Hall voltage? Obtain an expression for Hall voltage.

9-10. Obtain an expression for force on a magnetic dipole placed in a non-uniform magnetic field.

PROBLEMS

9.1 A one metre long wire carries a current of 10 amp. and makes an angle 30° with a uniform magnetic field B = 1.5 weber/m^2. What is the magnitude and direction of the force acting on the wire?

9.2 In Bohr model of hydrogen atom the electron ratates around the nucleus in a circular path of radius 5.1×10^{-11} metre at a frequency ν of 6.8×10^{15} revolutions/sec. The value of B set-up at centre of the orbit is 14 weber/m^2. Find the equivalent dipole moment.

9.3 An electron is moving with velocity $2\hat{i} + 3\hat{k}$ cm/sec. in an electric field of intensity $3\hat{i} + 6\hat{j} + 2\hat{k}$ and a magnetic field of $2\hat{j} + 3\hat{k}$ Tesla. Find the magnitude and directin of Loreatz force acting on the electron.

9.4 The current sensitivity of a B.G. is 4.54×10^2 mm/micro amp. on a scale kept at a distance 1 metre. Calculate the charge sensitivity of galvanometer if the time period of the coil is 6.2 sec.

9.5 The current sensitivity of a B.G. is 2.2×10^{-9} amp. for a deflection of 1 mm on a scale kept at a distance of 1 metre. Calculate the charge sensitivity of the galvanometer if the time period of the coil = 6.2 sec.

ANSWERS

1. 7.5 nt, perpendicular to the wire and $\vec{B}$ **2.** 9.0×10^{-24} amp. $metre^2$

3. $F = 9.6 \times 10^{-19}(2i + \hat{k})$, $F = 2.15 \times 10^{-18}$ Newtons, $\theta = \cos^{-1}\frac{2}{\sqrt{5}}$

4. 4.63×10^2 mm/micro coul. **5.** 2.17×10^{-9} coul/mm

SELF-TEST OBJECTIVE QUESTIONS

9.1 In a current carrying conductor is placed in a magnetic field, the force on the conductor will be calculated by

(*a*) Cross product (*b*) Dot product
(*c*) Simple product (*d*) none of the above

9.2 Most commonly used unit of magnetic flux is

(*a*) Wber-m^2 (*b*) Wber/m2
(*c*) Weber (*d*) Henry.

9.3 The direction of force of a current carrying conductor in a magnetic field is given by

(*a*) Cork screw rule (*b*) Lenz's law
(*c*) Left hand rule (*d*) Fleming's right hand rule

9.4 Damping of B.G. is kept small

(*a*) so that it may oscillate
(*b*) it may remain stable
(*c*) so that amplitude of first swing is small
(*d*) so that amplitude of first swiing is large

9.5 A shunt (with reference to current meter) is a resistance

(*a*) connected across the meter to increase its range
(*b*) connected across the meter to reduce its range
(*c*) connected in series to increase its range
(*d*) none of these

9.6 The range of a voltmeter can be increased by

(*a*) connecting a resistance in series
(*b*) connecting a low resistance in parallel
(*c*) connecting a resistance of high value in series with the meter movement
(*d*) none of these

9.7 In a moving coil metres, damping is provided by

(*a*) a separate pair of magnets called damping magnets

(*b*) The aluminium frame on which main coil is wound
(*c*) Neither of the two

9.8 The coil of the moving coil meter is wound on
(*a*) An iron frame (*b*) An aluminium
(*c*) An insulating material (*d*) Neither of the two

9.9 Which of the following statement is correct?
(*a*) Ammeter is nothing but galvanometer with resistance
(*b*) Voltmeter is noting but galvanometer with resistance
(*c*) Ammeter is nothing but galvanometer with low resistance
(*d*) Voltmeter is nothing but galvanometer shorted at two terminals

9.10 If m be the magnetic dipole moment of a current loop, then the torque τ can be expressed as
(*a*) $\tau = \vec{m} \cdot \vec{B}$ (*b*) $\tau = \vec{m} \times \vec{B}$

9.11 Physical meaning of $\nabla \cdot \vec{B} = 0$ so that
(*a*) an isolated monopole exists (*b*) an isolated monopole does not exist

9.12 $\nabla \cdot \vec{B} = 0$ is based on
(*a*) continuity equation (*b*) Faraday's law
(*c*) Gauss's law (*d*) Ohm's law

9.13 An electron with velocity v is injected into a uniform field $\vec{B}$. The force on it is
(*a*) $\perp$ to the direction of motion of the electron
(*b*) $\perp$ to the direction of magnetic field
(*c*) perpendicular to both the direction of motion of the electron and of the magnetic field
(*d*) none of these

9.14 In a strong magnetic field an electron is injected at rest. The force immediately acting on the electron is
(*a*) infinite (*b*) zero (*c*) very small (*d*) very large

9.15 The direction of motion of a conductor in a magnetic field is given by
(*a*) Fleming's rule (*b*) Maxwell's right hand rule
(*c*) Lenz's law (*d*) Kirchhoff's law

9.16 For a galvanomter the curret which will produce unit deflection on a scale at a specific distance is known as
(*a*) reduction factor (*b*) amplification factor
(*c*) tangent factor (*d*) figure of merit.

9.17 If T is the time period of swing of a coil, than ratio of charge sensitivity to current sensitivity is
(*a*) 2π (*b*) $2\pi / T$
(*c*) $2T / \pi$ (*d*) None of these

9.18 Magnetic flux density emerging out of a closed surface is
(a) infinite (b) zero
(c) depends upon the magnetic movement inside the closed boundary
(d) (b) & (c) both are correct

ANSWERS

1. (*a*)	**2.** (*c*)	**3.** (*c*)	**4.** (*d*)	**5.** (*a*)	**6.** (*c*)
7. (*b*)	**8.** (*b*)	**9.** (*c*)	**10.** (*b*)	**11.** (*b*)	**12.** (*c*)
13. (*c*)	**14.** (*b*)	**15.** (*a*)	**16.** (*d*)	**17.** (*b*)	**18.** (*b*)

CHAPTER 10

THE MAGNETIC FIELD DUE TO STEADY CURRENTS

We have introduced the concept of magnetic flux density and represented the magnetic field by a vector $\vec{B}$. Then calculated the forces and torques on current carrying wires. We know the cause of $\vec{B}$ that the current flowing through wire. How the $\vec{B}$ is related to current and the distance from the wire? We would like to know how various factors affect the magnitude and direction of $\vec{B}$. To be more specific, the objective is to formulate the equation from which $\vec{B}$ may be calculated if the current configuration is known.

10.1 MAGNETIC FIELD OF A MOVING POINT CHARGE (LAPLACE RULE)

It was found experimentally that the magnetic induction resulting from a charge q moving with velocity $\vec{v}$ at a distance r away from the charge where r is a vector pointing from the charge to the point where the field is being found, is related by,

$$B \propto \frac{qv \sin\theta}{r^2}$$

where B is the magnitude of the induction and θ is the angle between $\vec{v}$ and $\vec{r}$ as shown in Fig. 10.1. The above equation can be written in terms of equality by inserting a constant $\mu_0 / 4\pi$. Therefore,

$$B = \frac{\mu_0}{4\pi} \frac{qv \sin\theta}{r^2}$$

or
$$\vec{B} = \frac{\mu_0}{4\pi} q \left(\frac{\vec{v} \times \hat{r}}{r^2} \right) \qquad \ldots(10.1)$$

or
$$\vec{B} = \frac{\mu_0}{4\pi} q \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) \qquad \ldots(10.1a)$$

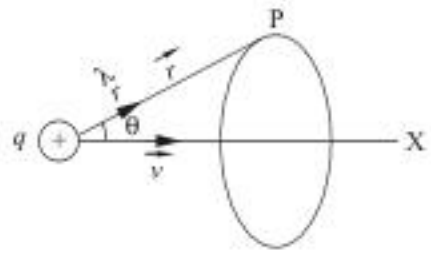


Fig. 10.1

where $\hat{r}$ is unit vector along r,
μ_0 is called permeability of free space.

We have chosen the permeability constant $\frac{\mu_0}{4\pi}$ as for Coulomb's law, we introduced $\frac{1}{4\pi\varepsilon_0}$ and for the same reason *i.e.* in rationalised MKS system the factors π should occur in problems having spherical, cylindrical or circular symmetry and should not occur when these symmetries do not exist.

In fact, certain equations that are derived from equation (10.1) but are used more often than it is, will be simple in form if we introduce $\frac{\mu_0}{4\pi}$. *We shall find the physical significance of the number just later.*

The direction of $\vec{B}$ is determined by the rules of vector product. From equation (10.1), $\vec{B}$ is perpendicular to $\vec{v}$ and $\vec{r}$, *i.e.* in the direction normal to the plane containing $\vec{v}$ and $\vec{r}$ which is plane of the paper.

10.2 THE BIOT-SAVART LAW

The first significant relationship between a current and its magnetic field was discovered in 1820 by Oersted who found that a magnetised needle tends to set itself at right angle to a straight wire in which there is a steady current. In the same year, Biot and Savart formulated the equation for the field due to current in a long straight wire. Finally, they produced a mathematical formula for the field due to a single current element rather than a finite length of wire, the equation can be used to calculate the field due to many different current configurations as shown in Fig. 10.2.

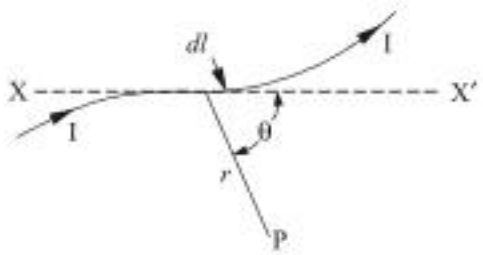


Fig. 10.2

In fact, we are often interested not in the field of moving charge, but in that of an element of current, as a length *dl* of wire carrying a current *I*. We can easily set up an equivalence with equation (10.1). Suppose cross-section of wire in A and ρ is the charge density of charge in wire. Assume this charge moves with velocity *v*.

The charge crossing any cross-section per second = ρ*v*A that is equal to current I

or $\rho v A = I$

If we multiply by *dl* on both sides, we get,

$$\rho v A dl = I dl$$

$$qv = I dl$$

(ρA*dl* is the total charge contained in wire of length *dl*).

So the magnetic induction $d\vec{B}$ due to *dl* length of wire carrying current I is given by the following relation at a point P shown in Fig. 10.2. After substituting $qv = Idl$ in equation (10.1), we get,

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{(d\vec{l} \times \hat{r})}{r^2} \qquad \ldots(10.2)$$

or, $$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{(d\vec{l} \times \vec{r})}{r^3} \qquad \ldots(10.2a)$$

The law described by equations (10.1) and (10.2) is called Biot-Savart law.

Principle of Superposition

The resultant field at P is found by integrating equation (10.2) as the principle of superposition also holds here as in the case of electric field,

$$\vec{B} = \oint d\vec{B}$$

or $$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{I}{r^3} (d\vec{l} \times \vec{r}) \qquad \ldots(10.3)$$

where $\oint$ stands for close path.

The direction of $d\overline{B}$ is normal to the line from the current element Idl to the observation point P. (By the rule discussed in last chapter.)

Right Hand Rule

Grasp the wire with right hand, the thumb pointing in the direction of current, the fingers will curl around the wire in the direction of $\overline{B}$.

10.3 THE FORCE BETWEEN TWO FINITE ELEMENTS OF CURRENT

Consider two current elements $I_1\ dl_1$ and $I_2\ dl_2$ flowing in different conductors as shown in Fig. 10.3. The magnetic induction due to I_2 current at a distance r on dl_1 is,

$$d\overline{B} = \frac{\mu_0}{4\pi} I_2 \frac{(d\vec{l}_2 \times \vec{r})}{r^3} \qquad \text{[From equation (10.2a)]}$$

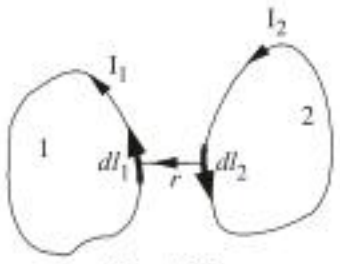


Fig. 10.3

Now the force on conductor carrying current I_1, say dF, due to current I_2 can be calculated by using the relation $(\vec{F} = I\vec{l} \times \overline{B})$. Thus the force between two elements of current $I_1 dl_1$ and $I_2 dl_2$ is given by,

$$d\vec{F} = \frac{\mu_0 I_1 I_2}{4\pi} \left(\frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{r})}{r^3} \right) \qquad \ldots(10.4)$$

This law for current elements is equivalent to Coulomb's law of force between static charges.

Biot-Savart Law and Coulomb's Law of Electrostatics

Eqs. 10.2 and 10.4 reveal that Biot-Savart law is similar to Coulomb's law in electrostatics. The field or force varies inversely as the square of the distance from the current element. It differs from Coulomb's law, since the direction of field is normal to the line from the current element (Idl) to the point of observation, instead of being directed along this line, as in Coulomb's law.

10.4 APPLICATIONS OF BIOT-SAVART LAW

We shall calculate the B in some important cases as given below:

1. Magnetic Field due to Steady Current in a Long Straight Wire

As an example of use of Biot-Savart law, let us calculate the magnetic field vector $\overline{B}$ at a point P a distance R from a long straight wire in which there is a current I as shown in Fig. 10.4.

The magnetic field $d\overline{B}$ due to the current element $Id\vec{l}$ (*i.e.* dl length of the wire carrying current I) of the wire at point P is given by Biot-Savart law *i.e.*

$$d\overline{B} = \frac{\mu_0}{4\pi} I \left(\frac{d\vec{l} \times \hat{r}}{r^2} \right) = \frac{\mu_0}{4\pi} \frac{I dl}{r^2} (\vec{j} \times \hat{r})$$

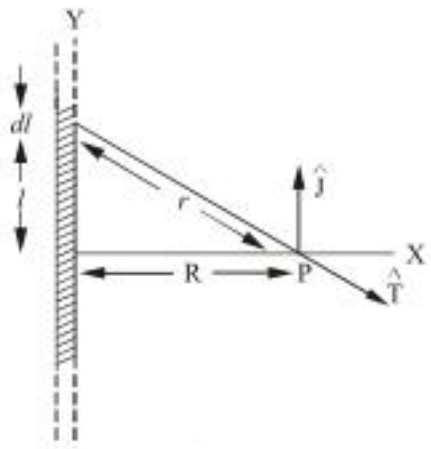


Fig. 10.4

where $\vec{j}$ is unit vector along y-axis and

$\hat{r}$ is unit vector along r.

or $$dB = \frac{\mu_0}{4\pi} \cdot \frac{I\,dl \sin\theta}{r^2}$$

The direction of $d\vec{B}$ at point P for all elements are the same, that is, into the plane of the figure at right angle to the page. The magnitude of $d\vec{B}$ due to whole wire is given by,

$$B = \int dB = \frac{\mu_0 I}{4\pi} \int_{l=-\infty}^{l=+\infty} \frac{\sin\theta\, dl}{r^2}$$

From Fig. 10.4,

$$r = (l^2 + R^2)^{1/2}$$

and, $$\sin\theta = \sin(\pi - \theta) = \frac{R}{r}$$

$$= \frac{R}{(l^2 + R^2)^{1/2}}$$

Putting these values, the expression for B becomes,

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{R\,dl}{(l^2 + R^2)^{3/2}}$$

To evaluate the integral on R.H.S., substitute,

$$l = R\tan\phi,\ dl = R\sec^2\phi\, d\phi$$

therefore,

$$B = \frac{\mu_0 I}{4\pi R} \int_{-\pi/2}^{+\pi/2} \cos\phi\, d\phi$$

$$= \frac{\mu_0 I}{2\pi R} \qquad \ldots(10.5)$$

The direction of $\vec{B}$ is specified by $(\hat{j} \times \hat{r}) = \hat{\theta}_0$, and is perpendicular to the plane containing $\hat{j}$ and $\hat{r}$. Hence,

$$\vec{B} = \frac{\mu_0 I}{2\pi R} \hat{\theta}_0 \qquad \ldots(10.5a)$$

Also the direction of $\vec{B}$ by R.H. rule *i.e.* grasp the wire with right hand, the thumb pointing in the direction of current, the finger will curl around the wire in the direction of $\vec{B}$, hence magnetic field lines are circular closed curves around the wire.

2. Magnetic Field Near a Straight Current Filament of Finite Length

Let us calculate the magnetic field due to a wire of length $2l$ at a point P as shown in Fig. 10.5. The geometry and notations to be used are shown in Fig. 10.5. The wire is placed along x-axis and origin of the coordinate system is at the middle. Further, let the point P be at a distance R from the x-axis.

Consider the contribution to the magnetic field by the current element Idx at P as shown in Fig. 10.5. According to equation (10.2) the magnetic field $d\vec{B}$ is given by,

$$d\vec{B} = \frac{\mu_0}{4\pi} I \left(\frac{d\vec{x} \times \vec{r}}{r^2} \right)$$

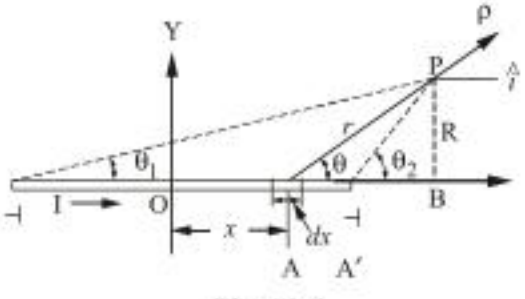


Fig. 10.5

From figure $r^2 = AB^2 + R^2$

and $AB = AA' + A'B$

where $AA' = l - x$ and $A'B = R\cot\theta_2$

Hence $AB = l - x + R\cot\theta_2$

So $r^2 = (l - x + R\cot\theta_2)^2 + R^2$, therefore,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,dx}{(l - x + R\cot\theta_2)^2 + R^2}(\hat{i}\times\hat{r})$$

where, $d\vec{x} = \hat{i}dx$, and $\hat{i}$ = unit vector along x-axis.

From the figure, it is seen that in triangle APB,

$$l - x + R\cot\theta_2 = R\cot\theta$$

then $dx = R\ \mathrm{cosec}^2\,\theta\,d\theta$

and $$\mathrm{cosec}\,\theta = \frac{r}{R} = \frac{\sqrt{(l - x + R\cot\theta_2)^2 + R^2}}{R}$$

and $$dx = \frac{(l - x + R\cot\theta_2)^2\,R^2}{R}d\theta$$

Substituting dx in the expression for $d\vec{B}$, we get,

$$d\vec{B} = \frac{\mu_0}{4\pi}\cdot\frac{I}{R}d\theta\,(\hat{i}\times\vec{r})$$

But the magnitude of the product $i\times r$ is $\sin\theta$ and direction is perpendicular to plane containing $\hat{i}$ and $\hat{r}$,

i.e. $\hat{i}\times\hat{r} = \hat{\theta}_0\sin\theta$

where $\hat{\theta}_0$ is unit vector perpendicular to the plane containing $\hat{i}$ and $\hat{r}$, hence,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I}{R}\sin\theta\,d\theta\,\hat{\theta}_0 \qquad \ldots(10.5\text{ b})$$

Magnitude $$dB = \frac{\mu_0 I}{4\pi R}\sin\theta\,d\theta$$

The total field is obtained by integrating between the limits θ_1 and θ_2 due to length $2l$ of the wire,

$$B = \int_{\theta_1}^{\theta_2} dB = \int_{\theta_1}^{\theta_2}\frac{\mu_0 I}{4\pi R}\sin\theta\,d\theta$$

$$= \frac{\mu_0 I}{4\pi R}\Big|-\cos\Big|_{\theta_1}^{\theta_2}$$

$$B = \frac{\mu_0 I}{4\pi R}(\cos\theta_1 - \cos\theta_2)$$

and directed along $\hat{\theta}_0$, hence,

$$\vec{B} = \frac{\mu_0 I}{4\pi R}(\cos\theta_1 - \cos\theta_2)\,\hat{\theta}_0 \qquad \ldots(10.5\text{ c})$$

Consequently, the magnetic field is circular and by symmetry has its centre on the axis. For the special case when the length becomes very long, thereby approaching the infinite case, *i.e.*,

$$l \to \infty, \quad \theta_1 \to 0, \quad \text{and } \theta_2 \to \pi$$

then,

$$\bar{B} = \frac{\mu_0 I}{2\pi R} \hat{\theta}_0 \qquad \ldots(10.5d)$$

which is same as given in equation (10.5).

3. Interaction between Two Parallel Long Currents

Fig. 10.6 shows two long parallel wires separated by distance d and carrying currents I_a and I_b. It is an experimental fact, noted by Ampere only one week after Oersted's magnetic phenomena of electric current. Wire a will produce a field of induction B_a at all nearby points. The magnitude of B_a, due to current I_a at a distance d *i.e.* on wire b is,

$$B_a = \frac{\mu_0 I_a}{2\pi d} \qquad \text{[From equation (10.5)]}$$

According to the R.H. rule the direction of B_a is downward as shown in Fig. 10.6. Wire b, which carries a current I_b, is now really placed in such field. Consider the l length of the wire b. It will experience a force $(I\vec{l} \times \vec{B})$ whose magnitude is,

$$F_b = I_b l B_a = \frac{\mu_0 l I_a I_b}{2\pi d} \quad \text{(After substituting } B_a \text{ given above)} \qquad \ldots(10.6)$$

Direction of F_b is determind according to vector rule F_b lies in the plane of the wires and points to the left.

One can start with wire b, compute the field of induction which it produces at the site of wire a, and then compute the force on wire a, *for parallel currents, F_a points to the right. The forces that the two wires exert on each other are equal and opposite, hence according to Newton's* law of action and reaction, *they will attract. For antiparallel currents, the two wires repel each other.* We can compare the above discussion with that of two point charges. There we explained that two charges act on each other through the intermediary of the electric field. Here two conductors in same manner by magnetic field. We think as,

current $\rightleftharpoons$ field

and not as in the action at a distance point of view in terms of,

current $\rightleftharpoons$ current

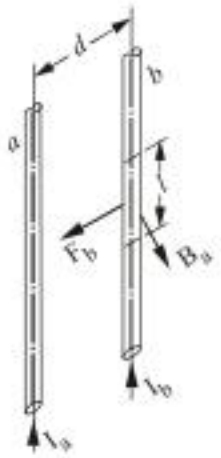


Fig. 10.6

Unit of Current in Rationalised MKS

The force of attraction between two very long parallel wires is used to define the ampere practically. The ampere is that current which when flowing through two long parallel straight conductors placed in vacuum at a distance of one metre from each other produces a force of attraction of 2×10^{-7} Newtons per metre length between the wires, *i.e.* when in equation (10.6),

$l = 1$ metre, $I_a = I_b = 1$ amp, $= 1$ metre

$F = 2 \times 10^{-7}$ Newtons

Hence $\mu_0 = 4\pi\times10^{-7}$ Newtons/amp^2

$$= 4\pi\times10^{-7}\ \frac{\text{Newtons metre}}{\text{Ampere}^2\ \text{metre}}$$

$$= 4\pi\times10^{-7}\ \frac{\text{Weber}}{\text{Ampere metre}}$$

Since, $1\ \text{Weber} = \dfrac{1\ \text{Newton metre}}{1\ \text{Ampere}}$

'Basically' the ampere is defined by equation (10.4). Thus if we take two current elements each of the unit length, and at unit distance apart, so oriented that the factor,

$$\frac{d\vec{l}_1\times(d\vec{l}_2\times\vec{r})}{r^3} = 1$$

Then the currents are defined to an ampere, *i.e.* $I_1 = I_2 = 1$ amp, if the resulting force is 10^{-7} Newtons. From this it follows by equation (10.4),

$$\mu_0 = 4\pi\times10^{-7}\ \text{Newton/ampere}^2$$

$$\mu_0 = 4\pi\times10^{-7}\ \text{Weber/amp.metre}$$

Be careful to note the following facts:

The value assigned to μ_0 is seen to be the result of definition rather than experiment. This, on the other hand, is not the case with ε_0, which we encountered in Coulomb's law of electrostatics. The difference is that we define the unit of current, the ampere, from the Biot-Savart law, whereas, we define the Coulomb, the unit of charge, as the charge flowing per second in a current of 1 ampere. Therefore, we are not free to choose the constant *i.e.* ε_0 in Coulomb's law at our own choice but must determine it by experiment.

4. Magnetic Field along the Axis of Circular Coil

Consider a circular coil of radius R and carrying a current I. The vector $d\vec{l}$ for a current element at the top of the coil points perpendicular towards the reader. The $d\vec{l}$ and $\vec{r}$ are perpendicular to each other and plane formed by $d\vec{l}$ and $\vec{r}$ is normal to the page as shown in Fig. 10.7.

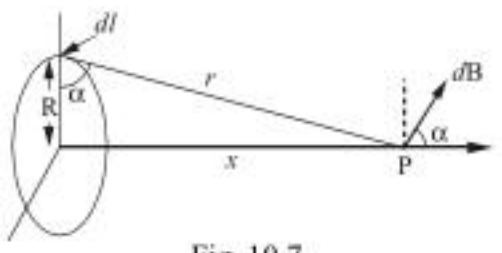


Fig. 10.7

The $d\vec{B}$ for this element is at right angle to this plane, this lies in the plane of the figure and perpendicular to r. The magnitude of dB from equation (10.2) is given by

$$dB = \frac{\mu_0}{4\pi}\cdot\frac{k\,dl}{r^2} \qquad \ldots(10.7)$$

Let us resolve $d\vec{B}$ into two components, one, $d\vec{B}\parallel$, along the axis of loop and another, dB $\perp$, at right angles to x-axis. As the coil is symmetrical about the x-axis the contribution $d\vec{B}$ due to element of coil on opposite side the dB $\perp$ will be equal in magnitude but opposite in direction and therefore, cancel out, we have only dB $\parallel$ component. The resultant B for the complete loop is given by,

$$B = \int dB$$

$$= \int \frac{\mu_0 I}{4\pi r^2} dl \cos\alpha$$

Since $dB_{\parallel} = dB\cos\alpha$ and dB is given in Eq. (10.7) and from figure.

$$r = \sqrt{R^2 + x^2}$$

$$\cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + x^2}}$$

therefore,

$$B = \int \frac{\mu_0 I}{4\pi} \cdot \frac{R dl}{(R^2 + x^2)^{3/2}}$$

$$= \frac{\mu_0 IR}{4\pi (R^2 + x^2)^{3/2}} \int dl$$

$\int dl$ is the circumference of the loop $2\pi R$.

$$B = \frac{\mu_0 IR^2}{2(R^2 + x^2)^{3/2}} \quad \ldots(10.8)$$

If coil having N turns, then $B = \frac{\mu_0 NIR^2}{2(R^2 + x^2)^{3/2}}$ $\ldots(10.8a)$

This shows the B decreases as the distance x increases. At the centre $x = 0$

$$B_{centre} = \frac{\mu_0 IR^2}{2R^3} = \frac{\mu_0 I}{2R} \quad \ldots(10.9)$$

If $x \gg R$ *i.e.* point P lies far from coil in Eq. (10.8), R in denominator can be neglected. Hence,

$$B = \frac{\mu_0 IR^2}{2x^3} \quad \ldots(10.10)$$

If coil having N turns, then,

$$B = \frac{\mu_0 INR^2}{2x^3} \quad \ldots(10.11)$$

Recalling the area of the coil is πR^2,

$$B = \frac{\mu_0 (NIA)}{2\pi x^3}, (A = \pi R^2)$$

and NIA according to definition given in the last chapter (see eq. 9.12) is magnetic dipole moment of the current coil represented by m

$$B = \frac{\mu_0 m}{2\pi x^3} \quad \ldots(10.12)$$

Variation of the field along the axis of coil and Principle of Helmholtz galvanometer

Differentiating Eqn. (10.8) with respect to x, we get,

$$\frac{dB}{dx} = \frac{\mu_0 IR^2}{2}\left(-\frac{3}{2}\right) \cdot 2x (R^2 + x^2)^{-5/2}$$

$$\frac{d^2 B}{dx^2} = -\frac{3\mu_0 IR^2}{2}\left[\left(R^2+x^2\right)^{-5/2} + x\left(-\frac{5}{2}\right)\cdot 2x\,(R^2+x^2)^{-7/2}\right]$$

or, $$\frac{d^2 B}{dx^2} = -\frac{3\mu_0 IR^2}{2}(R^2+x^2)^{-7/2}\left[R^2+x^2-5x^2\right]$$

$\frac{d^2 B}{dx^2}$ is zero when $R^2+x^2-5x^2=0$

or $$x = \pm\frac{R}{2} \qquad \ldots(10.13)$$

and at those points $\frac{d\bar{B}}{dx}$ is constant. Hence, at a distance of $\frac{R}{2}$ on either side of the coil, the rate of change of the field becomes constant. This is the principle of Helmholtz galvanometer. The variation is shown in Fig. 10.8.

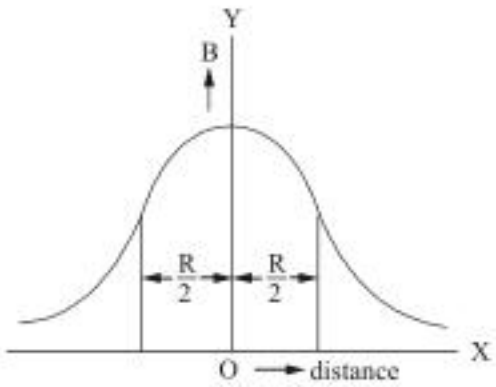


Fig. 10.8

Helmholtz Galvanometer

It is an improved form of the tangent galvanometer. It consists of two equal coils placed coaxially at a distance equal to the radius of the either coil as shown in Fig. 10.9 (a). The two coils are of the same radius and are connected in series so that the current through them is in the same direction. The field at any point on the axis of the coil is equal to the sum of the magnetic field due to individual coil.

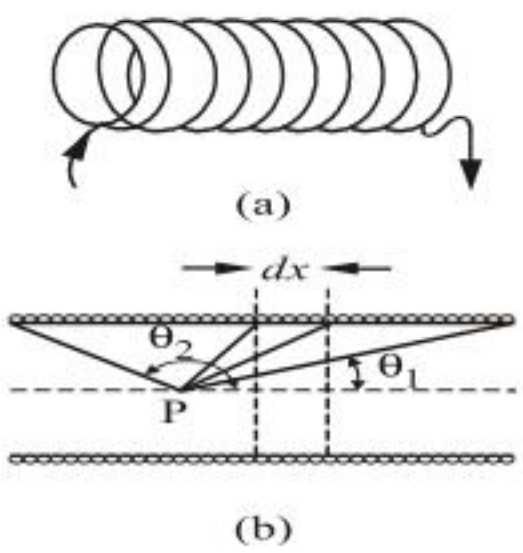


Fig. 10.9 (a)

Since the rate of change of field is constant at a distance $\frac{R}{2}$ from each coil as we discussed in previous article (Eq. 10.13). Therefore, *the field at a point mid-way between the coils is most uniform. As we pass away from this point, the increase in the field due to this coil is compensated by an equal decrease in the field due to the other coil. Thus the field is quite uniform over a considerable distance and magnetic needle rotates in this uniform magnetic field. And it is not necessary to use a short magnetic needle as in the case of tangent galvanometer.* So that we can use a large magnetic needle, which provides a better accuracy in measuring the current.

The magnetic field at the middle point between the two coils is obtained from equation (10.8). Since there are two coils, replacing N by 2N and putting X = R/2. Thus,

$$\bar{B} = \frac{\mu_0 2NIR^2}{2\left(R^2+\frac{R^2}{4}\right)^{3/2}}$$

$$= \frac{8}{\sqrt{125}} \left(\frac{\mu_0 NI}{R} \right) \qquad \ldots(10.13a)$$

If H be the horizontal component of the earth's magnetic field and needle shows a deflection θ, then by tangent law,

$$B = H \tan \theta$$

therefore,

$$\frac{8}{\sqrt{125}} \left(\frac{\mu_0 NI}{R} \right) = H \tan \theta$$

or $$I = \frac{\sqrt{125}}{8\mu_0 N} RH \tan \theta \qquad \ldots(10.13b)$$

In Fig. 10.9 (b) the curve representing the variation of magnetic fields of two coils with distance.

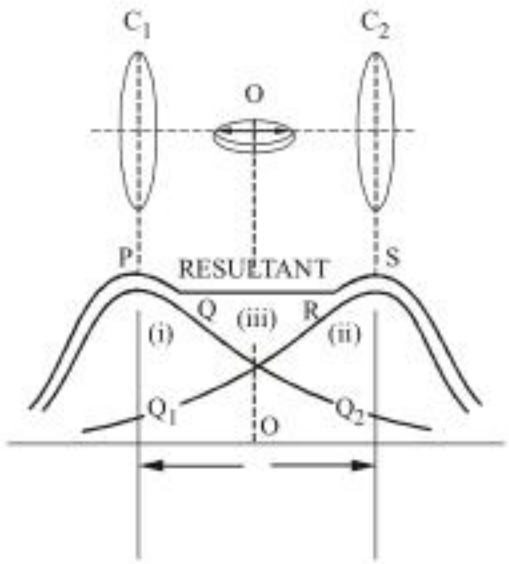


Fig. 10.9 (*b*)

The curve PQRS gives the resultant field and is obtained by adding the ordinates of two curves at every point. The straight portion QR of this curve indicates a uniform field near the point Q, *i.e.* middle point of the two coils.

5. Field on Axis of Solenoid

The result of Eqn. (10.10) may be extended to find vector $\vec{B}$ at a point P on the axis of a solenoid. A coil of wire wound in the form of cylinder is called solenoid, Fig. 10.10 (a). Assuming the uniform winding, so that the number of terms per unit length of the solenoid are constant, say, *n*.

Let us first calculate the field due to a small segment of length *dx* of the solenoid which subtends angle θ and $\theta + d\theta$ with the axis at the point P. From Fig. 10.10 (b) we have,

$$\frac{r d\theta}{dx} = \sin \theta$$

or $$dx = \frac{r d\theta}{\sin \theta}$$

The number of turns of the length $dx = \dfrac{nrd\theta}{\sin\theta}$ and the total current in this element $dx = \dfrac{\mathrm{I}nrd\theta}{\sin\theta}$

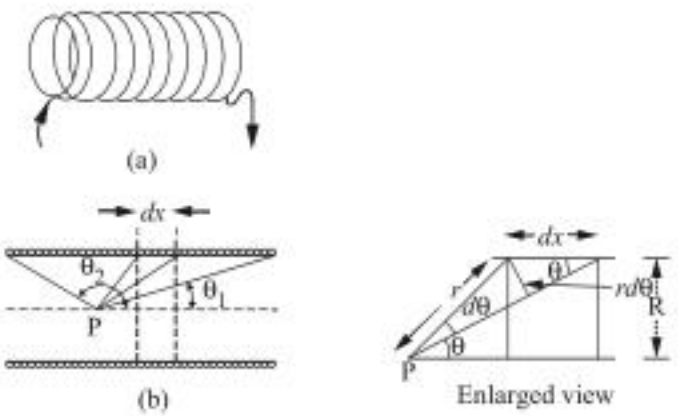


Fig. 10. 10 (*a*) Fig. 10.10 (*b*)

where I is the current flowing in the solenoid. Putting this in Eqn. (10.10) in place of I we get the contribution of the current rings in a length *dx* of the solenoid. The direction of the field is along the axis of coil that is *x*-axis hence *d*B due to *dx* element is given by,

$$d\mathrm{B} = \frac{\mu_0 \mathrm{R}^2}{2r^3}\frac{\mathrm{I}nrd\theta}{\sin\theta}, \text{ since } \sin\theta = \frac{\mathrm{R}}{r}$$

we get $$= \frac{\mu_0 \mathrm{I}n \sin\theta\, d\theta}{2}$$

The field due to whole solenoid is

$$\mathrm{B} = \frac{\mu_0 \mathrm{I}n}{2}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta$$

$$= \frac{\mu_0 \mathrm{I}n}{2}(\cos\theta_1 - \cos\theta_2) \qquad \ldots(10.14)$$

Note that if the fingers of the right hand are curled around the solenoid in the direction of circulations of the current, the thumb will point in the direction of B.

If the **solenoid is very long and P is somewhere near the middle,** $\theta_1 = 0$ and $\theta_2 = \pi$, then,

$$\mathrm{B} = \mu_0 \mathrm{I}n \qquad \ldots(10.15)$$

If the point P lies at one end of the solenoid, $\theta_1 = 0$, **and** $\theta_2 = \dfrac{\pi}{2}$ and thus,

$$\mathrm{B} = \frac{\mu_0 n\mathrm{I}}{2} \qquad \ldots(10.16)$$

This shows that $\vec{B}$ *at the end of a long solenoid is just one half of that at the centre,* the variation is shown in Fig. 10.10 (c) and it is clear that B at the centre of the solenoid remains nearly constant until we reach one of the ends.

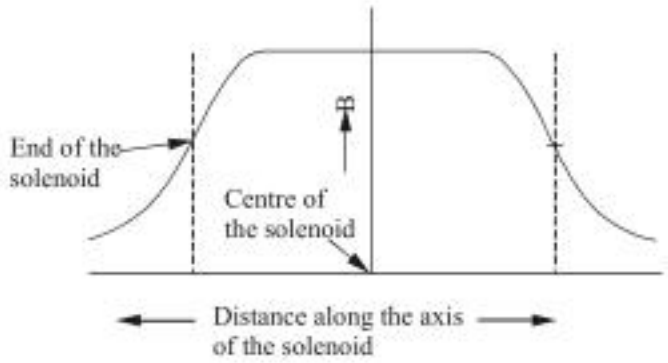


Fig. 10.10 (*c*)

Example 10.1 *Two long parallel wires separated by 2 cm in air carry current of 100 amp. Find the force on one metre length of conductor.*

Solution. Using relation (10.6), *i.e.*

$$F = \frac{\mu_0 l I_a I_b}{2\pi d}$$

As given $l = 1$ metre

$$I_a = I_b = 100 \text{ amp.}$$

$$d = 2\text{cm} = 2\times10^{-2} \text{ metre}$$

and putting $\mu_0 = 4\pi\times10^{-7}$ weber/amp.metre

$$F = 4\pi\times10^{-7}\frac{\text{weber}}{\text{amp.m}}\times\frac{1}{2\pi}\times\frac{1\text{m}\times100\times100\,\text{amp}^2}{2\times10^{-2}\,\text{metre}}$$

$$= 0.1\frac{\text{weber.amp}}{\text{metre}}$$

$$= 0.1 \text{ Newton}$$

Example 10.2 *Find the magnetic induction at the centre of square current loop. The geometry of the problem is shown in Fig. 10.11.*

Solution. Magnetic field at the centre of square current loop

Consider a square current loop of side a carrying current I. The magnetic field at P is obtained directly from the field due to a straight conductor of finite length a as given by equation (10.5c). Here the angle $\theta_1 = 45°$ and $\theta_2 = 135°$. Each current element of sides produces an equal contribution to the field at P. Therefore from equation (10.5c),

$$B = \frac{4\mu_0 I}{4\pi a/2}(\cos 45° - \cos 135°)$$

$$= \frac{2\sqrt{2}\mu_0 I}{\pi a}$$

$\overline{B}$ lies in a direction perpendicular to x-y plane *i.e.* z-axis, hence all contributions are along z-axis. Therefore, we can write,

$$\overline{B} = \frac{2\sqrt{2}\mu_0 I}{\pi a}\hat{k}$$

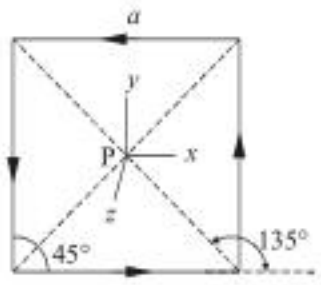


Fig. 10.11

Example 10.3 *Find an expression for the magnetic induction at the centre of a circular current loop.*

Solution. Magnetic field at the centre of a circular current loop

Consider a circular current loop of radius R carrying current I. Let us first calculate the magnetic field $\overline{B}$ due to small element $d\vec{l}$ from diagram shown in Fig. 10.12.

$$d\vec{l} = R\, d\theta\, \hat{\theta}_0$$

where $\hat{\theta}_0$ is a unit vector in the θ direction. Note that since $d\vec{l}$ is small means $d\theta$ is small. The unit vecotors, $\hat{r}$ and $\hat{\theta}_0$ are perpendicular to each other at all points along the path. Then, from equation (10.2), the field due to this element at the centre,

$$d\overline{B} = \frac{\mu_0}{4\pi}\frac{I\,(d\vec{l}\times\hat{r})}{R^2}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{R\, d\theta\, \hat{\theta}_0 \times \hat{r}}{R^2}\right)$$

$$= \frac{\mu_0}{4\pi} I \frac{R\, d\theta}{R^2}(\hat{\theta}_0 \times \hat{r})$$

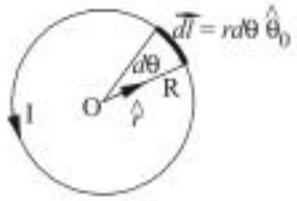


Fig. 10.12

Hence, total $\overline{B}$ due to coil at centre O,

$$B = \int_0^{2\pi} d\overline{B} = (\hat{\theta}_0 \times \hat{r})\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{I d\theta}{R}$$

$$= (\hat{\theta}_0 \times \hat{r})\frac{\mu_0}{4\pi}\frac{2\pi I}{R}$$

$$= \frac{\mu_0 I}{2R}\hat{k}.$$

$\hat{\theta}_0 \times \hat{r} = \hat{k}$ is a unit vector along z-axis because B is directed along the direction perpendicular to the plane containing $\hat{\theta}_0$ and $\hat{r}$ *i.e.* xy-plane here. Hence, B is directed out of the plane of the paper, *i.e.* z-axis.

Example 10.4 *A current of 10 amp. flows in each of the two conducting wires parallel to each other. The separation between the wires is 2 cm. Find the force per unit length of one of the wires. Will it be a force of attraction or repulsion?*

Solution. $F = \frac{\mu_0 I_1 I_2}{2\pi r}$ Newtons /metre

Here $I_1 = I_2 = 10$ amp, R = 2 cm = 0.02 m,

therefore,

$$F = \frac{(4\pi\times10^{-7}\ \text{Newtons/amp}^2)\,(10\ \text{amp})\,(10\ \text{amp})}{2\pi\times(0.02\,\text{m})}$$

$$= 10^{-3}\ \text{Newtons/m}.$$

Example 10.5 *Two similar coils of wire having a radius of 7 cm and 60 turns have a common axis and are 18 cm apart. Find the strength of the magnetic field at a point midway between them on their common axis, when a current of 0.1 amp. is passed through them.*

Solution. The field is given by equation (10.8).

$$B = \frac{\mu_0 n I R^2}{2 (R^2 + x^2)^{3/2}} \text{ Weber/metre}^2$$

The field due to both the coils is given by,

$$B = 2 \times \frac{\mu_0 n I R^2}{2 (R^2 + x^2)^{3/2}}$$

Here $I = 0.1$ amp., $n = 60$, $\mu_0 = 4\pi \times 10^{-7}$ weber/amp. metre

$R = 7$ cm $= 0.07$ m, $x = 9$ cm $= 0.09$ m

therefore,

$$B = \frac{(4\pi \times 10^{-7}) \times 60 \times 0.1 \times (0.07)^2}{[(0.07)^2 + (0.09)^2]^{3/2}} \text{ Wb/m}^2$$

$$= 2.5 \times 10^{-5} \text{ weber/metre}^2 .$$

Example 10.6 *If the magnetic field normal to the plane of a circular coil of n turns and radius r which carries a current I amp is measured on the axis of the coil at a small distance h from the centre of the coil, show that it is smaller than at the centre by the fraction*

$$\frac{3}{2}\frac{h^2}{r^2}.$$

Solution. Using equation (10.8) the field at a distance h from the centre of the circular coil of radius r, is given by,

$$B = \frac{\mu_0 n I r^2}{2 (r^2 + h^2)^{3/2}}$$

$$= \frac{\mu_0 n I r^2}{2r^3 (1 + h^{2/r^2})^{3/2}}$$

$$= \frac{\mu_0 n I}{2r}\left(1 + \frac{h^2}{r^2}\right)^{-3/2}$$

Since h is very small as compared with r, therefore, using binomial theorem, we get,

$$B = \frac{\mu_0 n I}{2r}\left(1 - \frac{3}{2}\frac{h^2}{r^2}\right)$$

at the centre $h = 0$.

Therefore, the magnetic field at the centre of the coil,

$$B_0 = \frac{\mu_0 n I}{2r}$$

The decrease in the magnetic field $= B_0 - B$

$$= \frac{\mu_0 nI}{2r} - \frac{\mu_0 nI}{2r}\left(1 - \frac{3}{2}\frac{h^2}{r^2}\right)$$

$$= \frac{\mu_0 nI}{2r} \times \frac{3}{2}\frac{h^2}{r^2}$$

Therefore, the fraction of the decrease of magnetic field

$$= \frac{B_0 - B}{B_0} = \frac{3}{2}\frac{h^2}{r^2}.$$

Example 10.7 *A long solenoid of length 1 metre and mean radius 10 cm consists of 1000 turns of wire. A current of 20 amps flows through it. Calculate the magnetic field on its axis*

(i) at its centre

(ii) at one of its ends.

Solution. The field at any point on the axis of a solenoid is given by equation (10.14).

$$B = \frac{\mu_0 nI}{2}(\cos\theta_1 - \cos\theta_2)$$

As given, $I = 20$ amp, $n = 1000$ turns/metre

$$\mu_0 = 4\pi \times 10^{-7} \text{ weber/amp. metre}$$

(*i*) Referring to Fig. 10.13(a) we have at the centre O,

$$\cos\theta_1 = \frac{0.05}{\left[(0.5)^2 + (0.05)^2\right]^{1/2}}$$

$$= \frac{0.5}{[0.2525]^{1/2}}$$

and, $$\cos\theta_2 = -\cos(\pi - \theta_1) = -\frac{0.5}{(0.2525)^{1/2}}$$

therefore,

$$B = \frac{(4\pi \times 10^{-7} \text{ weber/amp.metre})\,(100 \text{ turns/m}) \times (20 \text{ amp})}{2} \times \frac{1}{(0.2525)^{1/2}}$$

$$= 2.502 \times 10^{-2} \text{ Weber/metre}^2$$

(*ii*) Referring to Fig. 10.13(b), we have at an end P,

$$\cos\theta_1 = \frac{1}{\left[(1)^2 + (0.05)^2\right]^{1/2}}$$

$$= \frac{1}{\sqrt{(1.0025)}}$$

$$\cos\theta_2 = \cos\frac{\pi}{2} = 0$$

therefore, $$B = \frac{(4\pi \times 10^{-7} \times 1000 \times 20)}{2} \times \left(\frac{1}{\sqrt{(1.0025)}} - 0\right)$$

$$= 1.256 \text{ weber/metre}^2$$

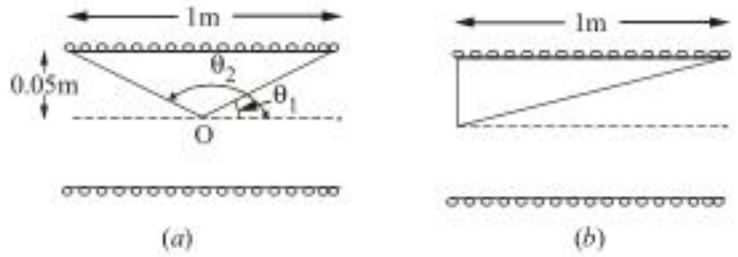


Fig. 10.13

10.5 THE SOLID ANGLE FORM OF BIOT-SAVART LAW AND THE MAGNETIC SCALAR POTENTIAL

The Biot-Savart law can be written as in the case of closed current loop,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times \hat{r}}{r^2}.$$

We now wish to show that the magnetic field near a current loop is proportional to the gradient of the solid angle subtended by the loop at the point where $\vec{B}$ is to be found. Let us displace the point P in Fig. 10.14 by an infinitesimal distance $d\vec{r}$ and find the value of $\vec{B} . d\vec{r}$

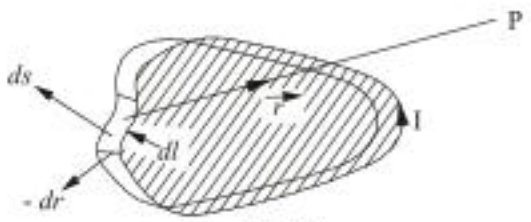


Fig. 10.14

$$\vec{B} \cdot d\vec{r} = \frac{\mu_0 I}{4\pi} \left(\oint \frac{d\vec{l} \times \hat{r}}{r^2} \right) \cdot d\vec{r}$$

The value of R.H.S. remains the same whether we first evaluate the integral then take the dot product with $d\vec{r}$

or we first take the product of $\frac{d\vec{l} \times \hat{r}}{r^2}$ with $d\vec{r}$ and then evaluate the integral around the whole circuit.

Hence we can write,

$$\vec{B} \cdot d\vec{r} = \frac{\mu_0 I}{4\pi} \oint \frac{(d\vec{l} \times \hat{r}) \cdot d\vec{r}}{r^2}$$

By the rules of triple scalar product *i.e.*,

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B}$$

$$(d\vec{l} \times \hat{r}) \cdot (d\vec{r} = (d\vec{r} \times d\vec{l}) \cdot \hat{r}$$

Therefore,

$$\vec{B} \cdot d\vec{r} = \frac{\mu_0 I}{4\pi} \oint \frac{(d\vec{r} \times d\vec{l}) \cdot \hat{r}}{r^2}$$

$$= \frac{\mu_0 I}{4\pi} \oint \frac{\left(d\vec{l} \times (-d\vec{r})\right) \cdot \hat{r}}{r^2} \qquad \left(\because \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}\right)$$

Now, $d\vec{l} \times (-d\vec{r}) \cdot \frac{\hat{r}}{r^2}$ can be interpreted in terms of solid angles. The interpretation is clear, when the point P is displaced by an amount dr, the solid angle Ω subtended by the circuit at P changes by $d\Omega$. If we keep P fixed and move the entire current or, in other words, each point of the circuit is displaced by an amount – dr, even then the change in solid angle is $d\Omega$.

The quantity $d\vec{l}\times(-d\hat{r})$ is a vector representing the small area $d\vec{S}$ shown as a ribbon surface in Fig. 10.14 *i.e.* $d\vec{S}=d\vec{l}\times(-d\hat{r})$. The projection of this area on a plane normal to r is given by

$$d\vec{l}\times(-d\hat{r})\cdot\hat{r}$$

Therefore,

$$\frac{d\vec{l}\times(-d\vec{r})\cdot\hat{r}}{r^2}=\frac{d\vec{S}\cdot\hat{r}}{r^2}=d\Omega$$

where $d\Omega$ represents the element of solid angle subtended by the area dS at P. Thus, the integral over the closed loop gives the entire solid angle subtended by ribbon. This solid angle is shown in Fig. 10.15. Here, it represents the decrease of the entire solid angle Ω subtended at P, that occurs when P moves a distance $d\hat{r}$ away from the loop or the loop moves $-d\vec{r}$ away from P. Therefore, we write,

$$\oint\frac{d\vec{l}\times(-d\vec{r})\cdot\hat{r}}{r^2}=-d\Omega$$

Since Ω is a scalar function,

$$d\Omega=\vec{\nabla}\Omega\cdot d\vec{r}$$

therefore, $\oint\limits_1\frac{d\vec{l}\times(-d\vec{r})\cdot\hat{r}}{r^2}=-\nabla\Omega\cdot d\vec{r}$

Hence,

$$\vec{B}\cdot d\vec{r}=-\frac{\mu_0 I}{4\pi}\nabla\Omega\cdot d\vec{r}$$

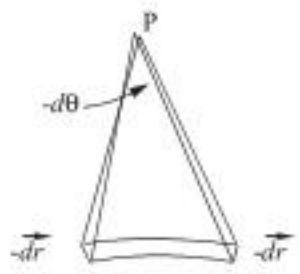


Fig. 10.15

or $$\vec{B}=-\frac{\mu_0 I}{4\pi}\nabla\Omega=-\nabla\left(\frac{\mu_0 I\Omega}{4\pi}\right) \qquad \ldots(10.17)$$

This Eqn. is the solid angle form of Biot-Savart law or the Ampere's law. The direction of $\vec{B}$ is that of $-\nabla\Omega$, so that $\vec{B}$ points away from the loop along its positive normal Eqn. (10.17) is the solid angle form of Ampere law. Now as we know that Ω is a scalar function. It is clear from eqn. (10.17) that the magnetic induction $\vec{B}$ is expressed in terms of scalar function Ω (x, y, z) of the coordinate of P which is analogous to the relation of electrostatics,

$$\vec{E}=-\nabla V$$

We can write Eqn. (10.17) as

$$\vec{B}=-\nabla V_m \qquad \ldots(10.18)$$

where, $$V_m=\mu_0\frac{I\Omega}{4\pi}$$

$$=\frac{\mu_0}{4\pi}\times\text{current}\times\text{solid angle} \qquad \ldots(10.19)$$

and for collection of loops,

$$V_m=\frac{\mu_0}{4\pi}\sum_k I_k\,\Omega_k \qquad \ldots(10.19a)$$

V_m is a scalar potential whose negative gradient gives the magnetic induction $\vec{B}$. This is generally called a 'scalar potential' since it is scalar quantity, to distinguish it from the vector potential, which we shall soon introduce. It is clear that, in cases, in which $\vec{B}$ can be derived from a scalar potential, we must have curl $\vec{B}=0$, since the curl of any gradient of scalar is zero.

Comment on Scalar Potential

The electrostatic potential is single valued. However, the potential V_m is not single-valued since the solid angle is not single-valued. In fact solid angle is not a single-valued function of position P. If point P makes a series of circuits through the loop, solid angle (Ω) will change to $\Omega + 4\pi n$ where n is some positive or negative integer. Unlike electrostatic potential, the difference between the scalar potential between any two points is also undefined unless the path followed between the two points is specified. This is so because each time the path links the circuit, the solid angle along the path changes by 4π and the difference in magnetic scalar potential changes by,

$$\frac{\mu_0 I}{4\pi}(\Omega + 4\pi - \Omega) = \mu_0 I.$$

Thus, the magnetic scalar potential V_m is multiple valued. *This method may be used for calculating the potential of conductors of negligible cross-sections. It cannot be applied for points within the conductors themselves.* It can be used for points well outside the current circuit under consideration. We can choose the value of Ω and hence of V_m which goes to zero at infinity. Hence V_m can be defined by an equation similar to the electric potential with reference point chosen at infinity.

$$V_m = -\int_{\infty}^{P} \vec{B} \cdot d\vec{l}$$

Unit. From relation $V_m = \frac{\mu_0 I\Omega}{4\pi}$, the MKS unit has dimensions of μ_0 times amperes. *The unit for V_m is not completely given. We have not interpreted V_m as representing energy for the reason that $\vec{B}$ does not directly represent a force on a real entity. If single magnetic pole really existed then $\vec{B}$ would be the force on unit pole at the point in question. Hence we say V_m is a sort of fictional potential energy per unit pole.*

10.6 THE MAGNETIC DIPOLE

Suppose we have a current I flowing in a positive direction around a very small loop of area A, the normal to the loop being $\hat{n}$. If θ is the angle betwen $\hat{n}$ and the vector from the loop to a point P, as shown in Fig. 10.16, the solid angle subtended by A at P will be $A\cos\frac{\theta}{r^2}$, so that the scalar potential will be by eqn. (10.19)

$$= \frac{\mu_0 I}{4\pi}\frac{A\cos\theta}{r^2}$$

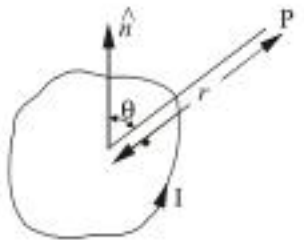


Fig. 10.16

This expression depends on position of the point P, just like the potential of an electric dipole (discussed in Chapter 5),

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\bar{p}\cdot\hat{r}}{r^2} \qquad \text{[see Eqn. (5.22)]}$$

Thus, a small loop carrying a current has magnetic induction $\bar{B}$, whose lines of force are like those of an electric dipole, we call such a small loop as magnetic dipole and define the product IA as the dipole moment. If this moment is represented by m, we have,

$$\text{Potential of magnetic dipole} = \frac{\mu_0 m\cos\theta}{4\pi r^2} = \frac{\mu_0}{4\pi}\frac{\vec{m}\cdot\vec{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{\bar{m}\cdot\bar{r}}{r^2} \qquad \ldots(10.20)$$

Magnetic Shell and its Equivalence with a Current Loop

In terms of this formulation, there is an interesting way in which we can regard the magnetic field of a current loop as carrying from a distribution of magnetic dipole over a surface spanning the loop shown in Fig. 10.17, we may divide the surface into many small loops, let a current I flow in each of them in positive direction. These currents will cancel each other on all the interior boundaries of the loops, but not over the outer boundary. Thus, the sum of the currents of all the small loops will give just the current originally present in the large loop. So that the magnetic field of the small loops must equal that of large loop. Each of the small loops of area A, however, produces a field like a dipole of moment IA. In other words, a distribution of dipoles over surface, whose dipole moment per unit area is I, distributed over any surface spanning the current loop will have the same magnetic induction at all points that the current loop itself has.

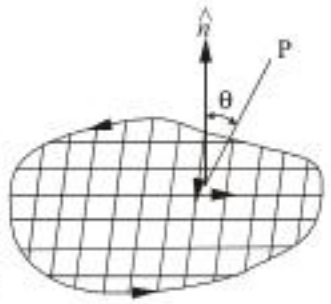


Fig. 10.17

Strength of magnetic shell

It may be defined as the magnetic moment per unit ara. Therefore, strength of the magnetic shell

$$= \frac{IA}{A} = I$$

Potential at a point very close to the surface

$$V_m = \frac{\mu_0 I}{4\pi} \times \text{solid angle}$$

In this case solid angle $= 2\pi$.

Hence,

$$V_m = \frac{\mu_0}{4\pi} I \cdot 2\pi = \frac{\mu_0 I}{2} \qquad \ldots(10.21)$$

At a point inside a closed magnetic shell,

Solid angle $= 4\pi$

Hence,

$$V_m = \mu_0 I \qquad \ldots(10.22)$$

10.7 AMPERE'S CIRCUITAL LAW OR AMPERE'S LAW

Ampere by using relation $\tau = m\,B \sin\theta$ (discussed in Chapter 9) in which τ is torque, m is the magnitude of the magnetic moment and θ is the angle between the direction of B and normal to thewire, measured B by bringing a compass needle near a straight wire. Even through m for a compass needle is not known, m may be regarded as constant independent of the position or orientation of the needle. Thus by measuring τ and θ one can obtain a relative measure of B for various distances r and for various currents I in the wire. The experimental results can be represented mathematically,

$$B \propto \frac{I}{r}.$$

We can convert this proportionality into an equality by inserting a proportionality constant $\frac{\mu_0}{2\pi}$ with the same reasoning as for Coulomb's law μ_0 is permeability constant for free space

$$B = \frac{\mu_0 I}{2\pi r}$$

or $\quad B(2\pi r) = \mu_0 I$

The L.H.S. in $\oint \vec{B} \cdot d\vec{l}$ for a path consisting of a circle of radius r centred on the wire. For all points on this circle $\vec{B}$ has the same constant magnitude of $\vec{B}$ and $d\vec{l}$ which is always tangent to the path of integration,

$$\oint \vec{B}\, d\vec{l} = \oint B \cdot dl = B \oint dl = B(2\pi r)$$

In this special case we can write the experimentally observed connection between

$$\vec{B} \text{ and } I, \oint \vec{B} \cdot d\vec{l} = \mu_0 I. \qquad \ldots(10.23)$$

Statement of Ampere's Law. The law states that the line integral of $\vec{B}$ around a close path is equal to μ_0 times the current enclosed by the path. If the path does not enclose the current, then,

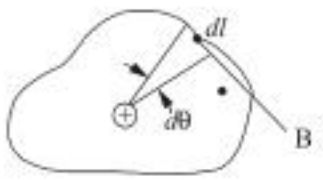
$$\oint \vec{B} \cdot dl = 0$$

10.8 PROOF OF AMPERE'S LAW— THE LINE INTEGRAL OF THE MAGNETIC FIELD VECTOR

We can prove the Ampere's law which is a fundamental relation between the current and the line integral of the magnetic field vector.

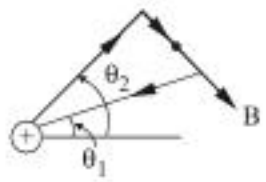
Line Integral for Irregular Path which encloses Current

Now we shall examine the line integral in the close path of integration around the current wire of an irregular shape as shown in Fig. 10.19 (a). First determine the value of line integral along a small section dl of total path, which is shown in amplified form in Fig. 10.19 (b). $\vec{B}$ along the arc $d\vec{l}$ is given in the similar way as above.



(*a*)

Fig. 10.19



(*b*)

Fig. 10.19

$$\int_{arc} \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \int_{\theta_1}^{\theta_2} d\theta = \frac{\mu_0 I}{2\pi}(\theta_2 - \theta_1)$$

Along the radial path $\vec{B}$ and $d\vec{l}$ are perpendicular to each other where $\vec{B} \cdot d\vec{l}$ is zero. So,

$$\int_{\substack{\text{radial} \\ \text{path}}} \vec{B} \cdot d\vec{l} = 0$$

Clearly for the element of the path shown, therefore,

$$\int_{path} \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi}(\theta_2 - \theta_1)$$

Since the result is independent of the distance R, then by taking the summation of the contribution to the total integral by all segments into which the total path has been divided, it is clear that $\theta_2 - \theta_1 = 2\pi$ *for closed path that encloses the current*

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I .$$

If the path is taken either radial or parallel to the current, $\vec{B}$ and $d\vec{l}$ are normal to each other. Since any arbitrary $d\vec{l}$ can be analysed into radial, tangential and parallel components, it is clear that only the tangential component contributes to the integral. Therefore, one chooses an arbitrary path for the integration around the current. The path could again be divided into segments as above only tangential components will be considered and the same result.

Value of $\oint \vec{B} \cdot d\vec{l}$ when path does not enclose current

Here too the path which does not enclose the current is broken into segments as shown in Fig. 10.20 (a). Based on the same consideration as in the case when path enclosed the current for the path shown in Fig. 10.20 (b) two tengential contributions exist.

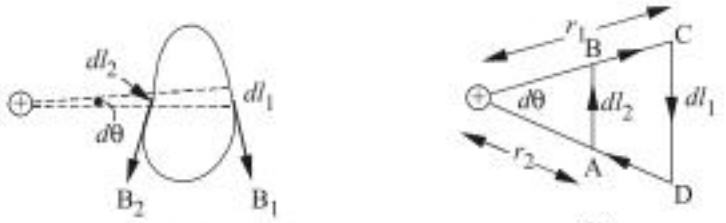


Fig. 10.20

Thus,

$$\int_{\substack{\text{total path}\\ ABCD}} \vec{B} \cdot d\vec{l} = \int_{\substack{\text{arc}\\ dl_1}} \vec{B} \cdot d\vec{l} + \int_{\substack{\text{arc}\\ dl_2}} \vec{B} \cdot d\vec{l}_2$$

But the two integrals on R.H.S. have equal and opposite values. However the value of integral vanishes for BC and AD, B and *dl* are perpendicular, hence $\vec{B} \cdot d\vec{l} = 0$. For complete path, we have

$$\int_{ABCD} \vec{B} \cdot d\vec{l} = \int_A^B \vec{B} \cdot d\vec{l} + \int_B^C \vec{B} \cdot d\vec{l} + \int_C^D \vec{B} \cdot d\vec{l} + \int_D^A \vec{B} \cdot d\vec{l}$$

$$= \int_A^B \vec{B} \cdot d\vec{l} + \int_C^D \vec{B} \cdot d\vec{l}$$

$$= \int_A^B \vec{B} \cdot d\vec{l} - \int_D^C \vec{B} \cdot d\vec{l}_1 \qquad \text{(since } dl_2 \text{ and } dl_1 \text{ are opposite)}$$

B at a distance $r_1 = \dfrac{\mu_0 I}{2\pi r_1}$

B at a distance $r_2 = \dfrac{\mu_0 I}{2\pi r_2}$

Hence,

$$\int_{ABCD} \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi r_2}\int_A^B dl_2 - \frac{\mu_0 I}{2\pi r_1}\int_D^C dl_1$$

$$= \frac{\mu_0 I}{2\pi r_2} AB - \frac{\mu_0 I}{2\pi r_1} DC$$

Now, from Fig. 10.20 (b),

$$\frac{DC}{AB} = \frac{r_1}{r_2}$$

or $$DC = \left(\frac{r_1}{r_2}\right) AB$$

Putting this value of DC in the above equation, we get,

$$\int_{ABCD} \vec{B}\cdot d\vec{l} = \frac{\mu_0 IAB}{2\pi r_2} - \frac{\mu_0 I r_1 AB}{2\pi r_1 r_2} = 0$$

Thus the results can be summarised,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \qquad \text{if path links current} \qquad \ldots(10.24)$$

$$= 0 \qquad \text{if path does not link current}$$

Ampere's Law and Gauss's Law of Electrostatics

Eqn. (10.24) is identical to Eqn. (4.21) in Chapter 4 for Gauss's law. Ampere's law is identical to Gauss's theorem of electrostatics for calculating magnetic field. As in electrostatics the differential form of Gauss's law is a fundamental equation of Maxwell electromagnetic theory. Similarly, Ampere's law will be also expressed as a fundamental equation of electromagnetic theory. We will just see it. However, we can apply the Ampere's law where $\oint \vec{B}\cdot d\vec{l}$ is easily evaluated, *i.e.* in problems having sufficient symmetry.

10.9 APPLICATIONS OF AMPERE'S LAW

When do we apply Ampere's law?

Like Gauss's law, Ampere's law is *always true* (for steady currents), but it is *not always useful*. The usefulness of Ampere's law depends on our ability to evaluate $\oint \vec{B}\cdot d\vec{l}$. For this, we construct a closed path, called *Amperian path*, near the current-distribution, for which the magnitude of field and angle between $\vec{B}$ and to the direction of path, have constant value; since then B-cos θ can be taken outside the integral sign. B is constant for uniform and symmetrical current-distribution and θ is constant for symmetrical path. *Thus, symmetry is crucial for the application of Ampere's law.* Therefore, *we apply Ampere's law only when we have uniform and symmetrical current-distribution and symmetrical Amperian path.* The standard current-configuration which can be handled by Ampere's law are:

1. infinite straight lines
2. infinite planes
3. infinite solenoids
4. toroids.

When Ampere's law applies, it's by far the fastest method; when it doesn't, we'll have to fall back on the Biot-Savart law.

We shall apply this law for calculating B in important cases given below:

1. $\vec{B}$ near a long wire. As we know that the line of magnetic induction for a long straight wire carrying a current I are concentric circles centred on the wire. The central dot in Fig. 10.21 represents a current I in the wire emerging from the page. The angle between $\vec{B}$ and $d\vec{l}$ is zero, hence $\vec{B} \cdot d\vec{l} = Bdl$, then applying the Ampere's law, we have, $\vec{B}$ at point P at a distance r

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$\vec{B} \cdot 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r} \qquad \ldots(10.25)$$

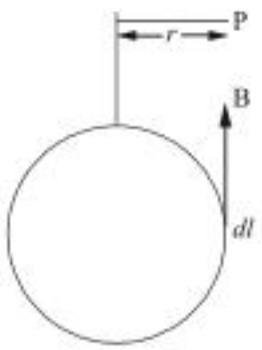


Fig. 10.21

In fact, we may regard this as an experimental result consistent and also derivable from Ampere's law. This expression is same as obtained by Biot-Savart law.

It is interesting to compare Eqn. (10.25) with the expression for electric field near a long line of charge,

$$E = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r}$$

Both are identical depending on constant multiple factor $\frac{\mu_0}{2\pi}$ and $\frac{1}{2\pi\varepsilon_0}$ and the factor responsible for the field I and λ. Finally each field varies as $1/r$. They are derived in the same manner. $\vec{E}$ by using Gauss'law and evaluating integral over Gaussian surface and $\vec{B}$ by using Ampere's law and evaluating for closed circular path. However, lines of $\vec{E}$ are perpendicular to the surface, while lines of $\vec{B}$ are everywhere tangent to the circular path.

2. Two Parallel Conductors

Whether we use Biot-Savart law or Ampere's law, B near a long wire at a distance r from the wire given by $\frac{\mu_0 I}{2\pi r}$. Hence, we can explain the force between the two currents in the same manner as discussed previously in Sec. 10.4 – application 3.

3. B for a Solenoid

A solenoid is a long wire wound in a close-packed helix and carrying a current I. For points near to a single turn, the behaviour of the wire is same as that of straight wire. The total field is the vector sum of the fields set up by all the turns. B is parallel to the solenoid axis at the interior points.

The field at external point is zero. For points such as P shown in Fig. 10.22 (*a*) the field set up by the upper part of the solenoid turns marked $\odot$ points to the left and tends to cancel the field setup by the lower part of the solenoid turns mark $\otimes$, which points to the right as shown in Fig. 10.22 (*a*).

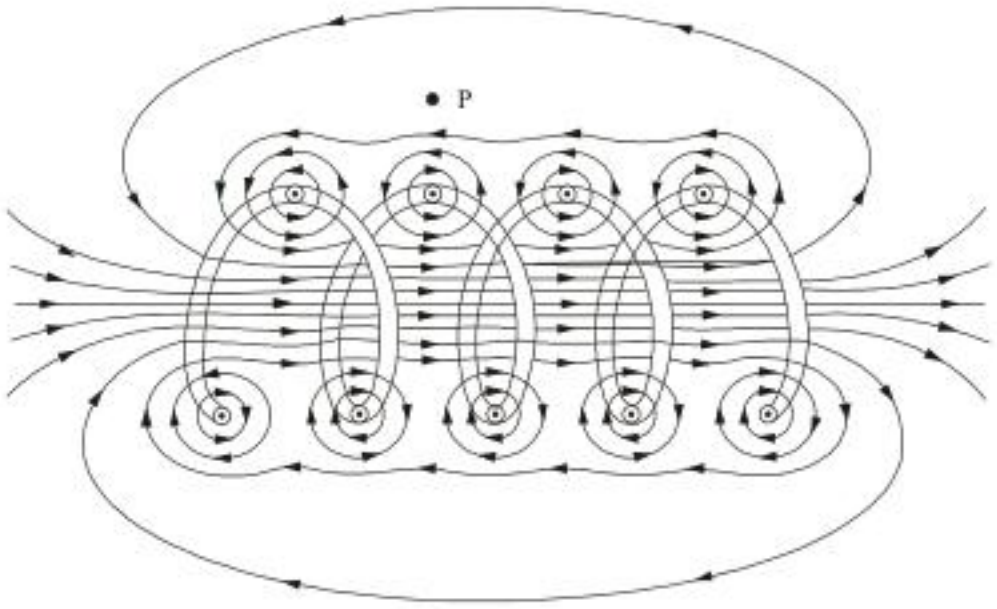


Fig. 10.22 (*a*)

Let us apply Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

to the path *abcd* shown in Fig. 10.22 (*b*). The total integral is sum of four integrals, one for each path of rectangle,

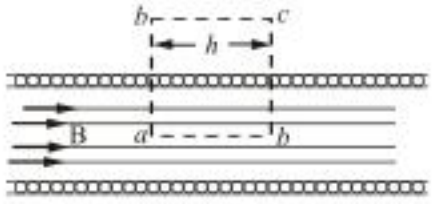


Fig. 10.22 (b)

$$\int_{abcd} \vec{B} \cdot d\vec{l} = \int_a^d \vec{B} \cdot d\vec{l} + \int_d^c \vec{B} \cdot d\vec{l} + \int_c^b \vec{B} \cdot d\vec{l} + \int_b^a \vec{B} \cdot d\vec{l}$$

for *ba*, B and *dl* are perpendicular to each other hence $\vec{B} \cdot d\vec{l} = 0$ and similarly for *dc* $\vec{B} \cdot d\vec{l} = 0$. The path *cb* is outside the solenoid $\vec{B} \cdot d\vec{l}$ is zero because B is zero outside the solenoid. The only contribution to the field is due to the path *ad* thus for entire path,

$$\int_{abcd}^{d} \vec{B} \cdot d\vec{l} = \int_a^d \vec{B} \cdot d\vec{l}$$

B and *dl* are parallel inside, hence,

$$\int_{abcd} \vec{B} \cdot d\vec{l} = B \int_a^d dl, \text{ since } \int_a^d dl = \text{length } ad = h$$

$$= Bh$$

The net current I that passes through the area bounded by the path of integration is not the same as the current I_0 in the solenoid because path of integration encloses more than one turn.

Let n be the number of turns per unit length, then

$$I = I_0(nh)$$

So, $$Bh = \mu_0 I_0 nh$$

or $$B = \mu_0 I_0 n \qquad \ldots(10.26)$$

The relation is obtained for an infinitely long ideal solenoid, it holds for actual solenoid for internal points near the centre of the solenoid. The relation does not depend on the diameter or the length of the solenoid and B is constant over the solenoid cross-section. If N are the total number of turns and l is the length of wire of the solenoid, then

$$n = \frac{N}{l}$$

$$B = \mu_0 I_0 \left(\frac{N}{l}\right) \qquad \ldots(10.27)$$

same expression as derived by Biot-Savart law.

4. A Toroid

Fig. 10.23 shows a toroid. A toroid may be visualised as a doughnut with wire wound uniformly around the doughnut. The toroid may be considered as a solenoid that has been bent into a circle with the ends joined. Since the axis of the resulting bent solenoid is circular, the lines of B are also circular.

Let us apply Ampere's law to the circular path of integration of radius r,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$B \cdot 2\pi r = \mu_0 I_0 N$$

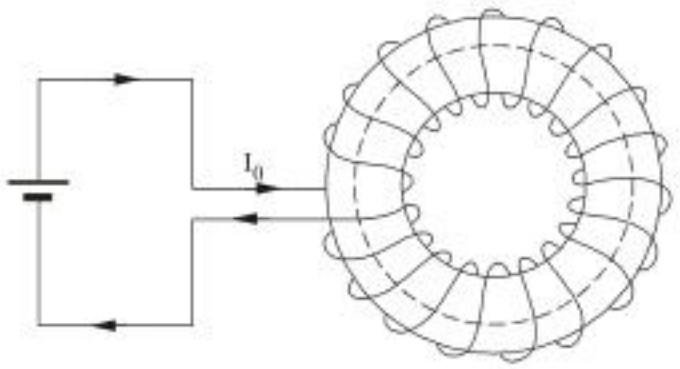


Fig. 10.23

where I_0 is the current in the toroid windings and N are the total number of turns. Therefore,

$$B = \frac{\mu_0}{2\pi} \frac{I_0 N}{r} \qquad \ldots(10.28)$$

In contrast to the solenoid, B is not constant over the cross-section of a toroid.

For External Point. As discussed in the case of solenoid, the field at outside point is zero. So here also B is zero because toroid is a bent circular solenoid.

Example 10.8 *Two parallel wires are separated by distance d carrying equal currents in opposite direction. Find the magnetic induction for points between the wires.*

Solution. Suppose P is the point at a distance x from an end as shown in Fig. 10.24. The resultant $B = B_1 + B_2$ where B_1 is the magnetic induction due to the wire one and B_2 due to the second,

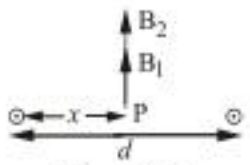


Fig. 10.24

$$B = \frac{\mu_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]$$

Example 10.9 *A long straight wire carries a current of 20 amperes. An electron is travelling at 10^7 m/sec. It is 2.0 cm from the wire. What force acts on the electron if its motion is directed (1) towards the wire, (2) parallel to the wire and, (3) at right angles to the direction given in (1) and (2).*

Solution. The magnitude of B produced by wire carrying currents is,

$$B = \frac{\mu_0 I}{2\pi r} = \frac{4\pi \times 10^{-7} \times 20}{2\pi \times 0.02/100}$$

$$= 2 \times 10^{-4} \text{ weber/metre}^2$$

Force on the electron $\vec{F} = q\,(\vec{v} \times \vec{B})$

(1) When electron moves towards the wire, v is perpendicular to the B due to wire so,

$$F = qvB = 1.6 \times 10^{-19} \times 10^7 \times 2 \times 10^{-4}$$

$$= 3.2 \times 10^{-16} \text{ Newtons.}$$

F is parallel to the current.

(2) When electron moves parallel to the wire, its velocity v will again be perpendicular to the magnetic field B. So the magnitude will be $= 3.2 \times 10^{-16}$ Newtons.

(3) When the electron is moving at right angle to the direction given above, then v and B will be parallel or anti-parallel.

Hence,

$$\vec{F} = q\,(\vec{v} \times \vec{B}) \text{ will be zero}$$

or $$F = 0$$

Example 10.10 *A solenoid is 1 metre long and 3 cm in mean diameter. It has 5 layers of windings of 850 turns each and carries a current of 5.00 amp. Calculate B at its centre.*

Solution. $B = \mu_0 I_0 n$

$$= 4\pi \times 10^{-7} \times 5 \times 5 \times 850$$

$$= 2.7 \times 10^{-2} \text{ weber/metre}^2$$

Example 10.11 *The electron circulates around the nucleus in a path of radius 5.1×10^{-11} m at a frequency of 6.8×10^{15} revolutions/second. Calculate B at the centre and the magnetic dipole moment.*

Solution. Current I = Charge flow/sec = ev

$$= 1.6 \times 10^{-19} \times 6.8 \times 10^{15}$$

$$= 1.1 \times 10^{-3} \text{ amp.}$$

B at the centre of the orbit is given by putting $x = 0$ in Eq. (10.8).

i.e. $$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}} = \frac{\mu_0 I}{2R} = \frac{4\times10^{-7}\times1.1\times10^{-3}}{2\times5.1\times10^{-11}}$$

$$= 14 \text{ weber/metre}^2$$

Magnetic dipole moment from Eq. (9.12) of Chapter 9,

$$m = NIA = 1\times1.1\times10^{-3}\times\pi\times(5.1\times10^{-11})^2$$

$$= 9\times10^{-24} \text{ amp.metre}^2$$

10.10 CHARACTER OF $\vec{B}$ LINES AND THE DIVERGENCE OF $\vec{B}$

In our study of electrostatics, we first found simple methods of obtaining the field by integrating the contribution resulting from all points charges according to Coulomb's law. We soon found, however, that this was a method of restricted usefulness. The really powerful tools of electrostatics come only when we apply general analytical methods, introducing the potential, Gauss's theorem, Poisson's equation, and similar relations. Here too, we shall introduce general analytical methods. We shall first show that div. $\vec{B}$ is always zero. The lines of magnetic field are closed circle around the axis of a current element. In other words, the lines of magnetic field produced by currents always close on themselves. This means, of course, that, if an infinitesimal region of space were examined, the total number of $\vec{B}$ lines entering the specified region must be equal to the total number of $\vec{B}$ lines leaving the specified region. As we have already established for steady current where the continuity prevails for the number of lines of electric currents is characterised mathematically by div. $\vec{j} = 0$. Same condition is satisfied by vector $\vec{B}$ and $\vec{B}$ lines are continuous, hence we must have div. $\vec{B} = 0$.

To be more specific, since the magnetic field lines do not have sources in contrast to the case of electric field where the lines emerge from positive charges and end on the negative charges, as we know the divergence of a vector signifies the starting or stopping of the vector, we should expect that

$$\text{div. } \vec{B} = 0. \qquad \ldots(10.29)$$

Proof:

We can prove div. $\vec{B} = 0$ from Biot-Savart law and using vector analysis. Noting that for small element of current $Id\vec{l} = Id\vec{l}$. According to Biot-Savart law, the field of complete circuit is given by,

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I d\vec{l}\times\hat{r}}{r^2} \qquad \text{(see eq. 10.2)}$$

taking the divergence of both sides,

$$\text{div. } \vec{B} = \nabla\cdot\vec{B} = \nabla\cdot\left(\frac{\mu_0}{4\pi}\int\frac{I d\vec{l}\times\hat{r}}{r^2}\right)$$

$$= \frac{\mu_0}{4\pi}\int\nabla\cdot\left(\frac{\vec{I}\times\hat{r}}{r^2}\right)dl$$

Since it makes no difference whether we first evaluate integral then take divergence or take the divergence first then evaluate the integral,

therefore, $\text{div}.\vec{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{\vec{I}\times\hat{r}}{r^3}\right) dl$

Since $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}$,

hence $\text{div}.\vec{B} = -\frac{\mu_0}{4\pi}\int \nabla \cdot \left[I \times \nabla\left(\frac{1}{r}\right)\right] dl$

But according to the vector analysis expansion,

$$\nabla \cdot \left(\vec{A}\times\vec{B}\right) = \left[\vec{B}\cdot\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\vec{B}\right]$$

therefore,

$$\text{div}.\vec{B} = -\frac{\mu_0}{4\pi}\int \left[\nabla\left(\frac{1}{r}\right)\cdot\nabla\times\vec{I} - \vec{I}\cdot\nabla\times\nabla\left(\frac{1}{r}\right)\right] dl$$

To interpret this result it is recalled that $\vec{B}$ is being specified at the field point, whereas the current is at the source points specified by dl. Say the field point is specified by the variable. (x, y, z) the source point might be specified by the variable (x', y', z'). But since the divergence operation is being taken with respect to the field point variables which are independent of the source point variables, then

$$\nabla\left(\frac{1}{r}\right)\cdot\nabla\times\vec{I} = 0 \cdot$$

The second term, which is the curl of the gradient of a scalar is identically zero, then it follows that always div. $\vec{B} = 0$.

10.11 AMPERE'S LAW IN CURL FORM OR CURL $\vec{B}$ AND STOKES' THEOREM

The Ampere circuital law is a simple relation between magnetic vector $\vec{B}$ and current I. The law is easily applied to calculate $\vec{B}$ when sufficient symmetry exists. *If one is interested to apply the law to problems of a general nature without a high degree of symmetry, it is necessary to relate the magnetic vector $\vec{B}$ with the current density*. The situation here is rather like that $\text{div}.\vec{E} = \frac{\rho}{t_0}$, which relates the divergence of $\vec{E}$ with the charge density to specify the conditions at a point. Ampere's law can be written by using the definition of I and current density.

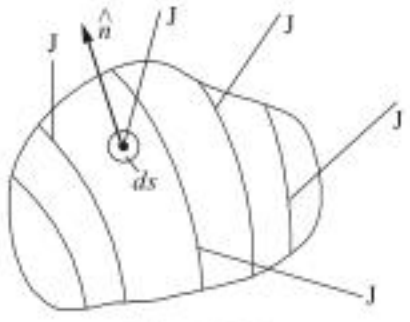


Fig. 10.25

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

$$= \mu_0 \int_s \vec{J}\cdot\hat{n}\, ds \qquad \ldots(10.30)$$

where the area s is that specifying the region through which the current exists, $\hat{n}$ is a unit vector normal to elementary area ds in Fig. 10.25. Applying the Stokes' theorem,

$$\oint \vec{B}\cdot d\vec{l} = \int_s (\nabla\times\vec{B})\cdot\hat{n}\, ds \qquad \ldots(10.31)$$

Comparing Eqn. (10.30) and (10.31), we get,

$$\text{Curl}\,\vec{B} = \nabla \times \vec{B} = \mu_0 \vec{J} \qquad \ldots(10.32)$$

Eqn. (10.32) is the point statement of Ampere's law corresponding to the integral statement, taking together the two corresponding forms are,

$$\left.\begin{aligned} &\oint \vec{B} \cdot d\vec{l} = \mu_0 I \\ \text{and} \quad &\nabla \times \vec{B} = \mu_0 J \end{aligned}\right\} \qquad \ldots(10.33)$$

These expressions bear the same general relationship to each other that Gauss'a law in electric field,

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

or, $$\text{div}.\,\vec{E} = \frac{\rho}{\varepsilon_0}$$

It is interesting to write down,

$$\left.\begin{aligned} &\text{div}.\,\vec{B} = \nabla \cdot \vec{B} = 0 \\ \text{and} \quad &\text{Curl}\,\vec{B} = \nabla \times \vec{B} = \mu_0 J \\ \text{in electrostatic,} \quad &\\ &\text{div}.\,\vec{E} = \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \\ \text{and} \quad &\text{Curl}\,\vec{E} = \nabla \times \vec{E} = 0 \end{aligned}\right\} \qquad \ldots(10.34)$$

A Further Comment on Scalar Potential

We see from Eqn. (10.33), $\nabla \times \vec{B} = \mu_0 J$, that in a region containing no current flow curl $\vec{B} = 0$, but this does not hold within a conductor. In empty space where curl $\vec{B} = 0$, we can introduce as we have done in Eqn. (10.19) a scalar potential but this cannot be done inside a conductor carrying a current. The scalar potential we introduced in Sec. 10.5 for a single loop of current was multiple-valued, that is, as we integrated $\vec{B}$ around a contour enclosing the current, the potential increased by $\mu_0 I$, so that if we integrated *n* times around the contour we should increase the potential by $n\mu_0 I$. In other words, for any single value of potential, all values obtained by adding or subtracting integral multiples of $\mu_0 I$ are equally legitimate. If now, we have many current loops, in our space, we can change the potential by integral multiples of μ_0 times of any one of the currents, by traversing suitable contours, with enough current loops, this means that we can obtain almost any value of the potential we desire, at a given point of space, if the current is distributed over the volume, so that we can enclose any amount by traversing a suitable contour within the volume, we can obtain any value of potential whatever at a given point of space. In this case, *the usefulness of the idea of potential breaks down completely. The integral of* $\vec{B}$ *is not independent of path, and there is no unique way of defining a scalar potential.*

10.12 THE VECTOR POTENTIAL

It is clear from our discussion of the preceding section that a scalar potential is not useful for discussing a magnetic field, except in the special case where the current flows in a single loop. Mathematically,

we have that curl $\vec{B}$ is not in general zero, so that the conditions for introducing a scalar potential do not exist. *There is, however, another quite different way of setting up a potential function, which is much more general, and equally useful. This is to set up what is called a 'vector potential'.* The fundamental characteristic of a potential is that it is a function that we differentiate to get the vector field. The reason why the potential is a useful device in electrostatics is that it is easy to compute it from charge distribution, by the solution of Poisson's equation.

The vector potential solution for the magnetic field of currents has the same useful features. The existence of vector potential is based on div. $\vec{B} = 0$, always satisfied by magnetic induction. By a well-known theorem of vector analysis, div. curl $\vec{A} = 0$, where $\vec{A}$ is any arbitrary vector, that is, the divergence of any curl is zero. It seems reasonable to assume from this that we can write,

$$\vec{B} = \text{curl } \vec{A} \qquad \ldots(10.35)$$

where $\vec{A}$ is a vector function of position, which is the vector potential. In other words, the vector $\vec{A}$ the curl of which is equal to the magnetic induction $\vec{B}$ is known as vector potential.

Eqn. (10.35) can be written as,

$$B_x \hat{i} + B_y \hat{j} + B_z \hat{k} = \left(\frac{\delta A_z}{\delta y} - \frac{\delta A_y}{\delta z}\right)\hat{i} + \left(\frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x}\right)\hat{j} + \left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)\hat{k}$$

Now equating the components on both sides of the equation,

$$\left.\begin{aligned} B_x &= (\nabla \times \vec{A})_x = \left(\frac{\delta A_z}{\delta y} - \frac{\delta A_y}{\delta z}\right) \\ B_y &= (\nabla \times \vec{A})_y = \left(\frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x}\right) \\ B_z &= (\nabla \times \vec{A})_z = \left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right) \end{aligned}\right\} \qquad \ldots(10.36)$$

where A_x, A_y, and A_z are the corresponding components of the vector potential A.

Divergence of $\vec{A}$. We shall consider in next section that div. $\vec{A} = 0$.

Let us prove it. As div. $\vec{B} = 0$, we have written,

$\vec{B} = \nabla \times \vec{A}$, it guarantees that,

$$\nabla \cdot B = \nabla \cdot (\nabla \times \vec{A}) = 0$$

The field A is called the vector potential. In cases where $\nabla \times \vec{B} = 0$, we have $\vec{B} = -\nabla V_w$, where V_w is a scalar potential. As we have discussed that the Scalar potential V_m was not completely specified by its definition, V_m is a multivalued function. If we found V_w for some problem, we can always find another potential that is equally good by adding a constant.

$$V_m' = V_w + C$$

The new potential V_m' gives the same magnetic field. So we can say that corresponding to a given value of magnetic field, there can be two or more scalar potentials differing by addition of a constant C, since gradient ∇C is zero we have V_w' and V_m represent the same phenomena *i.e.* the same magnetic field.

Similarly, we can have different vector potentials $\vec{A}$ which give the same magnetic field. Again, because $\vec{B}$ is obtained from $\vec{A}$ by differentiating as given in Eqn. (10.36) adding a constant to A does not change anything physical. But there is even more latitude for A. We can add to A any field which is the gradient of some scalar field, without changing the physical phenomena.

Suppose we have a vector potential A that gives correctly the magnetic field $\vec{B}$ and the other new vector potential A' will give the same field $\vec{B}$ if they have the same curl *i.e.*

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}'.$$

Therefore,

$$\nabla \times \vec{A} - \nabla \times \vec{A} = \nabla \times (\vec{A}' - \vec{A}) = 0$$

But if the curl of a vector is zero it must be the gradient of some scalar field, say ϕ. So,

$$A' - A = \nabla \phi$$

$$A' = A + \nabla \phi \qquad \ldots(10.37)$$

will be an equally satisfactory vector potential, leading to the same field $\vec{B}$.

It is usually convenient to vary $\vec{A}$ by imposing some condition on it (in the same way that we found it convenient to choose the potential V_m zero at large distances). We can, for instance, restrict $\vec{A}$ by choosing arbitrarily value of divergence of A, we can always do without affecting $\vec{B}$. This is because although A' and A have the same curl, and give the same $\vec{B}$, they do not need to have the same divergence. In fact,

$$\nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \phi$$

And by a suitable choice of ϕ, we can make $\nabla \cdot \vec{A}'$ anything we wish.

What should be the choice of $\nabla \cdot \vec{A}$? The choice should be made to get the greatest mathematical convenience and will depend on the individual problem. For magnetostatics, we will make the simple choice,

$$\nabla \cdot \vec{A} = 0$$

Then the complete definition of $\vec{A}$ is for magnetostatics.

$$\nabla \times \vec{A} = \vec{B}$$

and $$\nabla \cdot \vec{A} = 0 \qquad \ldots(10.38)$$

10.13 THE VECTOR POTENTIALS OF KNOWN CURRENTS

As we know that the magnetic field vector $\vec{B}$ is determined by the currents, the vector potential $\vec{A}$ is related to magnetic field B by the relation,

$$\vec{B} = \text{curl } \vec{A}.$$

So, $\vec{A}$ must also be determined by the currents. We want to find $\vec{A}$ in terms of the currents. We begin with the basic, equation,

$$\nabla \times \vec{B} = \mu_0 \vec{J}, \ \textit{i.e.}, \ \text{eq. (10.32)}$$

Substituting $\vec{B}$ in terms of vector $\vec{A}$, the equation reduces to

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J}$$

using the vector identity.

$$\nabla \times \nabla \times \mathrm{A} = \nabla\,(\nabla \cdot \vec{\mathrm{A}}) - \nabla^2\,\mathrm{A} = \mu_0\,\mathrm{J}$$

Now since $\nabla \cdot \vec{\mathrm{A}} = 0$, therefore,

$$\nabla^2 \vec{\mathrm{A}} = -\mu_0 \vec{\mathrm{J}}$$

or in terms of components,

$$\left.\begin{aligned} \nabla^2\,\mathrm{A}_x &= -\mu_0\,\mathrm{J}_x \\ \nabla^2\,\mathrm{A}_y &= -\mu_0\,\mathrm{J}_y \end{aligned}\right\} \quad \text{and} \quad \nabla^2\,\mathrm{A}_z = -\mu_0\,\mathrm{J}_z \qquad \ldots(10.39)$$

Each of the equations given in eq. (10.39) is mathematically identical to the equation of electrostatics *i.e.* Poisson's equation,

$$\nabla^2\,\mathrm{V} = -\frac{\rho}{\varepsilon_0}$$

Now we know that the solution of Poisson's equation gives the value of V (x, y, z), hence a very simple process for calculating electrostatics field of a charge distribution,

$$\mathrm{V}\,(x_1, y_1, z_1) = \frac{1}{4\pi\varepsilon_0} \int_{\mathrm{V}} \frac{\rho\,(x_2, y_2, z_2)}{r_{12}}\, dv_2 \qquad \text{[See Chapter 5 Eqn. (5.14)]}$$

where r_{12} is the separation between the points (x_2, y_2, z_2) and (x_1, y_1, z_1). So we see immediately from an analogy the general solution for equation,

$$\nabla^2\,\mathrm{A}_x = -\mu_0\,\mathrm{J}_x$$

As,
$$\mathrm{A}_x = \frac{\mu_0}{4\pi} \int_{\mathrm{V}} \frac{\mathrm{J}_x(x_2, y_2, z_2)}{r_{12}}\, dv_2$$

i.e. simply replacing $\frac{\rho}{\varepsilon_0}$ by $\mu_0\,\mathrm{J}_x$.

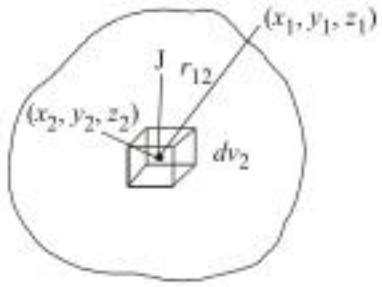


Fig. 10.26

The dv_2 and r_{12} are shown in Fig. 10.26. Similarly we have for A_y and A_z. And when three solutions are combined, we get the solution of $\nabla^2 \vec{\mathrm{A}} = -\mu_0 \vec{\mathrm{J}}$, *i.e.*

$$\mathrm{A}\,(x_1, y_1, z_1) = \frac{\mu_0}{4\pi} \int_{\mathrm{V}} \frac{\mathrm{J}\,(x_2, y_2, z_2)}{r_{12}}\, dv_2 \qquad \ldots(10.40)$$

One can verify by direct differentiation that this integral satisfies condition $\nabla \cdot \vec{\mathrm{A}} = 0$, so long as $\nabla \cdot \vec{\mathrm{J}} = 0$, which must happen for steady currents. So by using Eqn. (10.40), we can find vector potential $\vec{\mathrm{A}}$ due to the current density $\vec{\mathrm{J}}$.

If the current is flowing in a wire having diameter very small as compared with the dimension, equation (10.40) can be modified for a thin wire of area of cross-section S and length dl, $dv = \mathrm{S}dl$, hence,

$$J dv = \mathrm{JS}\, dl$$

as current density is defined as current per unit area, JS = I, hence

$$Jdv = Idl$$

Putting in equation (10.40), we have,

$$A = \frac{\mu_0}{4\pi} \oint \frac{Idl}{r_{12}} \qquad \ldots(10.41)$$

where r_{12} is the distance of dl to the point at which we are calculating potential and the integral is taken around a whole circuit.

10.14 APPLICATIONS OF VECTOR POTENTIALS

As an example, we shall calculate the magnetic field by using vector potential method.

1. The B of Straight Wire

Let us consider a long straight wire of radius a carrying a steady current I which is uniformly distributed throughout its cross-section and the wire is placed in coordinate system as shown in Fig. 10.27 along the z-axis. The current density vector $\vec{j}$ will have, in this case, only z-component. Its magnitude is given by,

$$J_z = \frac{I}{\pi a^2}$$

inside the wire and is zero outside. Since $J_z = 0$ and $J_y = 0$, we immediately have,

$$A_x = 0, A_y = 0$$

To get A_z we can use our solution for the electrostatics potential V of a wire with a uniform charge density.

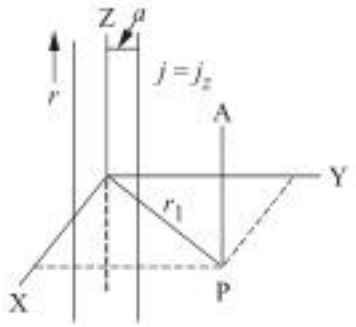


Fig. 10.27

For points outside an infinite charge cylinder, the electrostatic potential is,

$$V = -\frac{\lambda}{2\pi\varepsilon_0} \log r' \qquad \text{[see eq. (5.41)]}$$

where $r' = \sqrt{x^2 + y^2}$ and λ is the charge per unit length and is equal to the charge contained in a unit length of the cylinder, *i.e.*

$$\lambda = \pi a^2 \rho$$

therefore, $$V = -\frac{1}{2\pi\varepsilon_0} \rho\pi a^2 \log r'$$

The corresponding vector potential A_z is obtained by replacing $\frac{\rho}{\varepsilon_0}$ by $\mu_0 J_z$, *i.e.*,

$$A_z = -\frac{1}{2\pi} \pi a^2 \mu_0 J_z \log r'$$

putting $I = \pi a^2 J_z$,

$$A_z = -\frac{\mu_0 I}{2\pi} \log r'$$

Now, we can find B by using Eqn. (10.36),

$$B_x = \left(\frac{\delta A_z}{\delta y} - \frac{\delta A_y}{\delta z} \right)$$

Putting A_z given above and noting that $A_y = 0$, we get

$$B_x = -\frac{\mu_0 I}{2\pi} \frac{\delta}{\delta y} (\log r'), \; r' = \sqrt{x^2 + y^2}$$

After simplification,

$$B_x = -\frac{\mu_0 I}{2\pi} \frac{y}{r'^2}$$

Similarly,

$$B_y = \left(\frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x} \right) \text{ as } A_x = 0, \text{ and substituting } A_z$$

$$= \frac{\mu_0 I}{2\pi} \frac{\delta}{\delta x} (\log r') = \frac{\mu_0 I}{2\pi} \frac{x}{r'^2}$$

and, $$B_z = \frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y} = 0, \text{ since } A_x = A_y = 0.$$

The magnitude of the magnetic field B is given by,

$$B = \left(B_x^2 + B_y^2 + B_z^2 \right)^{1/2}$$

Substituting the values obtained above, we get,

$$B = \frac{\mu_0 I}{2\pi r'^2} (x^2 + y^2)^{1/2}$$

$$= \frac{\mu_0 I}{2\pi r'} \qquad \ldots(10.42)$$

This is the same result as obtained earlier by Biot-Savart law and Ampere's circuital law.

2. The field of a Small Current Loop—The Magnetic Dipole

Let us use the vector potential method to find the magnetic field of a small loop of current. By small loop, we mean simply that we are interested in the fields only at large distances compared with the size of the loop. Such a loop produces magnetic field like the electric field from an electric dipole. It will be shown below that any small loop is a magnetic dipole.

Consider a rectangular loop and choose out coordinates as shown in Fig. 10.28 (*a*). There are no currents in the *z*-direction so A_z is zero. There are currents in the *x*-direction on the two sides of length *a*. In each leg, the current density (and current) is uniform. So the solution for A_x is just like the electrostatic potential from two charged rods as shown in Fig. 10.28(*b*). Since the rods have opposite charges, the electric potential at large distances would be just the dipole potential at the point P, the potential would be,

$$V = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} \qquad \text{[See Chapter (5), Eq (5.23)]}$$

where p is the dipole moment of the charge distribution.

Fig. 10.28

The dipole moment in the present case is the total charge on one rod times the separation between them

$$P = \lambda\, a\, b$$

where λ is the linear density of charge on the rods.

If α is the area of cross-section of the wire, ρ is the charge per unit volume, then

$$\lambda = \text{volume of unit length of wire} \times \rho$$

$$= \alpha\rho$$

hence, $$P = \alpha\rho\, ab$$

and $$V = \frac{1}{4\pi\varepsilon_0}\frac{P\cos\theta}{r^2}$$

where θ is the angle between the direction of P and r and direction of P is same as E with negative y-direction,

$$\cos(180 - \theta) = \frac{y}{r}$$

$$\cos\theta = -\frac{y}{r}$$

therefore, $$V = -\frac{1}{4\pi\varepsilon_0}\frac{P}{r^2}\frac{y}{r}.$$

Putting the value of P given above,

$$V = -\frac{1}{4\pi\varepsilon_0}\frac{\alpha\, ab}{r^2}\frac{y}{r}$$

From this A_x is directly written by replacement of $\frac{\rho}{\varepsilon_0}$ by $\mu_0 J$ *i.e.*,

$$A_x = -\frac{1}{4\pi}\frac{\mu_0 J\alpha\, ab}{r^2}\frac{y}{r}$$

$J\alpha$ = current I

hence, $$A_x = -\frac{\mu_0 Iab}{4\pi}\frac{y}{r^3} \qquad \ldots(10.43)$$

Similarly, A_y can be calculated from the electric potential due to two oppositely charged rods lying parallel to y-axis separated by distance a as shown in Fig. 10.29 (a).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{P\cos\theta}{r^2} \qquad \text{(see Eq. (5.22)}$$

Since $\cos\theta = \frac{x}{r}$,

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\rho\alpha\, ab}{r^2}\frac{x}{r}$$

hence A_y is obtained by replacing $\frac{\rho}{\varepsilon_0}$ by $\mu_0 j$

$$A_y = \frac{\mu_0 Iab}{4\pi}\frac{x}{r^3} \qquad \ldots(10.43\text{ a})$$

Therefore, the vector potential

$$A = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$$

Putting A_x and A_y obtained above and also noting that $A_z = 0$,

we get, $A = \frac{\mu_0 Iab}{4\pi}\frac{(x^2+y^2)^{1/2}}{r^3}$

but $r^2 = x^2 + y^2 + z^2$

hence $x^2 + y^2 = r^2 - z^2$

or $(x^2+y^2)^{1/2} = r\sqrt{1-\frac{x^2}{r^2}}$ $\qquad \frac{z}{r} = \cos\theta_z$

hence $A = \frac{\mu_0 Iab}{4\pi}\frac{1}{r^2}\sqrt{1-\cos^2\theta_z}$

where θ_z is the angle which r makes with z-axis,

$$A = \frac{\mu_0}{4\pi} Iab\frac{\sin\theta_z}{r^2}$$

The quantity Iab which determines the strength of A and is product of area of the loop and current. This product is known as m, the magnetic dipole moment. Hence,

$$A = \frac{\mu_0}{4\pi}\frac{m\sin\theta_z}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{\bar{m}\times\hat{r}}{r^2} \qquad \ldots(10.44)$$

This will be the same at every point on a circle around the dipole. $\bar{A}$ is perpendicular to the plane containing the dipole and point of observation.

The magnetic field $\bar{B}$ can be calculated from Eqn. (10.36), as,

$$B_x = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)$$

$$= -\frac{\delta A_y}{\delta_z}, \text{ since } A_z = 0.$$

Substituting A_y from Eq. (10.43a), B_x reduces as given below.

$$B_x = -\frac{\partial}{\partial_z}\left(\frac{\mu_0 m}{4\pi}\cdot\frac{x}{r^3}\right) \quad \text{where } m = Iab \text{ and } r = (x^2 + y^2 + z^2)^{1/2}$$

Substituting this, B_x reduces to

$$B_x = \frac{\mu_0 m}{4\pi}\cdot\frac{3xz}{r^5}$$

Similarly,

$$B_y = \frac{\mu_0 m}{4\pi}\cdot\frac{3yz}{r^5}$$

The value of B_z is given in Eqn. (10.36), *i.e.*

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

Substituting the values from Eqn. (10.43) and Eqn. (10.43a),

$$B_z = \frac{\mu_0 m}{4\pi}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)\right]$$

$$= \frac{\mu_0 m}{4\pi}\left[\frac{2}{r^3} - \frac{3}{r^5}(x^2 + y^2)\right]$$

Since, $r^2 = x^2 + y^2 + z^2$

after substitution, we get

$$B_z = -\frac{\mu_0 m}{4\pi}\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)$$

therefore the magnitude of B,

$$B = \left[B_x^{\,2} + B_y^{\,2} + B_z^{\,2}\right]^{1/2}$$

On substituting the values obtained above, we get

$$B = \frac{\mu_0 m}{4\pi r^3}\sqrt{1 + 4\cos^2\theta} \qquad \ldots(10.45)$$

This expression is same as that of the electric dipole oriented along the z-axis. The current loop is, therefore, called a magnetic dipole.

3. Change in Magnetic Field at a Current Sheet

Consider a thin sheet of copper of uniform thickness t located in the xz-plane in which a current flows with constant density and direction everywhere within the metal. Let the current flows in the x-direction. Fig. 10.30 shows a portion of the sheet of infinite extent. If the current density inside the metal is j in amp/metre^2 then every metre of height *i.e.* of sheet in z-direction have an area t metre^2 and will include a ribbon of current amounting to jt ampere. We call this the surface current density or simply sheet current density and represented by j_s. j_s is a useful quantity and determines the change in the magnetic field from one side of the sheet to the other.

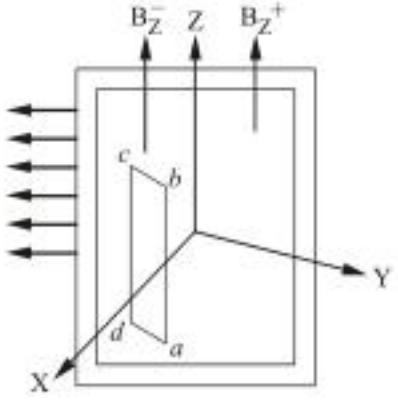


Fig. 10.30

The total magnetic field is the sum of the two fields, one due to sheet and the other from some external source present there.

Consider a rectangular path *abcd*, *ab* side is in front of the surface and *dc* behind with the short sides *bc* and *da* piercing the sheet. Let B_z^+ denote the *z*-component of the magnetic field just in front of the sheet and B_z^- be that just behind it the line integral of B around *abcd*, is given by,

$$\int_{abcd} \vec{B}\cdot d\vec{l} = \int_{ab} \vec{B}\cdot d\vec{l} + \int_{bc} \vec{B}\cdot d\vec{l} + \int_{cd} \vec{B}\cdot d\vec{l} + \int_{dc} \vec{B}\cdot d\vec{l}$$

Since we have considered the thin sheet and the point just near the sheet, hence *bc* and *da* sides are much shorter than *ab* and *cd*. Hence their contribution is neglected. Therefore,

$$\int_{abcd} \vec{B}\cdot d\vec{l} = \int_{ab} \vec{B}\cdot d\vec{l} + \int_{cd} \vec{B}\cdot d\vec{l}$$

$$= (B_z^+ - B_z^-)\cdot l,\ ab = cd = l$$

Field in front of the sheet and just behind it are just opposite to each other. Now using the Ampere's law

$$\int_{abcd} \vec{B}\cdot d\vec{l} = \mu_0 \times \text{current enclosed by rectangle}$$

$$= \mu_0 j_s l$$

or, $$(B_z^+ - B_z^{-1})\, l = \mu_0 j_s l$$

or, $$B_z^+ - B_z^- = \mu_0 j_s \qquad \ldots(10.46)$$

A current sheet of density j_s increases the component of $\vec{B}$ which is parallel to the surface and perpendicular to j_s. This increase is the same as the change in electric field at a sheet of charge.

If the sheet is the only current source (the other external source is contributing), then, of course, the field is symmetrical about the sheet. We have,

$$B_z^+ = \frac{1}{2}\mu_0 j_s \text{ and } B_z^- = -\frac{1}{2}\mu_0 j_s$$

Suppose there are two sheets carrying equal and opposite surface currents as shown in Fig. 10.31 with no other sources around. The direction of current flow is perpendicular to the plane of the paper, out on the left and in on the right. The field between the sheet is

$$\mu_0 j_s = \left(\frac{1}{2}\mu_0 j_s + \frac{1}{2}\mu_0 j_s\right)$$

and outside the sheets is zero

$$\frac{1}{2}\mu_0 j_s - \frac{1}{2}\mu_0 j_s = 0$$

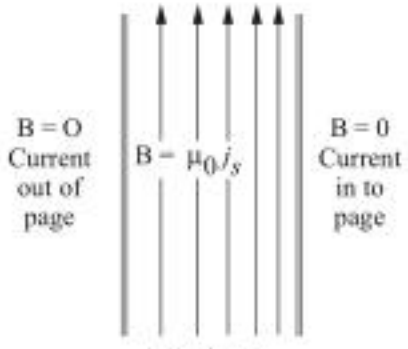


Fig. 10.31

REVIEW QUESTIONS

10.1 What is Biot-Savart law? Apply it to determine the magnetic field.
(1) Due to a long straight wire carrying current.
(2) Due to a wire of finite length carrying current.

10.2 What is Laplace rule? Derive from it Biot-Savart law. Obtain counterpart of Coulomb's law of force between charges for two finite current elements.

10.3 Discuss the interaction between two parallel long currents. Define ampere using the expression you arrive at.

10.4 Obtain an expression for the magnetic field along the axis of circular coil carrying current.

10.5 Give the theory and working of Helmholtz galvanometer. How do you get uniform magnetic field between the coils?

10.6 What is a solenoid? Obtain an expression for the field on the axis of solenoid.

10.7 Explain magnetic scalar potential. Show that for a current loop,

$$V_m = \frac{\mu_0}{4\pi} I\Omega$$

Symbols have usual meaning.

10.8 State and prove the Ampere's circuital law *i.e.*

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Illustrate it by determining the magnetic field of a current carrying long wire.

10.9 State and prove Ampere's law. Apply it to calculate magnetic field due to
(*a*) a solenoid and
(*b*) a Toroid carrying current.

10.10 Discuss the phenomena, interaction between two parallel currents by Ampere's law.

10.11 Establish the relation,

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

and fully explain the meaning of $\nabla \cdot \vec{B} = 0$.

10.12 What are vector and scalar magnetic potentials? How the idea of vector potential is introduced? Obtain an expression for magnetic vector potential due to an electric current flowing in a wire and hence calculate the magnetic field.

10.13 Discuss the equivalence of magnetic shell and circuit carrying electric current.

PROBLEMS

10.1 Two long parallel wires, each carrying a current of 10 amp. in the same direction, are separated by 10 cm. Calculate the force between the wires per unit length.

10.2 A square loop of wire of edge 10 cm. carries a current of 10 amp. Calculate the values of B at the centre of the loop.

10.3 Two long straight wires are kept in free space at a distance of 1.0 metre and 1.0 amp current is flowing in the same direction in each wire. Calculate the direction and magnitude of the mutual force per metre between the two wires. What will be the difference if the current flows in mutually opposite direction?

10.4 A current of 30 amps flows in a long wire, a rectangular loop ABCD carrying current 20 amp is situated 1.0 cm away from this wire. If the side of loop facing wire is 30 cm and other side is 8 cm, calculate the magnitude and direction of the resultant force acting on the loop due to magnetic field of the current in the straight wire.

10.5 The axis of circular coil of radius 5 cm having 10 turns and carrying a current of 5 amp. makes an angle θ with a uniform magnetic field $\vec{B}$. Find the torque on the coil.

10.6 A current of 30 amp. flows in a flat circular coil having 100 closely wound turns of radius 10 cm. What is flux density at the centre of the coil?

10.7 A closely wound solenoid of 1200 turns has an axial length 80 cm and radius of 1.5 cm. A current of 1.2 amp flows in the solenoid. Find the flux density at the middle of the axis.

10.8 A long solenoid of length 1 metre and radius of cross-section 1.5 cm has five layers of windings of 850 turns each. If the solenoid carries a current 5.0 amp. calculate the values of
(1) Magnetic induction and
(2) Magnetic flux ϕB for a cross-section of the solenoid at the centre of the solenoid.

10.9 A solenoid of length 20 cm and radius 2 cm is wound uniformly with 3000 turns of wire. It carries a current of 2 amperes. What is the value of B?
(1) On the axis of solenoid at the middle
(2) On the axis at one end

And what is flux ϕ_B
(1) Through the coil at the middle and
(2) Through one end.

10.10 If the coils of the Helmholtz galvanometer have diameter 14 cm and number of turns 275 each, find the magnetic field midway between the coils when their distance apart is 7 cm and a current of 0.1 amp flows through them.

ANSWERS

1. 2.0×10^{-4} Newtons/metre, attractive **2.** 1.13×10^{-4} weber/metre^2
3. 10^{-7} Newton/metre, attractive, the force becomes replusive
4. 3.2×10^{-3} Newtons **5.** $5\pi \times 10^{-2}$ B sin θ Newton/metre
6. 1.89×10^{-3} weber/metre^2 **7.** 2.26×10^{-3} weber/metre^2
8. 2.67×10^{-2} weber/metre^2, 0.885×10^{-6} webers
9. $12\pi \times 10^{-3}$ weber/metre^2, $6\pi \times 10^{-3}$ weber/m^2, $48\pi^2 \times 10^{-7}$ weber and $24\pi^2 \times 10^{-7}$ weber
10. 2.25×10^{-6} weber/metre^2

SELF-TEST OBJECTIVE QUESTIONS

10.1 Biot-Savart's law is

(*a*) $B = \frac{\mu I}{4\pi} \int \frac{\sin\theta}{r^2} dl$ (*b*) $B = \frac{\mu I}{2\pi} \int \frac{\sin\theta}{r^2} dl$

(*c*) $B = \frac{2\mu I}{\pi} \int \frac{\sin\theta}{r^2} dl$ (*d*) none of these

10.2 If the direction of the linear motion of a cork screw represents the direction of current through the conductor, the direction of rotation of cork screw gives the direction of magnetic field. This was stated by
(*a*) Lenz (*b*) Ampere
(*c*) Maxwell (*d*) Laplace

10.3 Two current carrying conductors are parallel to each other. The currents I_1 and I_2 are flowing in the same direction. They will experience
(*a*) no mutual force (*b*) attraction
(*c*) repulsion (*d*) none of the above

10.4 If the two conductors carry currents in opposite direction, these will
(*a*) attract each other (*b*) repel each other
(*c*) There is no force between them (*d*) none of the above

10.5 The unit of electric current, called the ampere, is that current which, when flowing through each of two .... conductors spaced one metre apart in vacuum and of infinite length give rise to a force between the conductors of 2×10^{-7} Newton per metre run,

(*a*) right angle (*b*) parallel
(*c*) inclined (*d*) opposite

10.6 A compass needle is placed near a box and experiences deflection. The box does not have a magnet inside. It may have

(*a*) a charge body inside
(*b*) capacitor fully inside
(*c*) current carrying conductor
(*d*) none of these

10.7 A wire placed in north-south direction carries current towards north. The north seeking pole of a compass needle placed near the wire is deflected towards east. The compass needle is

(*a*) above the conductor (*b*) below the conductor
(*c*) anywhere near it (d) none of the above

10.8 The magnetic flux density at the end of a one-metre-long solenoid carrying current of I amp and having N turn will be

(*a*) μNI (*b*) $\dfrac{\mu NI}{\sqrt{4R^2 + l^2}}$

(*c*) $\dfrac{\mu NI}{2}$ (*d*) $\dfrac{\mu NI}{\sqrt{R^2 + l^2}}$

10.9 A current is flowing in a circular conductor in a clockwise direction. The coil is in the plane of paper. The direction of magnetic field is

(*a*) away from you (*b*) towards you
(*c*) in the plane of paper (*d*) none of the above

10.10 At the centre of a current carrying single turn circular loop, magnetic field is

(*a*) $\bar{B} = \dfrac{I}{2\mu\pi R}$ (*b*) $\bar{B} = \dfrac{\mu I}{2R}$

(*c*) $\bar{B} = \dfrac{\mu I}{2\pi R}$ (*d*) $\bar{B} = \dfrac{\mu I}{2\pi R^2}$

10.11 The ratio of the intensity of magnetic field at the centre of a very long solenoid to that at the extreme ends is

(*a*) 2 (*b*) 1/2
(*c*) 4 (*d*) 1/4

ANSWERS

1. (a)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (c)
7. (a)	**8.** (c)	**9.** (b)	**10.** (b)	**11.** (a)	

CHAPTER 11

ELECTROMAGNETIC INDUCTION

In the preceding chapters the principles of static electric and magnetic fields were considered. In this chapter electric and magnetic fields that change with time are discussed, and so a number of new relations and concepts are introduced. Some of the more important of these are (1) Faraday's law, which gives the emf induced in a closed circuit due to a change of magnetic flux linking it; (2) a relation giving the emf induced in a conductor moving in a magnetic field. To be more specific, the present chapter will consider the phenomena associated with accelerated charges, or correspondingly, with time changing currents or time changing magnetic fields. Therefore, in the following sections we will relate the electric field to the changing magnetic field and examine some of the consequences of the relationship.

11.1 FARADAY'S EXPERIMENT ON ELECTROMAGNETISM

With the realisation that electric currents produce magnetic field, Faraday immediately began looking for a reverse effect and suggested that, somehow or other, magnets might also produce electric field.

Faraday's First Experiment

Fig. 11.1 shows a close circuit containing galvanometer. As there is no source of emf so there is no deflection in galvanometer.

Now suppose we move a bar magnet towards the wire keeping the wire stationary, and say north pole facing the wire, the galvanometer needle is deflected indicating that a current has been set in the circuit. However, *the deflection is observed only when the magnet is in motion.* If the magnet is moved away from the wire, the galvanometer needle is deflected in the opposite direction to that when the magnet moves towards the wire indicating that the current set up in the circuit is in opposite direction. It is also observed that if we move the magnet towards the wire, with its south pole facing the wire, the deflection is in opposite direction, again indicating that the current set up is now in reverse direction to that when the north pole faces the wire.

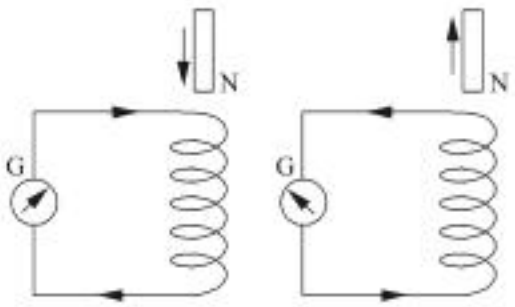


Fig. 11.1

There is also a deflection in the galvanometer when the magnet is held stationary and wire is moved towards or away from magnet. *Faraday discovered that moving the magnet towards the wire one way has the same effect as moving the wire towards the magnet the other way. Thus what actually matters is the relative motion*–between the circuit and the magnet. It is immaterial which of them is moved.

Faraday's Second Experiment

The arrangement is shown in Fig. 11.2. The magnet is replaced by a coil carrying current, we expect to observe the same effect because the coil carrying current produces magnetic field. P is a primary coil carrying current, S is a secondary coil and a galvanometer is connected in series with secondary. The motion of either of the coils shows a deflection in the galvanometer. He also observed that galvanometer shows a sudden deflection in one direction when current was started in primary and in the opposite direction when current was stopped.

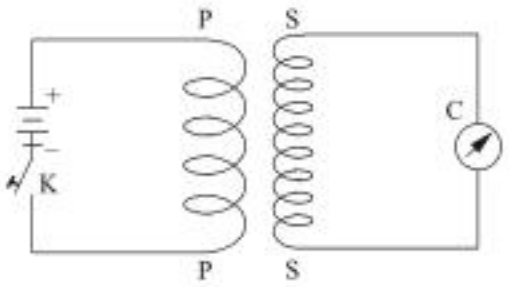


Fig. 11.2

Faraday's Conclusion

In both experiments, *Faraday found that electric effects exist only when there is something changing.* There was no steady deflection in the galvanometer, but the needle of galvanometer is suddenly deflected due to induced currents produced in the circuit in three different ways:

1. By moving a magnet near wire, towards or away from the wire.
2. By moving the wire towards or away from the magnet, or
3. By changing a current in the adjacent current.

11.2 LENZ'S LAW

So far we have not specified the direction of induced emf or current. The direction of the current and emf is given by Lenz's law. It states that the direction of induced emf (or current) in a close circuit is such that it opposes the very cause that produces it. This law is based on the principle of conservation of energy. To be more specific, this law says that the current induced is in a direction such that it produces a magnetic flux tending to oppose the original change of flux *i.e.* tending to keep the total flux through the circuit constant.

Thus when the applied flux density $\vec{B}$ is decreasing in magnitude, the current induced in the loop is in such a direction as to produce a field which tends to increase $\vec{B}$. On the other hand, when $\vec{B}$ is increasing, the current induced in the loop is in such a direction as to produce a field which tends to decrease $\vec{B}$. Thus the induced current in the loop is always in such a direction as to produce flux opposing the change in $\vec{B}$. Lenz's law applies only to closed circuit.

Consider the Faraday's experiment as shown in Fig. 11.1. As we push the magnet towards the circuit, an induced current is set up in the circuit. According the Lenz's law, the direction of this current should be such as to oppose the motion of the magnet towards the circuit. As we know that a current through a coil produces magnetic field like a magnetic dipole, therefore the two faces of the coil will behave as two poles of a magnet. Thus if the current produced in our coil is to oppose the motion of the magnet towards the coil, the face of the coil towards the magnet should become north pole, similarly in other cases.

11.3 FARADAY'S LAWS OF ELECTROMAGNETIC INDUCTION

We have seen that there is a motional e.m.f. in circuit moving in a magnetic field in such a way that the magnetic flux through the circuit changes. An e.m.f. in a stationary circuit when the magnetic flux through the circuit is also produced changes because the sources of the magnetic field are moving. It is an experimental observation that an e.m.f. appears in a circuit when the magnetic flux through the circuit changes from any cause. Such an e.m.f. is called an induced emf and is related to the induced electric field caused by the changing magnetic field. From those observations Faraday enunciated two important laws:

1. Whenever the magnetic flux associated with any closed circuit changes, an induced current flows through the circuit which lasts only so long as the change lasts. An increase in the magnetic flux produces inverse current, while a decrease of such flux a direct current.
2. The magnitude of the induced emf produced in a coil is directly proportional to rate of change of the magnetic flux through the coil.

$$\xi \propto \frac{d\phi_B}{dt}$$

where ϕ_B is the magnetic flux through the circuit and given by

$$\phi_B = \int_s \vec{B} \cdot d\vec{S}$$

Combining the two laws, we have,

$$\xi = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{S} \qquad \ldots(11.1)$$

The negative sign gives the direction of induced emf. It is one of the fundamental laws of electromagnetism, on a par with Ampere's and Gauss' law.

Unit. Dimensionally eq. (11.1) has the form,

$$\text{Induced emf} = \frac{\text{magnetic flux}}{\text{Time}} = \frac{\text{weber}}{\text{sec}}$$

Since, $\quad \text{Weber} = \dfrac{\text{Newton.meter}}{\text{Ampere}}$

Hence, $\quad \dfrac{\text{Weber}}{\text{sec}} = \dfrac{\text{Newton.meter}}{\text{Ampere/sec.}} = \dfrac{\text{Joules}}{\text{Couls}} = \text{Volts}$

Thus, the RMKS units of induced emf is the volt.

11.4 AN ELEMENTARY QUANTITATIVE STUDY OF INDUCTION *i.e.* DEDUCTION OF FARADAY'S LAW FROM LORENTS' FORCE

The actual value of the induced emf can be deduced by analysing the motion of a conducting rod in a uniform magnetic field $\vec{B}$ by the use of *Lorentz force and the law of conservation of energy*. Consider a circuit as shown in Fig. 11.3. Let a current I flows in the rod AB (of length l) resting on two parallel horizontal rails on which the rod can easily slide. Now apply a uniform magnetic field $\vec{B}$ perpendicular to the plane of rails, *i.e.* plane of the paper and points into it.

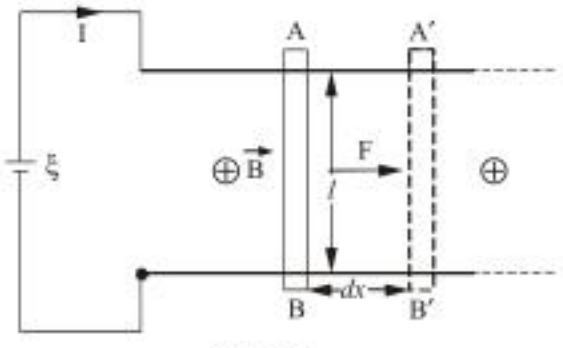


Fig. 11.3

The current I flowing through the rod AB placed in the magnetic field $\vec{B}$ will cause a force on the rod AB,

$$\vec{F} = I\vec{l} \times \vec{B}$$

in the direction as shown in Fig. 11.3 *i.e.*, perpendicular to both $\vec{l}$ and $\vec{B}$. The magnitude is given by,

$$F = BIl$$

Since the bar can slide over the rails, the force will cause AB to displace along the direction of F through a distance dx to A′ B′ during the interval of time dt. The work done for this displacement is,

$$dW = \vec{F} \cdot d\vec{x}$$
$$= BIl\, dx$$

Now if ξ is the emf of the battery, I the current in the circuit, the energy spent during the interval of time dt is $\xi I dt$. A part of this energy is used in overcoming the resistance r of the circuit *i.e.*, $(I^2 r\, dt)$ and the other part is used in displacing the conductor. *Hence according to law of conservation of energy,*

$$\xi I dt = I^2 r dt + BIl dx$$

Solving for I,

$$I^2 r dt = \xi I dt - BIl dx$$

or $$I = \frac{\xi - Bl dx/dt}{r}$$

Here l dx is the amount by which the area enclosed by the circuit A and B increases during the time dt. Therefore, the product Bldx gives the increase (change) in flux $d\phi_B$ *in time dt and Bldx/dt gives the rate of change of flux.* Hence the emf induced in the circuit resulting from the change of flux is given by,

$$\xi_{ind} = -\frac{d\phi_B}{dt} = -\frac{Bl dx}{dt}$$
$$= -Blv \qquad \ldots(11.2)$$

The negative sign indicates that induced emf opposes the main emf of the circuit. This property of induced emf is the same as the *force of friction in mechanical motion*. The induced emf opposes the main emf of the circuit in the same way as the force of friction in mechanical motion opposes the impressed force causing the motion. *The force of friction changes its direction if the direction of the external force is reversed. Here also the induced emf changes its direction if the direction of main emf is reversed.*

11.5 HOW THE INDUCED EMF (OR INDUCED CURRENT) IS GENERATED BY MOVING A CONDUCTOR IN A MAGNETIC FIELD?

In the preceding article the emf induced by a change in flux linkage is discussed. In some situations it is convenient or desirable to consider specifically the emf induced in a conductor moving in a magnetic field. We can easily understand this feature of magnetic induction from the $e\vec{v} \times \vec{B}$ force on a particle of electric charge e moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$. When we move the conductor (wire) across the field, the electrons in the wire feel the force $e\vec{v} \times \vec{B}$. The electrons have sidewise velocity along the direction of motion of conductor because they move with the conductor. *This* $\vec{v}$ *with a vertical* $\vec{B}$ *results in a force on the electrons directed along the wire. Consequently the electrons are pushed along the length of conductor.* They in turn complete the circuit. What counts is the push integrated around the complete circuit. We call this net integrated push the induced emf in the circuit. More precisely, the induced emf is defined as the tangential force per unit charge in the conductor integrated over length, once around the complete circuit *i.e.*,

$$\xi_{ind} = \oint \vec{E} \cdot d\vec{l}$$

where $\vec{E}$ is the electric field intensity *i.e.* the force per unit charge is given by $\vec{E} = \frac{\vec{F}}{e} = \vec{v} \times \vec{B}$. Therefore the emf induced between two points 1 and 2 on the conductor is given by,

$$\xi_{ind} = \int_1^2 \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

11.6 INDUCTION AND RELATIVE MOTION

Faraday discovered that moving the magnet towards the wire one way has the same effect as moving the wire towards the magnet the other way. But when the magnet is moved, we no longer have any $\vec{v} \times \vec{B}$ force on the electrons in the wire. This is the new effect that Faraday found. Today we understand it from a relativity argument. Thus what actually matters is the relative motion between wire and magnet. We shall discuss few experiments.

A Conducting Rod Moves Through a Uniform Magnetic Field

Fig. 11.4 (a) shows a thin rod or a wire supposed to be moving at constant velocity v in a direction perpendicular to its length, in the space where a uniform magnetic field $\vec{B}$ exists. Let $\vec{B}$ be produced by a solenoid which is at rest in the frame of reference F with coordinates (x, y, z). There is no electric field in the frame F in the absence of the rod.

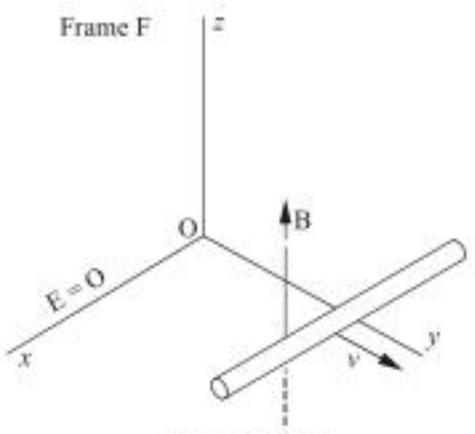


Fig. 11.4 (*a*)

The rod, being a conductor, contains charged particles. Those will be in motion when the rod is moved with the same velocity with which the rod is moving. Then a particle of charge q will experience a force,

$$\vec{F} = q\left(\vec{v} \times \vec{B}\right)$$

with $\vec{B}$ and $\vec{v}$ as directed in Fig. 11.4 (b). F is in the positive x-direction if q is positive. In the case of negatively charged electrons that are in fact the movable carriers in most conductors, the $\vec{F}$ will be in a direction opposite to that as shown in Fig. 11.4 (b).

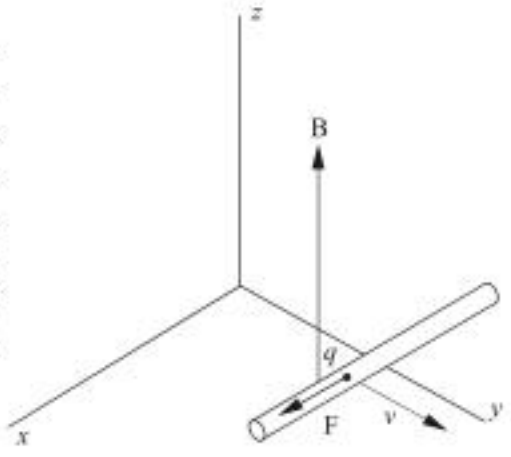


Fig. 11.4 (*b*)

The force $\vec{F}$ pushes electrons towards one end of the rod, consequently the other end develops a positive charge. This goes on until these separated charges themselves create an electric field inside the rod. Finally a steady state is reached by moving the rod at constant velocity $\vec{v}$ when the force $\vec{F}$ is equal to the force due to electric field created inside the rod in the opposite direction. Then the motion of charge relative to the rod ceases, *i.e.* the field created $\vec{F}$ is given by,

$$q\vec{E} = -\vec{F} \qquad \ldots(1)$$

Thus we can say that the electric field is produced by moving the rod in magnetic field $\vec{B}$ given by,

$$q\vec{E} = -q\left(\vec{v} \times \vec{B}\right)$$

$$\vec{E} = -\left(\vec{v} \times \vec{B}\right) \qquad \ldots(2)$$

where v is the velocity with which rod is moving in uniform magnetic field B. Thus the charge distribution causes an electric field outside the red, as shown in Fig. 11.4 (b).

Source of B is moving and rod is stationary

Let us observe this system from a frame F′ that moves with the rod. Forgetting the rod for the moment, we see in this frame F′ indicated in Fig. 11.4 (c) a magnetic field $\vec{B}'$ (not much different from $\vec{B}$ if v is small) together with a uniform electric field given by,

$$\vec{E}' = (-\vec{v}') \times \vec{B}'$$

where v' is the velocity with which frame F′ in which the magnetic field is situated is moving.

If the magnitudes of v and v' are same and $v << c$ then B′ in not much different from B.

In fact, when the frame is moving relative to rod which is kept stationary, hence from relativity, we can write,

$$\vec{v} = -\vec{v}'$$

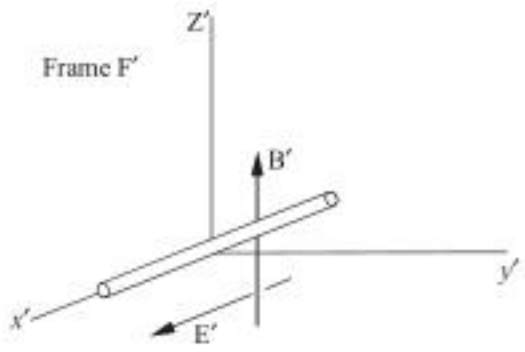


Fig. 11.4 (c)

therefore, $\vec{E}' = (-\vec{v}) \times \vec{B}$ ...(3)

Now if the rod is put into the system, there will be redistribution of charge on the surface of the rod so as to make the electric field zero inside the rod as in the case of any conductor in an electric field.

Conclusion

An observer in frame F says: inside the rod there has developed an electric field $E = -(v \times B)$ exerting a force $q\vec{E} = -q\,(\hat{v} \times \vec{B})$ which just balances the force that would otherwise cause any charge q to move along the rod. An observer in F′ says: inside the rod there is no electric field, and although there is a uniform magnetic field here, no force arises from it, because no charges are moving each account is correct.

Thus, $E = E'$, hence our statement that moving the magnet towards the wire one way has the same effect as moving the wire towards the magnet other way.

11.7 THE FLUX RULE

The amount of the induced emf is given by a simple rule discovered by Faraday. The rule is that when the magnetic flux that passes through (this is the normal component of $\vec{B}$ integrated over the area of the coil) is changing with time, the emf induced is equal to the rate of change of the flux. We will call this as the "flux rule". *We see that, by moving the wire near the magnet, by moving a magnet near the wire or by changing a current in a wire the associated magnetic flux changes in each case,* hence the induced emf is produced and induced current flows through the galvanometer which shows a deflection.

11.8 TIME VARYING FIELD

So far we have discussed induced emf by relative motion of magnets and coils (rods or wire). We have a general rule that for any circuit whose parts move in a fixed magnetic field the induced emf is the time derivative of the flux, regardless of the shape of the circuit.

On the other hand, *what happens if the loop is stationary and the magnetic field is changed.* We cannot deduce the answer to this question from the same argument. Faraday discoverd by an experiment that the flux rule is still correct no matter how the flux changes. Force on electric charge is given in

complete generality by $\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$, *there are no new special forces due to changing magnetic fields. Any force on charges at rest in a stationary wire come from the* $\vec{E}$ *term. Faraday's observations led to the discovery that electric and magnetic fields are related by a new law, in a region where the magnetic field is changing with time, electric fields are generated.* It is this electric field which drives the electrons around the wire and is responsible for the emf in a stationary circuit when there is a changing magnetic flux.

So, we can say that the induced emf depends only on the rate of change of flux of $\vec{B}$ *and not on anything else.* Regardless of how this change is made, whether by motion of circuit or motion of source, or change in the strength of the source.

Let us consider a closed circuit C enclosing an open surface S and placed in a magnetic field $\vec{B}$ which is varying with time as shown in Fig. 11.5. Then the magnetic flux linked with small element dS of this surface is $\vec{B} \cdot \hat{n}\, d\mathrm{S}$. Then the magnetic flux linked with the circuit C is given by,

$$\phi_B = \oint_S \vec{B} \cdot \hat{n}\, d\mathrm{S}$$

where $\hat{n}$ is unit vector along the positive normal to dS and the induced emf ξ is given by,

$$\xi = \oint \vec{E} \cdot d\vec{l}$$

where E is induced electric field at the element dl of the circuit as shown in Fig. 11.5. The electric field due to induction should be distinguished from an electric field due to charges*.
We can state as a universal relation Faraday's law of induction. If C is some closed curve, stationary in coordinates (x, y, z) if S is a surface spanning C, and B (x, y, z) is the magnetic field measured in (x, y, z) at any time t, then,

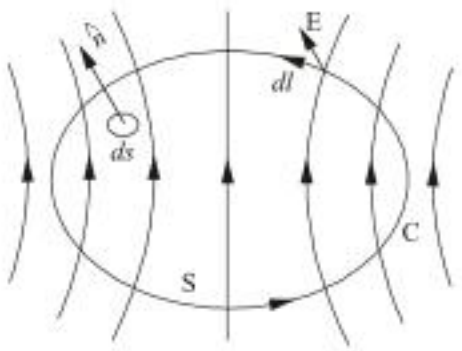


Fig. 11.5

$$\xi = \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_s \vec{B} \cdot \hat{n}\, d\mathrm{S} = -\frac{d}{dt}\phi_B \quad \ldots(11.4)$$

The above equation is the Faraday's law in integral form when the coil (or circuit) is stationary while B changes the derivatives with respect to time in relation $\frac{d}{dt}\int_s \vec{B} \cdot \hat{n}\, d\mathrm{S}$ does not depend on the path of integration and $\frac{d}{dt}$ can be taken under the sign of integration. Such induction is called transformer induction, hence,

$$\text{emf} = \oint_c \vec{E} \cdot d\vec{l} = -\int_s \frac{\delta \vec{B}}{\delta t} \cdot \hat{n}\, d\vec{S} \quad \ldots(11.5)$$

Using the Stoke's theorem,

$$\oint_c \vec{E} \cdot d\vec{l} = \int_s (\nabla \times \vec{E}) \cdot \hat{n}\, d\mathrm{S}$$

---

* Whereas the field due to charges E_c can be described as the gradient of an electric potential, the field due to induction cannot. E_c is lamellar field (See art 11.9).

Comparing the two, we have,

$$\int_s (\nabla \times \vec{E}) \cdot \hat{n}\, dS = -\int_s \frac{\delta \vec{B}}{\delta t} \cdot \hat{n}\, dS$$

Since two integrals are equal for any surface of integration, the integrand must be equal,

$$\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} \qquad \ldots(11.6)$$

If B does not depend on position

$$\text{Then Curl } \vec{E} = -\frac{d\vec{B}}{dt} \qquad \ldots(11.7$$

If B also depends on position as well as time, then,

$$\text{Curl } \vec{E} = -\frac{\delta \vec{B}}{\delta t}$$

This is the Faraday's law expressed in differential forms.

So the flux rule applies whether the flux changes because the field changes or because the circuit moves (or both). Thus we can say changing flux of $\vec{B}$ sets up an induced electric field $\vec{E}$ at various points around the loop. When the circuit also moves, we have,

$$\xi = \oint (\vec{v} \times \vec{B})\, d\vec{l} - \int_s \frac{\delta \vec{B}}{\delta t} \cdot \hat{n}\, dS \qquad \ldots(11.8)$$

11.9 DIFFERENCE BETWEEN INDUCED ELECTRIC FIELDS AND FIELDS DUE TO STATIC CHARGES

The induced electric fields that are set up by induction process are not associated with charges but with a changing magnetic flux. Although both kinds of electric fields exert forces on charges, there is difference between them. *This difference is that lines of $\vec{E}$ associated with changing magnetic flux forms closed loops whereas the lines of $\vec{E}$ associated with charges cannot form a close loop but only originate on a positive charge and end on a negative charge.*

To make it more clear, let us consider the line integral of $\vec{E}$ around a closed path. As we have discussed in electrostatics that line integral of $\vec{E}$ due to static charges is zero. For example, if V_a and V_b be the potentials at two points *a* and *b*, then the potential difference,

$$V_b - V_a = \frac{W_{ab}}{q_0} = -\int_a^b \vec{E} \cdot d\vec{l}$$

If *a* and *b* are the same point, then

$$\oint \vec{E} \cdot d\vec{l} = 0$$

However, when field is by induction *i.e.* by changing magnetic flux $\oint \vec{E} \cdot d\vec{l}$ is precisely not zero but according to Faraday's law given by,

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\delta \phi_B}{\delta t}$$

As we have discussed in equation (3.38) that when line integral E around a close path is zero field is conservative. *Thus the electric fields associated with stationary charges are "conservative",* but

those associated with changing magnetic fields (*i.e.* induced fields by Faraday's law) are "non-conservative". Further, *electric potential which exists only for the conservative forces, has no meaning for electric fields produced by induction.*

Hence,

$$\left.\begin{aligned} \nabla \times \vec{\mathrm{E}} &= -\frac{\delta_{\mathrm{B}}}{\delta t} \\ \text{or} \quad \text{Curl } \vec{\mathrm{E}} &= -\frac{\delta_{\mathrm{B}}}{\delta t} \end{aligned}\right\} \text{ for induction field}$$

and $\nabla \times \vec{\mathrm{E}} = 0,$ for static charges field

However, the field $\vec{\mathrm{E}}$ due to induction can have an associated vector potential. Since $\nabla \cdot \vec{\mathrm{B}} = 0$ everywhere and $\vec{\mathrm{B}}$ can be expressed as a curl of a vector A, *i.e.*

$\vec{\mathrm{B}} = \text{Curl } \vec{\mathrm{A}}$, where $\vec{\mathrm{A}}$ is a vector potential to be discussed in the next article.

Summary:

To summarise, the above four induction relations are:

$$\left.\begin{aligned} &(i)\ \xi = \oint (\vec{v} \times \vec{\mathrm{B}}) \cdot d\vec{l} - \int_{\mathrm{s}} \frac{\delta \vec{\mathrm{B}}}{\delta t} \cdot \hat{n} d\mathrm{S} && \text{General case when } \vec{\mathrm{B}} \text{ is changing and conductor is in motion} \\ &(ii)\ \xi = \oint (\vec{v} \times \vec{\mathrm{B}}) \cdot d\vec{l} && \text{Motion only (\textit{i.e.} motional induction)} \\ &(iii)\ \xi = -\int_{\mathrm{s}} \frac{\delta \vec{\mathrm{B}}}{\delta t} \cdot \hat{n}\, d\mathrm{S} && \text{B changes only (transformer induction)} \\ &(iv)\ \xi = -\frac{d}{dt} \int_{\mathrm{s}} \vec{\mathrm{B}} \cdot \hat{n} d\mathrm{S} && \text{Flux linkage (flux changes, it is immaterial how it changes)} \end{aligned}\right\}$$

...(11.19)

Example 11.1 *An all-metal aeroplane drives down vertically at 300 km/sec. at a place where the horizontal component of the earth's field is 0.4 oersted. If the wing span is 30 metres, what will be the resulting potential difference between the tips?*

Solution. As given, the vertical distance covered in one sec.

$$300 \text{ km} = 300 \times 10^3 \text{ m} = 3 \times 10^5 \text{ m}$$

The distance between the wing tips = 30 m

Therefore, the area swept by the wing span in one sec,

$$\frac{d\mathrm{A}}{dt} = 3 \times 10^5 \times 30 \text{ m}^2/\text{sec}.$$

The potential difference induced between the wing tips,

$$\xi = \frac{d\phi_{\mathrm{B}}}{dt} = \frac{d}{dt} - (\vec{\mathrm{B}} \cdot \vec{\mathrm{A}}) = -\frac{d}{dt}(\mathrm{HA}) = \mathrm{H}\frac{d\mathrm{A}}{dt} \quad \text{(numerically)}$$

$$= (0.4 \times 10^{-4} \text{ weber/m}^2) \times (3 \times 10^5 \times 30 \text{ m}^2/\text{sec})$$

(Since) 1 weber/m^2 = 10^4 gauss (or oersted)

$$= 360 \text{ volts}.$$

Example 11.2 *A vertical copper disc of diameter 20 cm makes 10 revolutions per second about a horizontal axis passing through its centre. A uniform magnetic field of 100 gauss acts perpendicular to the plane of the disc. Calculate the potential difference between its centre and rim in volts.*

Solution. The magnetic flux linked with the disc, is given by,

$$\phi_B = BA$$

By Faraday's law the induced emf is given by,

$$\xi = -\frac{d\phi_B}{dt} = -\frac{d}{dt}(BA) = B\frac{dA}{dt} \quad \text{(numerically)}$$

Given $B = 100 \text{ gauss} = 100 \times 10^{-4} \text{ weber/m}^2 = 10^{-2} \text{ weber/m}^2$

$r = 10 \text{ cm} = 0.1 \text{ metre}$

$\frac{dA}{dt}$ = Area swept out by disc in unit time

$= \pi r^2 \times$ number of revolutions per second

$= 3.4 \times (1)^2 \times 10 = 3.14$

therefore, $\xi = B\frac{dA}{dt} = (10^{-2} \text{ weber/m}^2) \times (3.14 \text{ m}^2/\text{sec})$

$= 3.14 \times 10^{-3} \text{ volts} = 3.14 \text{ millivolts.}$

Example 11.3 *The rails of a railway track are 1.5 metres apart and assumed to be insulated from one another. Calculate the emf in volts that will exist between the rails if a train in passing at 100 km/ hour. Assume that horizontal component of earth's magnetic field is 0.36 oersted and* $\tan\theta = 1.036$ *where* θ *is the angle of dip.*

Solution. If V and H are the vertical and horizontal components of earth's magnetic field respectively, then,

$$V = H \tan\theta = (0.36 \times 10^{-4} \text{ weber/m}^2) \times (1.036)$$

Velocity of the train,

$$v = \frac{\text{distance}}{\text{time}} = \frac{100 \times 10^3 \text{ m}}{60 \times 60 \text{ sec}} = \frac{1000}{36} \text{ m/sec.}$$

Distance between the rails, $d = 1.5$ m.

The area swept out by the train in unit time $= \frac{dA}{dt} = d \cdot v$

$$= (1.5 \text{ m}) \times \left(\frac{1000}{36} \text{ m/sec}\right) = \frac{1000 \times 1.5}{36} \text{ m}^2/\text{sec}$$

Therefore, the induced emf, $\xi = -\frac{d\phi_B}{dt} = V\frac{dA}{dt}$ (numerically)

$$= 0.36 \times 10^{-4} \times 1.036 \text{ weber/m}^2 \left(\frac{1000 \times 1.5}{36} \text{ m}^2/\text{sec}\right)$$

$$= 1.554 \times 10^{-3} \text{ volts} = 1.554 \text{ millivolts.}$$

Example 11.4 *A long solenoid of diameter 5 cm. has 100 closely and evenly wound turns per cm of its length. At the centre of the solenoid, a small coil of 50 turns and 2 cm. diameter is placed such that the magnetic field produced by the solenoid is parallel to the axis of the coil. The current in the solenoid is changed from 2 amp. to zero and then raised to 2 amp. in the opposite direction at a steady rate over a period of 0.1 second. Calculate the emf induced in the coil during this change.*

Solution. The magnetic induction B at the centre of the solenoid

$$B = \mu_0 n I \text{ weber/m}^2$$

Here, $\mu_0 = 4\pi \times 10^{-7}$ weber/amp.m,

$n = 100$ turns/cm

$= 10^4$ turns/m

and, $I = 2$ amp.

Therefore, $B = (4\pi \times 10^{-7} \text{ weber/amp.m}) \times 10^4 \text{ turns/m}) \times (2 \text{ amp})$

$= 2.512 \times 19^{-2}$ weber/m^2

Area of the central coil = number of turns × area of each turn

$$= 50 \times 3.14 \times \left(\frac{1}{100} \text{m}\right)^2 = 1.57 \times 10^{-2} \text{ m}^2$$

The initial flux through the coil,

$$\phi_B = B.A = 2.512 \times 10^{-2} \times 1.57 \times 10^{-2} \text{ webers}$$

$$= 3.94 \times 10^{-4} \text{ webers}$$

As the current is changed from + 2 amp. to – 2 amp, the flux will change from $+ 3.94 \times 10^{-4}$ webers to $- 3.94 \times 10^{-4}$ webers, therefore, the change in flux linked with the coil in time interval of 0.1 sec. is

$$\phi = 3.94 \times 10^{-4} (-3.94 \times 10^{-4}) - 7.88 \times 10^{-4} \text{ webers}$$

According to Faraday's, the magnitude of emf induced,

$$\xi = \frac{d\phi}{dt} = \frac{7.88 \times 10^{-4}}{0.1} = 7.88 \times 10^{-3} \text{ volts} = 7.88 \text{ mV}$$

11.10 INDUCED ELECTRIC FIELD AND VECTOR POTENTIAL

We still have one problem to solve. How do we calculate the induced electric field in cases in which relative motion considerations do not work?

The magnetic flux is given by,

$$\phi_B = \int_S \vec{B} \cdot d\vec{S}$$

and we know that $\vec{B} = \text{curl } \vec{A} = \nabla \times \vec{A}$

hence,

$$\phi_B = \int_S \text{curl } \vec{A} \cdot d\vec{S} = \int_S (\nabla \times \vec{A}) \cdot \hat{n}\, dS$$

Now using the Stoke's theorem,

$$\oint_{\substack{\text{boundary}\\ \text{of } dS}} \vec{A}\cdot d\vec{l} = \int_{s} (\nabla\times\vec{A})\cdot\hat{n}\,dS$$

putting in the above equation, we get,

$$\phi_B = \int_{s} \vec{B}\cdot\hat{n}\,dS = \oint_{\substack{\text{boundary}\\ \text{of } dS}} \vec{A}\cdot d\vec{l} \qquad \ldots(11.10)$$

This equation is quite general. It comes from the general relation $\vec{B} = \text{curl}\,\vec{A}$ and the use of Stoke's theorem and holds when currents are variable. Using the Faraday's law, we have,

$$\xi = \oint \vec{E}\cdot d\vec{l} = -\frac{\delta\phi_B}{\delta t}$$

putting ϕ_B from (11.10),

$$\xi = \oint \vec{E}\cdot d\vec{l} = -\oint \frac{\delta\vec{A}}{\delta t}\cdot d\vec{l}$$

so, that, $\oint \left(\vec{E} + \frac{\delta\vec{A}}{\delta t}\right)\cdot d\vec{l} = 0$

For any closed path, as we have discussed that when $\oint \vec{E}\cdot d\vec{l} = 0$, $\vec{E}$ can be expressed as negative gradient of a scalar. The vanishing of this integral implies the existence of a potential V whose negative gradient gives the sum $\vec{E} + \frac{\delta\vec{A}}{\delta t}$. Thus,

$$\vec{E} + \frac{\delta A}{\delta t} = -\nabla V$$

or $$\vec{E} = -\frac{\delta\vec{A}}{\delta t} - \nabla V \qquad \ldots(11.11)$$

The quantity $-\nabla V$ appears to represent any electrostatic field that is present. For steady currents $\vec{A}$ is constant and equation (11.11) reduces to the fundamental equation of electrostatics,

$$\vec{E} = -\nabla V$$

As here we are not dealing with static field ∇V is disregarded,

$$\vec{E} = -\frac{\delta\vec{A}}{\delta t} \qquad \ldots(11.12)$$

This equation expresses Faraday's law for fixed circuit in terms of fields and the directly calculable vector potential A and is the solution of our problem.

11.11 FOUCAULT'S OR EDDY CURRENTS

Whenever the number of lines of force linked with a conductor changes, emf is induced. However, it is not necessary that the conductor be in the form of a wire or coil. The induced effect is observed even if the conductor is in the form of sheet or plate in accordance with Lenz's law. Although the induced emf in such cases is small but since the resistance of the conductor is negligible hence the induced current will be very strong which will produce strong magnetic field.

Foucault observed that when a mass of metal is moved in a magnetic field, induced currents are produced resulting in the dissipation of energy in the form of heat. Such currents are called Foucault's currents or eddy currents. Sometimes these are also referred as induction currents.

The phenomenon of induction currents in continuous material medium due to varying magnetic field is well understood with Faraday's law.

$$\text{Curl } \vec{E} = \nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}$$

Multiplying the above equation by conductivity σ of the medium and then taking the curl of the equation, we get,

$$(\text{curl } \sigma \text{ curl } \vec{E}) = -\sigma \frac{\delta}{\delta t} \text{curl } \vec{B}$$

Assuming σ to be constant for material,

$$\text{curl curl } \sigma \vec{E} = -\sigma \frac{\delta}{\delta t} \text{curl } \vec{B}$$

But by Ohm's law,

$$\vec{J} = \sigma \vec{E} \quad \text{(see eq. 8.28)}$$

hence, $$\text{curl curl } \vec{J} = -\sigma \frac{\delta}{\delta t} \text{curl } \vec{B} \qquad \ldots(1)$$

But by equation (3.29),

$$\text{curl curl } \vec{J} = \text{grad div } \vec{J} - \nabla^2 \vec{J}$$

Further, assuming that there is no accumulation of charge, therefore,

$$\text{div } \vec{J} = 0$$

we get,

$$\text{curl curl } \vec{J} = -\nabla^2 \vec{J} \qquad \ldots(2)$$

Substituting in equation (1) the value of equation (2), we get

$$-\nabla^2 \vec{J} = -\sigma \frac{\delta}{\delta t} \text{curl } \vec{B}$$

or $$\nabla^2 \vec{J} = \sigma \frac{\delta}{\delta t} \text{curl } \vec{B} \qquad \ldots(3)$$

But, by Ampere's law,

$$\text{curl } \vec{B} = \mu_0 \vec{J}$$

which is expression when medium is vacuum. When medium of permeability μ is considered, then we have,

$$\text{curl } \vec{B} = \mu \vec{J}$$

Therefore, equation (3) reduces,

$$\nabla^2 \vec{J} = \sigma \frac{\delta}{\delta t} (\mu \vec{J})$$

$$= \sigma\mu \frac{\delta \vec{J}}{\delta t} \qquad \ldots(4)$$

But, $\vec{J} = \frac{\vec{I}}{A}$, where A is the area of cross-section of the conductor, equation (4) can be written as,

$$\nabla^2 I = \sigma\mu \frac{\delta I}{\delta t} \qquad \ldots(11.13)$$

which is the equation for the currents induced in the conducting medium by the changing magnetic field. These induced currents are known as Foucault's, eddy or induction currents.

11.12 INDUCTORS AND INDUCTANCE

Inductor. An inductor is a device for storing energy in a magnetic field. An inductor is generally called as inductance. In usual practice a coil or solenoid is treated as inductor. This is done for uniform usage when we speak, for example, of an inductor of 1 henry inductance, a capacitor of 1 microfarad capacitance or a resistor of 1 ohm resistance. Inductor may be regarded as the magnetic counterpart of a capacitor, which stores energy in an electric field.

Self-inductance

We have already calculated the magnetic field due to a current in conductor and the value of magnetic flux due to magnetic induction $\vec{B}$ is given by,

$$\phi_B = \int_S \vec{B} \cdot d\vec{S}$$

Finally we have induction law in terms of flux rule, *i.e.*

$$\xi = \frac{d\phi_B}{dt}$$

Induction law holds whenever there is change in flux. When the current changes there is a change in the flux. Let us consider that the current i is changing in a single coil, flux will also change in the same proportion, *i.e*,

$$\phi_B \propto i$$

$$\phi_B = Li \qquad \ldots(11.14)$$

or $$\xi = -L\frac{di}{dt} \qquad \ldots(11.15)$$

where L is constant which depends on the geometry of the circuit and the permeability of the medium in which it is immersed. It is known as the coefficient of self-inductance or the self-inductance of the circuit. Self-inductance can be defined either by equation (11.14) or (11.15). Then, we define, the inductance L of an inductor is equal to the ratio of its total magnetic flux linking and the current i through the inductor, *i.e.*,

$$L = \frac{\phi_B}{i}$$

If i = 1 amp, *then it is equal to the total flux through the circuit when unit current flows through it. This definition is satisfactory for a medium with a "constant permeability," such as air. When the permeability of the medium is not constant, then it is defined by eqn. (11.15), i.e.,*

$$L = \frac{\xi}{-\frac{di}{dt}}$$

i.e. self-inductance is equal to the ratio of the induced emf in the circuit to the rate of decay of current.

If $$-\frac{di}{dt} = \frac{1\text{amp}}{\text{sec.}}$$

Then L is the induced emf in the circuit when the rate of decay of current is unity. However both those definitions lead to the same result when permeability of the medium is constant.

Unit

The unit of inductance is the henry, thus from equation (11.14),

$$\text{Henrys} = \frac{\text{Webers}}{\text{Ampere}}$$

But $\text{Weber} = \dfrac{\text{Newton.Meter}}{\text{Ampere}}$

Hence, $\text{Henrys} = \dfrac{\text{Newton.Meters}}{\text{Ampere}^2}$, it is "absolute" unit.

From equation (11.15),

$$\text{Henrys} = \frac{\text{Volts}}{\text{amp/sec.}} = \frac{\text{Volts sec}}{\text{Ampere}}$$

$$= \frac{\text{Newton.metre}}{\text{Coul.}} \frac{\text{Sec}}{\text{Amp}}$$

since $\text{volt} = \dfrac{\text{Newton.metre}}{\text{Coul.}}$

Hence $\text{Henrys} = \dfrac{\text{Newton.metres}}{\text{Ampere}^2}$ as $\dfrac{\text{Coul}}{\text{sec}}$ Ampere

which is same as derived above.

Practical unit is Henry,

$$1 \text{ Henry} = \frac{1 \text{ volt}}{1 \text{ amp/sec}}$$

$$1 \text{ Milli henry} = 10^{-3} \text{ Henry}$$

$$1 \text{ Micro henry} = 10^{-6} \text{ Henry}$$

i.e. 1 henry is the inductance of a circuit in which an emf of 1 volt is induced, when the current in it changes at the rate of one amp. per second.

Physical Significance of Self-Inductance

The self-inductance plays the same role in an electrical circuit as the mass or moment of inertia in mechanical motion. When circuit is on the self-inductance by inducing an emf slows down the growth of the current. Similarly, when circuit is broken an induced emf in the same direction as original emf thus slows down the decay of current in the circuit. If we wish to change the current in a coil we must overcome this inertia by connecting the coil to some external voltage source such as a battery or a generator as shown in Fig.11.6. In such circuit, the current i depends on the voltage V according to the relation,

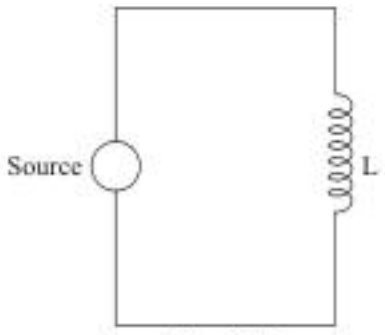


Fig. 11.6

$$V = L\frac{di}{dt}$$

This equation has the same form as the Newton's law of motion for a particle in one dimension *i.e.* externally applied voltage corresponds to externally applied force F and the current corresponds to the velocity v of the particle. We can make the following correspondence:

Particle	*Coil*
F (force)	V (Potential difference)
v (velocity)	i (current)
x (displacement)	q (charge)
$F = m\dfrac{dv}{dt}$	$V = L\dfrac{di}{dt}$
mv (momentum)	Li (flux)
$\dfrac{1}{2}mv^2$ (KE)	$\dfrac{1}{2}Li^2$ (magnetic energy)

11.13 CALCULATION OF SELF-INDUCTANCE

(1) Solenoid

Let l be the length of a long solenoid with an air core and total number of turns are N. When a current i flows through it the magnetic field inside it is given by equation (10.27), *i.e.*

$$B = \mu_0 \frac{Ni}{l}$$

Now if A be the area of each turn then magnetic flux through each turn,

$$= \mu_0 \frac{NiA}{l}$$

total flux through the solenoid

$$= \mu_0 \frac{N}{l} iAN$$

or $$\phi_B = \mu_0 \frac{N^2}{l} Ai$$

when the current i varies, the flux ϕ_B changes giving rise to the induced emf

$$\xi = -\frac{d}{dt}\left(\mu_0 \frac{N^2}{l} Ai\right)$$

$$= -\frac{\mu_0 N^2 A}{l} \frac{di}{dt}$$

$$= -L \frac{di}{dt}, \qquad \text{From definition of L}$$

Therefore, $$L = \frac{\mu_0 N^2 A}{l}$$

or $$L = \mu_0 n^2 l A \qquad \ldots(11.16)$$

where n are the number of turns per unit length.

If the solenoid is wound over a core of constant permeability μ, then,

$$L = \frac{\mu_0 N^2 A}{l} \qquad \ldots(11.17)$$

where $\mu = \mu_0 \mu_r$ and μ_r is the relative permeability.

(2) Two Parallel Wires

Consider the circuit consists of two parallel wires A and B separated by distance d. The current is going through A and returning by B. Suppose radius of each wire be r.

Consider the unit length as shown dotted in Fig. 11.7. The field due to the current in the two wires,

$$B = \frac{\mu_0 i}{2\pi x} + \frac{\mu_0 i}{2\pi (d - x)}$$

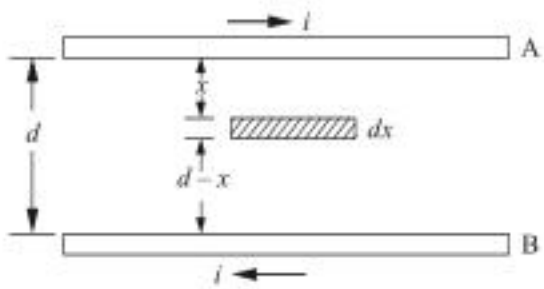


Fig. 11.7

$\left(\text{The magnetic field due to a straight wire} = \frac{\mu_0 i}{2\pi r}\right)$

The flux $d\phi_B$ through the dotted area,

$$d\phi_B = B\,dx$$

$$\text{total flux } \phi_B = \int_0^d d\phi_B = \int_r^{d-r} \left(\frac{1}{x} + \frac{1}{d-x}\right) \frac{\mu_0}{2\pi}\, dx$$

$$= \frac{\mu_0 i}{2\pi} \int_r^{d-r} \left(\frac{1}{x} + \frac{1}{d-x}\right) dx$$

$$= \frac{\mu_0 i}{2\pi} \left[\log x - \log (d-x)\right]_r^{d-r}$$

$$= \frac{\mu_0 i}{2\pi} \left[2 \log_e \frac{d-r}{r}\right]$$

$$= \frac{2\mu_0 i}{2\pi} \log_e \frac{d-r}{r}$$

Now when i varies the induced emf according to Faraday's law is given by,

$$\xi = -\frac{d\phi_B}{dt}$$

$$= -\frac{2\mu_0}{2\pi} \log_e \frac{d-r}{r} \frac{di}{dt}$$

$$= -L \frac{di}{dt}$$

hence, $$L = \frac{\mu_0}{\pi} \log_e \frac{d-r}{r} \qquad \ldots(11.18)$$

$$= \frac{\mu_0}{\pi} \times 2.3026 \log_{10} \frac{d-r}{r} \qquad \ldots(11.19)$$

(3) Two Coaxial Cylinders

The field B due to the current i in the inner cylinder at a distance r as shown in Fig. 11.8 is given by,

$$B = \frac{\mu_0 i}{2\pi r}$$

The flux through the area ABCD where AB = 1 metre

$$\phi_B = \int_a^b \frac{\mu_0 i}{2\pi r}\, dr = \frac{\mu_0 i}{2\pi} \left|\log_e r\right|_a^b$$

$$= \frac{\mu_0 i}{2\pi} \log_e \frac{b}{a}$$

Now when i varies flux changes,

$$\xi = -\frac{d\phi_B}{dt}$$

$$= -\frac{\mu_0}{2\pi} \log_e \frac{b}{a} \frac{di}{dt}$$

$$= -L \frac{di}{dt}$$

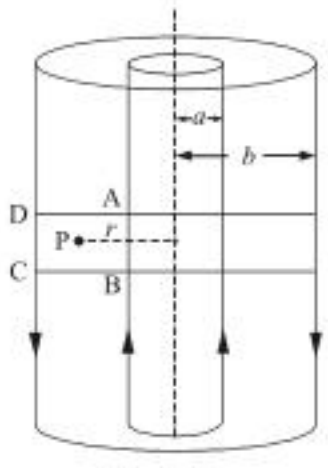


Fig. 11.8

Therefore, $$L = \frac{\mu_0}{2\pi} \log_e \frac{b}{a} \qquad \ldots(11.20)$$

This is the self-inductance of two coaxial cylinders per unit length.

4. Toroidal Coil of Rectangular Cross-section

A toroidal coil of rectangular cross-section is shown in Fig. 11.9. Let N be the number of turns uniformly distributed over its circumference. Suppose a and b be its interval and external radii respectively.

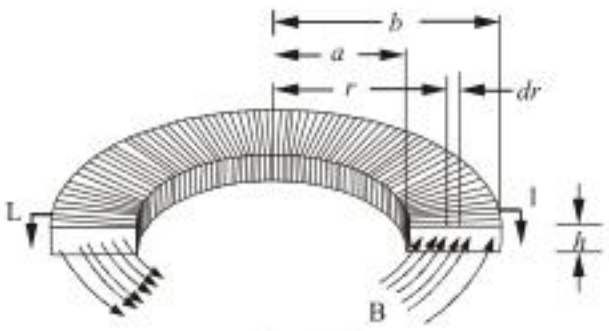


Fig. 11.9

The lines of B for the toroid are concentric circle. Consider a circular path of radius r concentric with the toroid. Let a current i flows through the winding. Applying Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \times \text{current enclosed by path}$$

$$B\,2\pi r = \mu_0 \, Ni$$

$$B = \frac{\mu_0 \, Ni}{2\pi r}$$

Consider a rectangular strip of width dr and height h at a distance r from the centre. Then the magnetic flux linked with this elementary strip is,

$$d\phi_B = B\,dA$$

$$= \frac{\mu_0 \, Ni}{2\pi r} h\,dr, \; (dA = h\,dr)$$

Therefore, the flux linked with one turn of the toroid is

$$\phi_B = \int_a^b d\,\phi_B = \frac{\mu_0 \, Nih}{2\pi} \int_a^b \frac{dr}{r}$$

$$= \frac{\mu_0 \, Nih}{2\pi} \log_e \frac{b}{a}$$

total flux with N turns is

$$= N\phi_B$$

$$= \frac{\mu_0 \, N^2 \, ih}{2\pi} \log_e \frac{b}{a}$$

$$= Li, \quad \text{according to definition of self-inductance}$$

Therefore, $$L = \frac{\mu_0 \, N^2 \, h}{2\pi} \log_e \frac{b}{a} \qquad \ldots(11.21)$$

If coil is wound over a core of permeability μ, then,

$$L = \frac{\mu \, N^2 \, h}{2\pi} \log_e \frac{b}{a} \qquad \ldots(11.22)$$

where $\mu = \mu_r \mu_0$ and μ_r is the relative permeability.

(5) Self-inductance of a Toroidal Coil of Circular Cross-section

Fig. 11.10 shows a toroid. Let us apply Ampere's law to the circular path of mean radius R and let total number of turns be N in it and i be the current flowing. Then magnetic field B is given by,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \, Ni$$

Since lines of $\vec{B}$ for the toroid are concentric circles *i.e.* angle between $\vec{B}$ and $d\vec{l}$ is zero, therefore,

$$\oint B\,dl = \mu_0 \, Ni$$

or $$B = \frac{\mu_0 Ni}{2\pi R}$$

tota flux linked with toroid,

$$\phi_B = BNA$$

where A is the cross-sectional area of each turn of the coil. Substituting B from the above,

$$\phi_B = \frac{\mu_0 Ni}{2\pi R} \cdot NA = \frac{\mu_0 N^2 iA}{2\pi R}$$

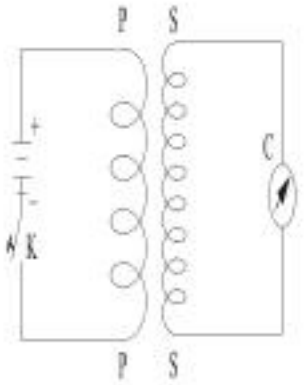


Fig. 11.10

If current is changing, then,

$$\phi_B = Li$$

therefore, $$L = \frac{\mu_0 N^2 A}{2\pi R} \quad \ldots(11.23)$$

For a toroid wound on a core of constant permeability μ

$$L = \frac{\mu N^2 A}{2\pi R} \quad \ldots(11.24)$$

where $\mu = \mu_0 \mu_r$ and μ_r is the relative permeability.

11.14 ENERGY STORED IN MAGNETIC FIELD

As we have already discussed [art. 11.12—Physical significance] that the inductance of a circuit opposes any change in the current. Suppose in a circuit current is increasing, self-inductance will try to decrease this current. Therefore, the work must be done to overcome the induced emf and to drive the current against it. This work done is stored up as energy in the system.

To derive a quantitative expression for the storage of the energy in the magnetic field consider the circuit which shows a source of emf ξ connected to a resistance R and inductance L. From, Kirchhoff's law,

$$\xi = iR + L\frac{di}{dt}$$

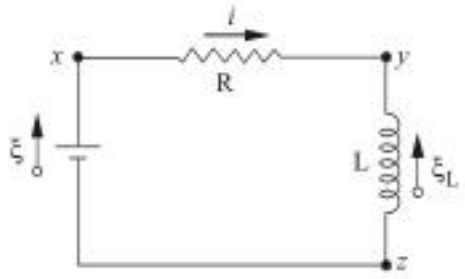


Fig. 11.11

which is derived on the principle of conservattion of energy. If we multiply the above equation by i, we get,

$$\xi i = i^2 R + Li\frac{di}{dt}$$

We can interpret the above equation as follows:

1. If a charge dq passes through the source of emf (battery) in time dt, then the work done by source in time dt is $\xi\, dq$. The rate of doing work is $\xi \frac{dq}{dt} = \xi i$. Therefore, *the LHS of above, i.e. ξ i is the rate at which the source of emf delivers energy to the circuit.*
2. The first term on R.H.S. is the rate at which *energy appears as heat in the resistance R.*
3. *The energy that does not appear as heat, according to the rule of conservation of energy, must be stored in the magnetic field. Therefore, $Li\frac{di}{dt}$ represents the rate at which the energy is*

stored in the magnetic field, i.e.,

$$\frac{dU_B}{dt} = Li\frac{di}{dt} \quad \text{or} \quad dU_B = Li\,di$$

therefore, the energy stored in increasing the current from O to i_0 is

$$U_B = \int_0^{i_0} Li\,di = \frac{1}{2}Li_0^2 \qquad \ldots(11.25)$$

This is the total stored magnetic energy in an inductance L carrying current i_0. This relation is identical to the expression for the energy associated with a capacitor C carrying charge q *i.e.,*

$$U_E = \frac{1}{2}\frac{q^2}{C}$$

Thus $\frac{1}{2}Li_0^2$ is the energy required from an external source to overcome the emf in the inductance when the current rises to its steady value.

Energy Stored up in Solenoid

The energy stored is given by equation (11.25), *i.e.*

$$U_B = \frac{1}{2}Li^2$$

where i is the current in the solenoid. We have calculated in the last section the self-inductance of a solenoid given by,

$$L = \frac{\mu_0 N^2 A}{l}$$

and magnetic induction $\vec{B}$ is given by,

$$B = \frac{\mu_0 Ni}{l}$$

or $$i = \frac{Bl}{\mu_0 N}$$

putting L and i, we get the energy stored up in the magnetic field of the solenoid,

$$U_B = \frac{1}{2}\frac{\mu_0 N^2 A}{l}\left(\frac{Bl}{\mu_0 N}\right)^2$$

$$= \frac{1}{2\mu_0}B^2 Al \text{ joules} \qquad \ldots(11.26)$$

Energy Density

Al is the volume of the solenoid. *The energy is stored within this volume because B outside of the solenoid is zero.* Moreover, the stored energy must be uniformly distributed throughout the volume of the solenoid because the magnetic field is uniform enerywhere inside. The energy density is defined as the energy per unit volume. The energy stored per unit volume of the magnetic field,

$$= \frac{U_B}{Al} = \frac{1}{2}\frac{B^2}{\mu_0} \text{ Joules/metre}^3 \qquad \ldots(11.27)$$

This equation gives the energy density stored at any point where the magnetic induction is $\vec{B}$. *Though the equation is derived for solenoid, but it holds in general or, in other words, it is true for all magnetic field configurations. It is analogous to that of energy in an electric field i.e.* $\frac{\varepsilon_0 E^2}{2}$.

11.15 MEASUREMENT OF SELF-INDUCTANCE BY RAYLEIGH'S METHOD

This method of measuring the self-inductance of a coil employs a Wheatstone bridge. The self-inductance coil L is placed in one of the arms of the Wheatstone bridge in series with fractional resistance R and a B.G. is connected in arm BD as shown in Fig. 11.12.

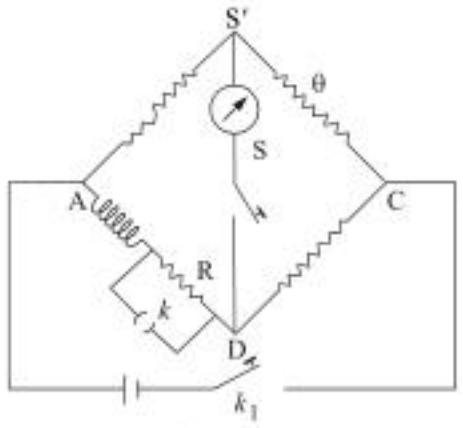


Fig. 11.12

At first the resistance R is shunted out with the key K and the battery key K_1 was pressed and then the galvanometer key K_2 and the balance for steady current is obtained. Now, if the galvanometer key K_2 is closed first and then the battery key K_1, a ballistic throw is observed due to an induced emf $L\frac{di}{dt}$ in the arm AD of the bridge which gives rise to a proportionate current in all parts of the bridge. Thus when K_1 is pressed after K_2

$$\text{Induced current} = k\,L\frac{di}{dt}$$

where k is constant of proportionality depending on the galvanometer and the bridge resistances. Therefore the charge flowing through galvanometer during a small interval of time dt

$$= k\,L\frac{di}{dt}\,dt$$

Therefore, the total charge that flows through the galvanometer during the period t in which the current in arm AD rises from 0 to i_0 is,

$$q = \int_0^t k\,L\frac{di}{dt}\,dt = \int_0^{i_0} k\,L\,di$$

or $$q = k\,L\,i_0$$

But, $$q = \frac{C}{BA}\frac{T}{2\pi}\theta\left(1+\frac{\lambda}{2}\right)$$, see theory of B.G.

Hence, $$k\,L\,i_0 = \frac{CT}{2\pi BA}\theta\left(1+\frac{\lambda}{2}\right) \quad \ldots(1)$$

The galvanometer constant $\frac{C}{BA}$ is determined by calibrating the galvanometer by introducing a small resistance of the order of 0.001, 0.01 or 0.1 ohm in the arm AD by adjusting R. This small resistance will not appreciably affect the value of the steady current i_0 in the arm AD but introduces a small additional emf Ri_0 in this arm. The current due to this emf $= k\,R\,i_0$ if θ_0 be the steady deflection in galvanometer due to this current, then,

$$k\,R\,i_0 = \frac{C}{BA}\theta_0 \qquad \ldots(2)$$

Dividing (1) by (2),

$$\frac{L}{R} = \frac{T}{2\pi}\frac{\theta}{\theta_0}\left(1+\frac{\lambda}{2}\right)$$

or
$$L = R\frac{T}{2\pi}\frac{\theta}{\theta_0}\left(1+\frac{\lambda}{2}\right) \qquad \ldots(11.28)$$

This gives the value of L in terms of known quantities.

Example 11.5 *Calculate the inductance of a solenoid of 2000 turns wound uniformly over a length of 50 cm on a cylindrical paper tube 4 cm in diameter. The medium is air.*

Solution. From equation (11.16),

$$L = \frac{\mu_0 N^2 A}{l}$$

$$= \frac{4\times10^{-7}\times2000\times2000\times\pi\,(2\times10^{-2})^2}{0.5}$$

$$= 12.6\times10^{-3} \text{ henrys}$$

$$= 12.6 \text{ milli henrys}$$

Example 11.6 *A solenoid having a core of 4 cm² in cross-section, half air* $\mu = 1$ *and half iron* $(\mu = 500)$ *is 20 cm long. If the number of turns on it are 1000, calculate the self-inductance of the coil.*

Solution. Using equations (11.16) and (11.17), we get,

$$L = L_1 + L_2$$

where L_1 is the inductance due to part as air core and L_2 that of iron core.

$$L = \mu_0 n^2 l\,A_1 + \mu\mu_0 n^2 l\,A_2,\ A_1 = A_2 = \frac{1}{2}4\text{cm}^2 = 2\text{cm}^2$$

$$= \mu_0\left[\frac{1000}{(20\times10^{-2})}\right]^2 \times 20\times10^{-2}\left[2\times10^{-4}+500\times2\times10^{-4}\right]$$

$$= \frac{4\pi\times10^{-7}\times1000\times1000\times20\times10^{-2}\times2\times10^{-4}}{20\times20\times10^{-4}}(1+500)$$

$$= 0.6298 \text{ henry}$$

Example 11.7 *Calculate the self-inductance of a toroid of rectangular cross-section of total number of turns 1000. Given a = 5.0 cm, b = 10 cm, and h = 1.0 cm.*

Solution. Using equation (11.21),

$$L = \frac{\mu_0 N^2 h}{2\pi}\log_e\frac{b}{a}$$

$$= \frac{2.3026\times4\pi\times10^{-7}\times10^6\times1\times10^{-2}}{2\pi}\log_{10}\frac{10\times10^{-2}}{5\times10^{-2}}$$

$$= 1.4 \text{ milli henry}$$

Example 11.8 *An air cored toroidal coil has 500 turns and a mean diameter of 32 cm with a cross-sectional area of 2 cm². Calculate (1) the inductance of the coil, (2) the average emf induced when a current of 2 amp is reversed in a time of 0.02 sec.*

Solution. From equation (11.23), inductance L is given by,

$$L = \frac{\mu_0 N^2 A}{2\pi R}$$

here as given,

$$L = \frac{4\pi \times 10^{-7} \times 500 \times 500 \times 2 \times 10^{-4}}{2\pi \times 16 \times 10^{-2}}$$

$$= 6.25 \times 10^{-5} \text{ henry}$$

$$\text{Induced emf } \xi = -L \frac{di}{dt}$$

The negative sign gives the direction, hence magnitude,

$$\xi = L \frac{di}{dt}$$

$$= 6.25 \times 10^{-5} \times \frac{2-(-2)}{0.02} \text{ volts}$$

$$= 0.0125 \text{ volt}$$

Example 11.9 *A coil has an inductance of 5.0 henry and resistance of 20 ohms. If a 100 volt emf is applied, calculate the energy stored in the magnetic field after the current has built up to its maximum value.*

Solution. Maximum current is given by,

$$i = \frac{\xi}{R} = \frac{100}{20} = 5.0 \text{ amp.}$$

The energy stored is given by equation (11.25) *i.e.*

$$U_B = \frac{1}{2} L i^2$$

$$= \frac{1}{2} \times 5 \times 5^2 \text{ joules}$$

$$= 6.25 \text{ joules}$$

Example 11.10 *Compare the energy required to set up in a cube of 10 cm sides (a) a uniform electric field of 10^5 volts/metre and (b) a uniform magnetic field of 1 weber/metre².*

Solution. (a) In the electrostatics field, where v_0 is the volume of the cube, we have by equation (4.41)

$$U_E = \frac{1}{2} \varepsilon_0 E^2 v_0$$

$$= \frac{1}{2} (8.9 \times 10^{-12}) \times (10^5)^2 (0.1)^3 \text{ joules}$$

$$= 4.5 \times 10^{-5} \text{ joules}$$

(b) Energy in magnetic field is given by equation (11.26),

$$U_B = \frac{1}{2} \frac{B^2}{\mu_0} v_0$$

$$= \frac{1}{2} \times \frac{1 \times (0.1)^3}{4\pi \times 10^{-7}}$$

$$= 400 \text{ joules}$$

Thus, much larger amount of energy can be stored in magnetic field than in an electric field. Conversely, much more energy is required to set up a magnetic field than is required to set up an electric field of similar magnitude.

11.16 MUTUAL INDUCTANCE

Fig. 11.13 is an arrangement of two fixed coils. A controllable current i_1 is passed through the primary P. The current in P will produce magnetic field say B_1 at any instant when current is constant i_1. If ϕ_{21} denotes the flux of B_1 through the secondary,

$$\phi_{21} = \int_S \vec{B}_1 \cdot d\vec{S}_2$$

The value of ϕ_{21} depends on B_1 which in turn depends on i_1 provided the shape and relative position of P and S is fixed, we have,

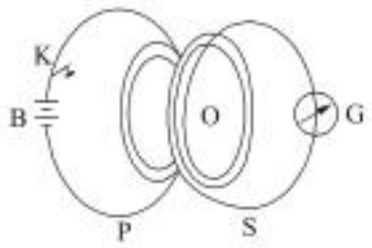


Fig. 11.13

$$\phi_{21} \propto i_1$$

$$\frac{\phi_{21}}{i_1} = \text{constant} \qquad \ldots(11.29)$$

Suppose i_1 changes with time the flux ϕ_{21} will also change and there will be an induced emf in secondary. According to Faraday's law,

$$\xi_{21} = -\frac{d\phi_{21}}{dt}$$

$$= -\text{const}\,\frac{di_1}{dt} \qquad \ldots(11.30)$$

The constant in equation (11.29) and (11.30) is same and denoted by M_{21}, is called the coefficient of mutual inductance.

$$\xi_{21} = -M_{21}\frac{di_1}{dt}$$

or magnitude $\xi_{21} = M_{21}\frac{di_1}{dt}$

The negative sign indicates the direction of emf induced. The value of M_{21} depends on the geometry of our arrangement of coils P and S.

Let us now pass the current through the secondary and calculate the induced emf in the primary coil. Proceeding in the same manner, we get,

$$\xi_{21} = M_{21}\frac{di_2}{dt}$$

Thus for any two circuits, in fact,

$$M_{21} = M_{21} = M$$

The coefficient of mutual inductance can be defined in terms of any of the equations (11.29) or (11.30).

(1) By equation (11.29), *it is the flux linked with a circuit due to a unit current flowing through the other nearby circuit.*

$$M_{21} = \frac{\phi_{21}}{i_1}$$

If $i_1 = 1, M_{21} = \phi_{21}$

(2) By equation (11.30), *it is emf induced in a circuit, when the rate of decay of current in the other circuit is unity, i.e.,*

$$M_{21} = \frac{\xi_2}{-\frac{di}{dt}}$$

when, $-\frac{di_1}{dt} = 1, M_{21} = \xi_{21}$

The definition given by equation (11.29) is satisfactory when the permeability of the medium is constant. When permeability of the medium is not constant, then we define mutual induction by equation (11.30). However, both these definitions lead to the same result when the permeability of the medium is constant.

Units

The units of mutual inductance are the same as those of the self-inductance. The practical unit of the mutual inductance of the two circuits, when the induced emf in one circuit is one volt due to rate of change of current one ampere/sec. in the other circuit and is called henry. Absolute unit is given in the same way as for self-inductance

$$\text{henrys} = \frac{\text{Newtons.metres}}{\text{ampere}^2}$$

11.17 MUTUAL INDUCTANCE BETWEEN TWO ARBITRARY CIRCUITS

Suppose we have two circuits as shown in Fig. 11.14. As we know that the current I_1 in circuit 1 will produce a magnetic flux. Let us take the part of this flux associated with circuit 2, *i.e.*,

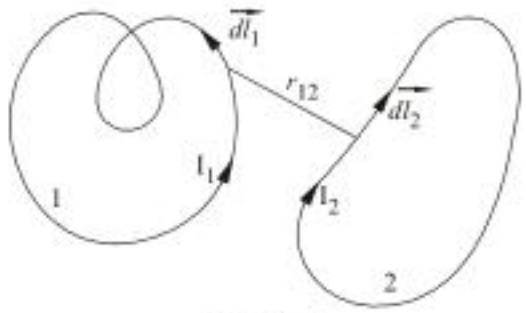


Fig. 11.14

$$\phi_{21} = \int_{S_2} \vec{B}_1 \cdot \hat{n} dS_2$$

where B_1 is the magnetic field at a point on S_2 due to current I_1. Since,

$$\nabla \cdot \vec{B}_1 = 0 \text{ and } \vec{B}_1 = \nabla \times \vec{A}_1$$

where $\vec{A}_1$ is the vector potential at the circuit 2 due to current in circuit 1.

$$\phi_{21} = \int_{S_2} (\nabla \times \vec{A}_1) \cdot \hat{n} dS_2$$

Using the Stoke's theorem, *i.e.*

$$\int_{S_2} (\nabla \times A_1) \cdot \hat{n} dS_2 = \oint_2 \vec{A}_1 \cdot d\vec{l}_2$$

hence, $\phi_{21} = \oint_2 \vec{A}_1 \cdot d\vec{l}_2$

But, $A_1 = \dfrac{\mu_0 I_1}{4\pi} \oint_1 \dfrac{dl_1}{r_{12}}$ (see eq. 10.41)

According to the definition of vector potential given in equation (10.41), r_{12} is the separation from dl_1 on circuit 1 to the point on the circuit 2 where potential is A_1.

Putting the value of A_1

$$\phi_{21} = \oint_2 \left(\frac{\mu_0 I_1}{4\pi} \oint_1 \frac{d\vec{l}_1}{r_{12}} \right) \cdot d\vec{l}_2$$

since I_1 does not depend on the variables of integrations and integrations are taken with respect to stationary circuit. Hence,

$$\phi_{21} = \frac{\mu_0 I_1}{4\pi} \oint\oint_{12} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_1^2} \qquad \ldots(11.31)$$

$$= M_{21} I_1$$

where, $M_{21} = \dfrac{\mu_0}{4\pi} \oint\oint_{12} \dfrac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{12}}$ $\ldots(11.32)$

which according to equation (11.29) is mutual inductance. *Equation (11.32) is the Neumann equation.* We see from this integral that M_{21} depends only on the circuit geometry. Proceeding in the same manner, we can calculate,

$$\phi_{12} = M_{12} I_2$$

with $M_{12} = \dfrac{\mu_0}{4\pi} \oint\oint_{21} \dfrac{d\vec{l}_2 \cdot d\vec{l}_1}{r_{21}}$ $\ldots(11.33)$

$r_{12} = r_{21}$ is simply the separation between dl_1 and dl_2. Equations (11.32) and (11.33) are identical. Hence,

$$M_{12} = M_{21}$$

therefore, there is no distinction between M_{12} and M_{21} and we call the mutual inductance M of any two circuits. *This relation is also called the theorem of reciprocity.*

We should be careful to note that while discussing the flux linkage in the above case we have considered that at a time current was flowing in one coil, *i.e.* ϕ_{21} due to current I_1 in coil I and ϕ_{12} due to current I_2 in coil 2.

If there are currents in two coils simultaneously, the magnetic flux linking in either coil will be the sum of two fluxes which would exist separately. Because the law of superposition applies for magnetic field. The flux in either coil will therefore be proportional not only due to the change in current in the other coil but also to the change in the current of the coil itself.

11.18 MUTUAL INDUCTANCE OF CONCENTRIC SOLENOIDS

Suppose we have a long solenoid called the primary, and n_p are the number of turns per unit length on it. Another secondary coil of n_s turns is wound on primary whose area of cross-section is A_p as shown in Fig. 11.15. The magnetic inductance due to the current i in primary is obtained by Ampere's law *i.e.*

$$B = \mu_0 n_p i$$

magnetic flux = B × area of cross-section

$$\phi_B = \mu_0 n_p A_p i$$

this flux is linked with each turn of the secondary. Suppose there is no leakage of the flux. Therefore, the total flux linked with n_s turns of the secondary say ϕ'_B is given by

$$\phi'_B = n_s \cdot \phi_B = \mu_0 n_s n_p A_p i$$

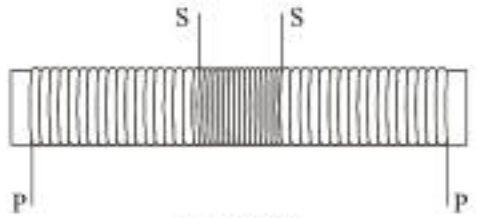


Fig. 11.15

Then induced emf $= -\dfrac{d\phi' B}{dt}$

$$= -\mu_0 n_p n_s A_p \frac{di}{dt}$$

$$= -M \frac{di}{dt}$$

therefore, $M = \mu_0 n_p n_s A_p$ ...(11.34)

If the primary and secondary are wound over a core of material having permeability μ, then,

mutual inductance $= \mu n_p n_s A_p$ and $\mu = \mu_1 \mu_0$ ...(11.35)

11.19 RELATION BETWEEN MUTUAL INDUCTANCE AND SELF-INDUCTANCE

Consider the toroid of N_1 turns called primary with a secondary toroidal winding of N_2 as shown in Fig. 11.16. A current i_1 in the first winding produces a magnetic induction,

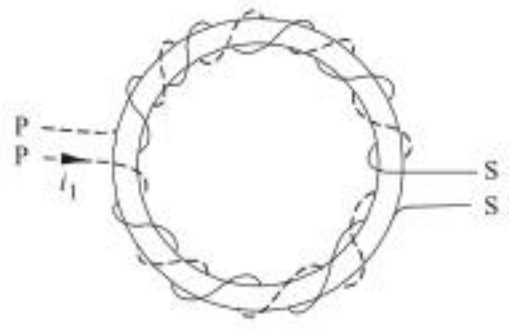


Fig. 11.16

$$B = \frac{\mu_0 N_1 i_1}{2\pi R}$$

and consequently fluxes, ϕ_{11} through primary winding

$$= BN_1 A = \frac{\mu_0 N_1^2 A i_1}{2\pi R}$$

ϕ_{21} through the secondary winding

$$= BN_2 A = \frac{\mu_0 N_1 N_2 A i_1}{2\pi R}$$

From these fluxes it follows,

$\phi_{11} = L_1 i_1$, by definition of self-inductance

$\phi_{21} = M_{21} i_1$, by definition of mutual inductance

$$L_1 = \frac{\mu_0 N_1^2 A}{2\pi R}$$

$$M_{21} = \frac{\mu_0 N_1 N_2 A}{2\pi R}$$

Reversing the procedure and considering a current i_2 in secondary, we get

$$L_1 = \frac{\mu_0 N_2^2 A}{2\pi R}$$

$$M_{21} = \frac{\mu_0 N_1 N_2 A}{2\pi R}$$

L_1 is the self-inductance of 1st coil and L_2 that of the second coil. We see $M_{12} = M_{21}$ is the mututal inductance between the two coils, product of L_1 and L_2

$$L_1 \cdot L_2 = \frac{\mu_0 N_1^2 A}{2\pi R} \cdot \frac{\mu_0 N_2^2 A}{2\pi R} = \frac{\mu_0^2 N_1^2 N_2^2 A^2}{4\pi^2 R^2} = M_{12}^2$$

Hence, $M_{12} = \sqrt{L_1 L_2}$ ...(11.36)

This relation represents a limit that is imposed on the mutual inductance between two circuits. In deriving this relation, we have assumed that there is no flux leakage but practically there is always some leakage. Then the mutual inductance is always less than given in equation (11.36). A coupling coefficient K is often introduced and defined by,

$$M = K\sqrt{L_1 L_2}, |K| \le 1 \qquad \ldots(11.37)$$

11.20 ENERGY STORED IN TWO INTERACTING CIRCUITS

Let us consider two circuits of self-inductances L_1 and L_2 as shown in Fig. 11.17. They are mutually coupled and if M is the mutual coefficient of the circuit, the fluxes are linked. The induced emf in circuit 1 is denoted by ξ_1 and that of circuit 2 by ξ_2, then,

$$\xi_1 = L_1 \frac{dI_1}{dt} \pm M \frac{dI_2}{dt}$$

$$\xi_2 = L_2 \frac{dI_2}{dt} \pm M \frac{dI_1}{dt}$$

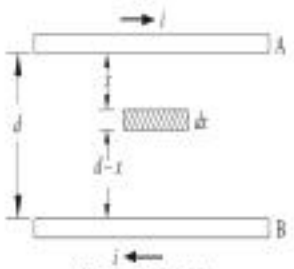


Fig. 11.17

If fluxes due to I_1 and I_2 are in the same direction in each coil then coupling is positive *i.e.* positive M. For negative coupling we have negative M. The energy can be calculated from the basic definition of emf that the work done for producing emf, *i.e.*

$$\xi_1 = \frac{dW_1}{dq_1} \text{ and } \xi_2 = \frac{dW_2}{dq_2}$$

Hence the work done to build up current I_1 and I_2.

$$dW = \xi_1 dq_1 + \xi_2 dq_2$$

$$= \left(L_1 \frac{dI_1}{dt} \pm M \frac{dI_2}{dt} \right) dq_1 + \left(L_2 \frac{dI_2}{dt} \pm M \frac{dI_1}{dt} \right) dq_2$$

$$= \left(L_1 \frac{dI_1}{dt} \pm M \frac{dI_2}{dt} \right) I_1 \, dt + \left(L_2 \frac{dI_2}{dt} \pm M \frac{dI_1}{dt} \right) I_2 \, dt$$

$$= L_1 I_1 \, dI_1 \pm M I_1 \, dI_2 + L_2 I_2 \, dI_2 \pm M I_2 \, dI_1$$

$$= L_1 I_1 \, dI_1 + L_2 I_2 \, dI_2 \pm d \, (M I_1 I_2)$$

Integrating this we get energy stored in the system,

$$U_B = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M I_1 I_2$$

The first two terms on the right hand side of this equation are energies due to self-inductance and called self-energies arising from the interaction of each circuit with its own field. The third is interaction energy arising from mutual inductance, the positive value corresponds to positive coupling and negative value for negative coupling.

11.21 THE ENERGY OF STEADY CURRENTS

The energy of small current loop is given by,

$$U = \vec{m} \cdot \vec{B} \qquad \text{(see eq. 9.14)}$$

Though we have derived it for a plane rectangular loop but it holds for a small loop of any shape. We can find the energy of a circuit of any shape by considering it as made up of small current loops shown in Fig. 11.18. Total energy is sum of the energies of the small current loops. If the area of each little loop is dS, its energy is $i dS\, B \cos\theta$ where $B \cos\theta$ is the component normal to dS and m the magnetic dipole moment is product of current and area.

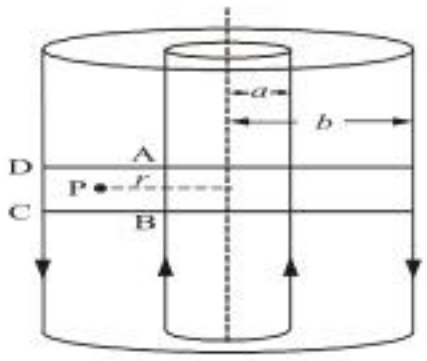


Fig. 11.18

The total energy $= \Sigma i\, B \cos\theta\, dS$. Now going to the limit of infinitesimal loop, the sum becomes an integral, therefore, the total energy given by,

$$= i \int_S B \cos\theta \, dS = i \int_S \vec{B} \cdot \hat{n} \, dS$$

when $\hat{n}$ is the unit vector normal to dS.

If we put $\vec{B} = \nabla \times \vec{A}$, where $\vec{A}$ is vector potential the total energy $i \int_S (\nabla \times A) \cdot \hat{n} \, dS$ by Stoke's theorem,

$$U = i \oint_{\text{circuit}} \vec{A} \cdot d\vec{l} \qquad \ldots(11.39)$$

Actually A is the vector potential due to those currents (other than the i in the wire) which produce the field $\vec{B}$ at the wire.

Any distribution of steady currents can be imagined to be made up of filaments that run parallel to the lines of current flow for each pair of circuit the energy is given by equation (11.39) where the integral is taken around one circuit, using the vector potential A from the other circuit. For the total energy we want the sum of all such pairs. If instead of keeping track of the pairs, we take the complete sum over all the filaments, we would be counting the energy twice, so that total energy

$$\vec{U} = \frac{1}{2}\int \vec{J} \cdot \vec{A}\, dv \qquad \ldots(11.40)$$

Since $i = \vec{J}a$, where a is area of the coil and

$adl = dv$, and J is circuit density.

The expression is similar to electrostatics relation,

$$U = \frac{1}{2}\int \rho V\, dv \qquad \text{(see eq. 5.46)}$$

So in terms of vector potential energy of steady current is given by equation (11.40).

11.22 INDUCTANCES IN SERIES AND IN PARALLEL

Inductances are often connected in series and in parallel such as resistors and capacitors. We could proceed with a derivation based simply on $\xi = -L\frac{di}{dt}$ and obtain formulas for the effective inductance of two inductances in series or in parallel. We cannot ignore the practical fact that an inductor always has a certain internal resistance. A perfect inductance without resistance is much more difficult to realise than a perfect capacitance or a perfect resistance.

Series Inductors

Referring to Fig. 11.19 (a) the equation for emf is,

Fig. 11.19 (*a*)

$$\text{Voltage drop} = R_1 i + L_1\frac{di}{dt} + M\frac{di}{dt} + L_2\frac{di}{dt} + M\frac{di}{dt} + R_2 i$$

$$= (R_1 + R_2)\, i + (L_1 + L_2 + 2M)\frac{di}{dt}$$

The circuit thus resembles a resistor $(R_1 + R_2)$ in series with an inductance $L_1 + L_2 + 2M$.

But we know coupling coefficient,

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

where $-1 \le K \le 1$.

Positive Coupling

If we neglect the resistance the magnitude of induction becomes

$$L_1 + L_2 + 2K\sqrt{L_1 L_2}$$

For positive coupling that is the fluxes in L_1 and L_2 are in the same direction or current in the same direction.

Negative Coupling

The magnitude of inductance is,

$$L_1 + L_2 - 2K\sqrt{L_1 L_2}$$

for negative coupling.

So we have the effective inductance of the series circuit shown above,

$$L_{eff} = L_1 + L_2 \pm 2K\sqrt{L_1 L_2} \qquad \ldots(11.41)$$

depending upon the coupling.

Parallel Inductors

Referring to Fig. 11.19, the equations for the pair of inductors are,

$$i = i_1 + i_2$$

$$V = L_1 \frac{di_1}{dt} \pm M\frac{di_2}{dt} + R_1 i_1$$

and $$V = L_2 \frac{di_2}{dt} \pm M\frac{di_1}{dt} + R_2 i_2$$

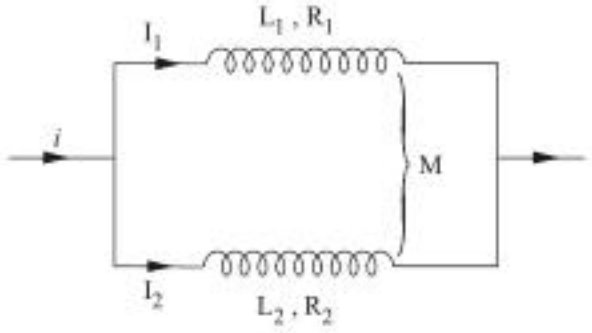


Fig. 11.19 (*b*)

Positive signs are used when the positive current senses i_1 and i_2 are considered, that is, the coupling is positive. If we assume that the resistance terms are negligible,

$$V = L_1 \frac{di_1}{dt} \pm M\frac{di_2}{dt}$$

$$V = L_2 \frac{di_2}{dt} \pm M\frac{di_1}{dt}$$

$$i = i_1 + i_2$$

Solving these equations, we obtain,

$$V = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}\frac{di}{dt}$$

So to our approximation the pair behaves as having inductance,

$$L_{eff} = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M} \qquad \ldots(11.42)$$

Example 11.11 *A straight solenoid has 50 turns per cm in the primary and 200 turns in the secondary. The area of cross-section of the solenoid is 4 cm². Calculate the mutual inductance.*

Solution. From equation (11.35),

$$M = \mu_0 N_p N_s A_p,$$

As given $N_p = 50$ turns per cm

$= 50 \times 100$ turns per metre

Hence, $M = 4\pi \times 10^{-7} \times 5000 \times 200 \times 4 \times 10^{-4}$ henry

$= 50.27 \times 10^{-5}$ henry

Example 11.12 *Two coils, a primary of 400 turns and a secondary of 20 turns, are wound on an iron ring of mean diameter 20 cm and cross-section 2 cm radius. Calculate their mutual inductance, μ for iron is 800.*

Solution. From equation (11.35 *a*),

$$M = \mu\mu_0 N_p N_s A_p$$

As given, N_p, the total number of turns in primary per unit length,

$$N_p = \frac{400}{2\pi \times 10 \times 10^{-2}} \text{ turns/metre}$$

hence, $$M = 800 \times 4\pi \times 10^{-7} \times \frac{400}{2\pi \times 10 \times 10^{-2}} \times 20 \times \pi \times (2 \times 10^{-2})^2$$

$= 1.6076 \times 10^{-2}$ henry

11.23 MEASUREMENT OF MUTUAL INDUCTANCES

The two coils PP and SS are joined in the circuit as shown in Fig. 11.20. First the points C and D are joined. The key K is pressed and the throw θ_1 in the B.G. is observed. Let the mutual inductance of the two coils = M, the maximum current in the primary circuit = I_0 and the resistance of the secondary circuit = R. According to definition of mutual inductance,

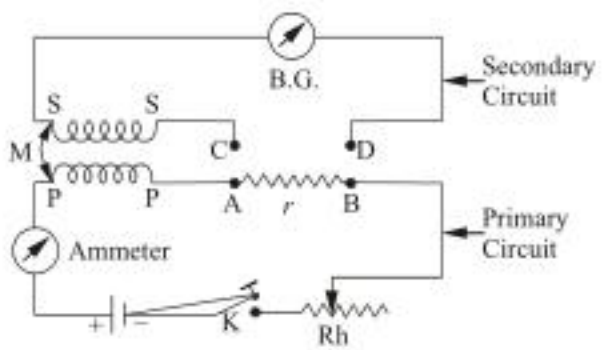


Fig. 11.20

Induced emf across the secondary coil

$$= M\frac{dI}{dt}$$

Total charge through the secondary

$$= \frac{M\int_0^{I_0} \frac{dI}{dt}\,dt}{R}$$

or $$q = \frac{MI_0}{R}$$

hence, $$\frac{MI_0}{R} = \frac{CT}{2\pi n AB}\theta_1\left(1+\frac{\lambda}{2}\right) \quad \ldots(1)$$

where symbols on right hand side have the usual meaning. Now, C and D are disconnected, and C is connected to A and D to B, press the key and note the steady deflection θ_2 in galvanometer. The current in the primary circuit I_0 remains unchanged. The potential difference across $r = I_0 r$.

Current through the secondary circuit

$$= \frac{I_0 r}{R}$$

therefore, $$\frac{I_0 r}{R} = \frac{C}{n AB}\theta_2 \quad \ldots(2)$$

Dividing (1) by (2),

$$\frac{M}{r} = \frac{T}{2\pi}\frac{\theta_1}{\theta_2}\left(1+\frac{\lambda}{2}\right)$$

or $$M = r\cdot\frac{T}{2\pi}\frac{\theta_1}{\theta_2}\left(1+\frac{\lambda}{2}\right) \quad \ldots(11.43)$$

Hence M is known in terms of known quantities and can be calculated.

11.24 SIMPLE R.L. CIRCUIT GROWTH AND DECAY OF CURRENT HELMHOLTZ EQUATION

Fig. 11.21 shows a circuit containing an inductance L in series with a resistance R which is connected to a source of constant emf ξ that can be suddenly introduced or removed by special key K.

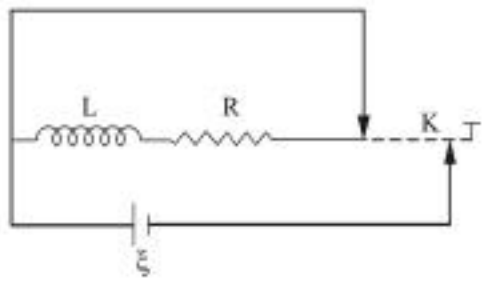
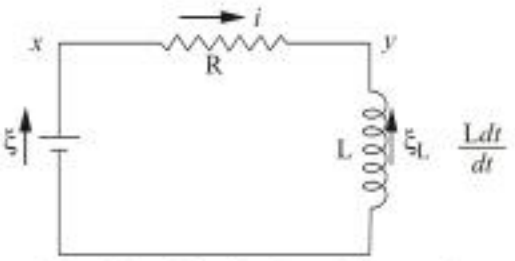


Fig. 11.21

Growth of Current

When the key K is depressed, the current in R starts to increase. If the inductor (L) was not present the current would rise rapidly to a maximum steady value $I_0 = \frac{\xi}{R}$. Because of the inductance L, according to Faraday's and Lenz's law, a self-induced emf $-L\frac{dI}{dt}$ appears in the circuit which opposes the rise of current that means it opposes the battery emf ξ. For the direction of current shown in Fig. 11.22 x will be at higher potential than y, therefore the drop is potential of –IR across the resistance. The Kirchhoff's 2nd law gives,



Circuit just after when key is depressed
Fig. 11.22

$$L\frac{dI}{dt} + RI = \xi$$

or $$\xi - IR = L\frac{dI}{dt}$$

or $$\frac{dI}{\xi - IR} = \frac{dt}{L}, \text{ or } -\frac{R dI}{\xi - IR} = -\frac{R}{L}dt$$

If current rises from 0 to I in time t. Hence, integrating the above equation within the said limits, we have

$$\int_0^I -\frac{R dI}{\xi - IR} = \int_0^t -\frac{R dt}{L}$$

or $$\left|\log_e (\xi - IR)\right|_0^I = -\frac{Rt}{L}$$

or $$\log_e \frac{\xi - IR}{\xi} = -\frac{R}{L}t$$

or taking antilog

$$\frac{\xi - IR}{\xi} = e^{-\frac{R}{L}t}$$

or $$I = \frac{\xi}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

or $$I = I_0\left(1 - e^{-\frac{R}{L}t}\right) \qquad \ldots(11.44)$$

$I_0 = \frac{\xi}{R}$ is the maximum value of the current. This is the equation for the growth of the current and gives the value of current at any moment. The I approaches I_0 exponentially as shown in Fig. 11.23.

The rate of growth of current is obtained by differentiating equation (11.44),

$$\frac{dI}{dt} = \frac{R}{L} I_0 e^{-\frac{R}{L}t}, \text{ by equation (11.44)}$$

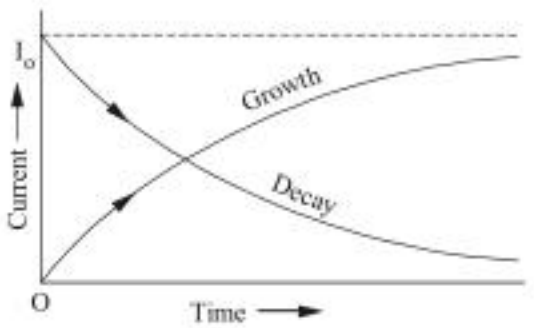


Fig. 11.23

$$= \frac{R}{L}(I_0 - I) \qquad \ldots(11.45)$$

It is evident from equation (11.45) that as current I approaches the maximum value I_0 the rate of growth of current becomes less rapid. Hence the self-induced emf which is $-L\frac{dI}{dt}$ becomes smaller. Thus a time delay is introduced and the current approaches the value I_0 asymptotically.

We can rewrite equation (11.44),

$$I = I_0\,(1 - e^{-t/\lambda}) \qquad \ldots(11.46)$$

where λ the ratio L/R is called "inductive time constant".

$\lambda = \frac{L}{R}$, *we call it time constant because it has dimensions of time,*

$$\lambda = \frac{L}{R} = \frac{\text{Henry}}{\text{Ohm}} = \frac{\text{Volt}}{\frac{\text{amp}}{\text{sec}}\,\text{ohm}}, \quad \left(L = -\frac{\xi}{\frac{dI}{dt}}\right)$$

$$= \frac{\text{volt.sec}}{\text{amp.ohm}} \quad (\text{Volt} = \text{amp.ohm})$$

$$= \text{sec.}$$

Physical significance of time constant can be understood from equation (11.46). If we put $t = \frac{L}{R}$ in equation (11.46), we get,

$$I = I_0\,(1 - e^{-1})$$

$$\frac{I}{I_0} = 1 - \frac{1}{e} = \frac{e-1}{e} = \frac{1.718}{2.718} = 0.63$$

$$\frac{I}{I_0} = \frac{2}{3} \text{ approx.} \qquad \ldots(11.47)$$

Thus the time constant λ of the circuit is the time during which the current rises to $\frac{2}{3}$rd of its final value. The rate of growth of current depends on time constant L/R as shown in Fig. 11.24. If λ is greater the rise in current is slow.

Decay of Current

Now if the key K is released, *i.e.*, the source of emf is withdrawn, the equation of circuit obtained by putting $\xi = 0$, is given by,

$$L\frac{dI}{dt} + IR = 0$$

or

$$\frac{dI}{I} = -\frac{R}{L}dt$$

The change in current from I_0 will again produce self-induced emf which will again oppose the decay of current.

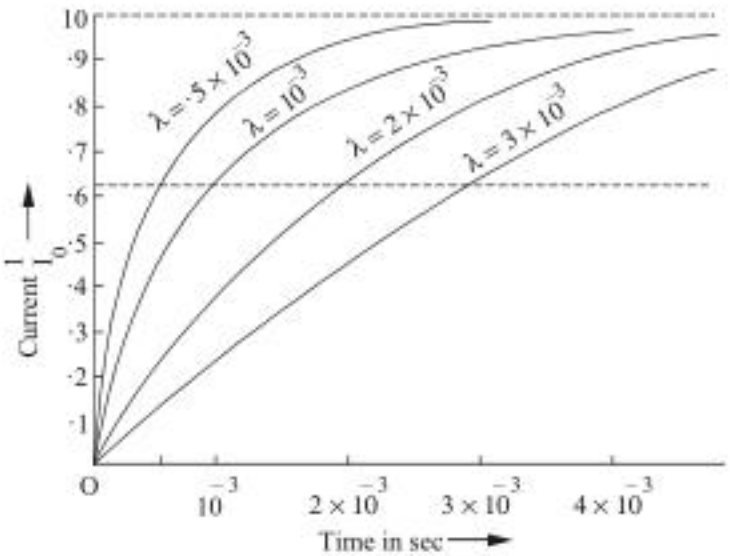


Fig. 11.24

Thus, in this case the current starts to decrease from final value I_0, say its value after time t is I. Integrating the above equation within the said limits.

$$\int_{I_0}^{I} \frac{dI}{I} = \int_{0}^{t} \frac{R}{L}\, dt$$

$$\log \frac{I}{I_0} = -\frac{R}{L} t$$

$$I = I_0 e^{-\frac{R}{L}t} \qquad \ldots(11.48)$$

This represents the decay of current in the circuit. The equation (11.48) is complementary to equation (11.44). The decay of current is shown in Fig. 11.23.

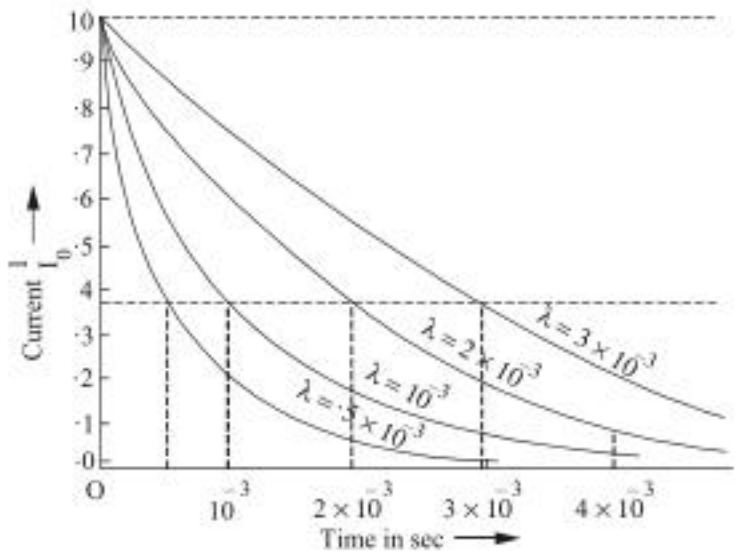


Fig. 11.25

Here again the rate of decay of current depends on time constant $\lambda = \frac{L}{R}$, and if λ is greater than more slowly the current decreases to zero as shown in Fig. 11.25.

Example 11.13 *Calculate the value of the time constant in L-R circuit when the current rises to 63.2 per cent of its steady value in one second.*

Solution. By eqn. (11.44) we have,

$$I = I_0\left(1 - e^{-\frac{R}{L}t}\right)$$

$$= I_0\left(1 - e^{-\frac{t}{\lambda}}\right)$$

From given values $t = 1$ sec.

$$0.632 = \left(1 - e^{-\frac{1}{\lambda}}\right)$$

or $$e^{-\frac{1}{\lambda}} = 1 - 0.632 = 0.368 = e^{-1}$$

or $$\frac{1}{\lambda} = 1$$

or $$\lambda = \frac{L}{R} = 1 \text{ sec}.$$

Example 11.14 *A solenoid has an inductance of 50 henry and a resistance of 30 ohms. It is connected to a 100 volt battery. How long will it take for the current to reach one half of its final value?*

Solution. By eqn. (11.44) we have,

$$I = I_0\left(1 - e^{\frac{R}{L}t}\right)$$

From given values,

$$\frac{1}{2} = \left[1 - e^{-\frac{30}{50}t}\right]$$

or $$e^{\frac{-3}{5}t} = \frac{1}{2}$$

or $$e^{\frac{3}{5}t} = 2$$

or $$\frac{3}{5}t = \log_e 2 = 2.3026 \log_{10} 2$$

$$t = \frac{5}{3}(0.69) = 1.2 \text{ sec}.$$

Example 11.15 *A relay, having an inductance of 5.0 henry and a resistance of 200 ohms, operates with a current of 1.5 mA. How long will relay take to operate, when a potential difference of 0.5 volt is established across the terminals? What will be the rate of increase of current at the instant of operation?*

Solution. $I = \frac{\xi}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

$$0.0015 = \frac{0.5}{200}\left(1 - e^{-\frac{200}{5}t}\right)$$

$$e^{-40t} = \frac{200 \times 0.0015}{0.5} - 1 = 0.4$$

$$t = \frac{2.3026 \log_{10} 0.4}{40} = 0.0229 \text{ sec.}$$

$$\frac{di}{dt} = \frac{\xi}{\mathrm{L}} e^{-\frac{\mathrm{R}}{\mathrm{L}}t} = \frac{0.5}{5} \times \frac{2}{5} = 0.04 \text{ amp/sec.}$$

Example 11.16 *A 3.0 henry inductor is placed in series with 10 ohm resistance, an emf of 3.0 volts being suddenly applied to the combination. At 0.30 second after contact being made, (a) what is the rate at which the energy is being applied by the battery? (b) at what rate does energy appear as joule heat in resistor? (c) at what rate the energy is being stored in the magnetic field?*

Solution. By eq. 11.44

$$\mathrm{I} = \frac{\xi}{\mathrm{R}}\left(1 - e^{-\frac{\mathrm{R}}{\mathrm{L}}t}\right)$$

putting the given values,

$$= \frac{3}{10}\left(1 - e^{-\frac{10}{3} \times 0.30}\right)$$

$$= \frac{3}{10}(1 - e^{-1}) = 0.189 \text{ amp.}$$

(a) The rate $\mathrm{P_E}$ at which energy is delivered by the battery,

$$\mathrm{P_E} = \xi \mathrm{I} = 3 \times 0.189 = 0.567 \text{ watt.}$$

(b) The rate at which energy appears in joule heat in the resistor is given by,

$$\mathrm{P_J} = \mathrm{I}^2 \mathrm{R} = (0.189)^2 \times 10 = 0.357 \text{ watt.}$$

(c) The rate at which energy is stored in the magnetic field is obtained as follows:

$$\text{Since } \mathrm{U_B} = \frac{1}{2}\mathrm{LI}^2$$

rate $$\mathrm{P_B} = \frac{d\mathrm{U_B}}{dt} = \mathrm{L\,I}\frac{d\mathrm{I}}{dt}$$

and $$\frac{d\mathrm{I}}{dt} = \frac{\xi}{\mathrm{L}} e^{-\frac{\mathrm{R}}{\mathrm{L}}}$$

$$= \frac{3}{4} e^{-1} = 0.37 \text{ amp./sec.}$$

therefore, $\mathrm{P_B} = 3 \times 0.189 \times 0.37$

$$= 0.210 \text{ watt}$$

Note that as required by the "principle of conservation of energy".

$$\mathrm{P_E} = \mathrm{P_J} + \mathrm{P_B}$$

$$0.567 \text{ watt} = 0.357 \text{ watt} + 0.210 \text{ watt}$$

$$= 0.567 \text{ watt.}$$

11.25 R.C. CIRCUIT

We shall now discuss the circuit consisting of a source of emf ξ, a capacitor of capacitance C and a resistance R as shown in Fig. 11.26.

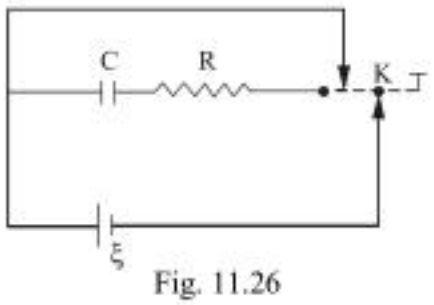


Fig. 11.26

Charging of a Condenser

When the key K is pressed, the capacitor will become charged. Because of the capacitor the current in the circuit will vary with time.

Let the potential difference across the capacitor at any instant be V. This potential difference will oppose the emf ξ of the cell. The current at any time t is given by,

$$I = \frac{\xi - V}{R}$$

or $$\xi = V + RI$$

But $$I = \frac{dQ}{dt} \text{ and } V = \frac{Q}{C}$$

where Q is charge on the capacitor. Therefore,

$$R\frac{dQ}{dt} + \frac{Q}{C} = \xi$$

or $$\frac{dQ}{dt} = \frac{\xi}{R} - \frac{Q}{RC} = \frac{\xi C - Q}{RC}$$

or $$-\frac{dQ}{\xi C - Q} = -\frac{dt}{RC}$$

Integrating this we get,

$$\int_0^Q -\frac{dQ}{\xi C - Q} = \int_0^t -\frac{dt}{RC}$$

$$\left|\log_e \xi C - Q\right|_0^Q = -\frac{t}{RC}$$

$$\log_e \frac{\xi C - Q}{\xi C} = -\frac{t}{RC}$$

or $$\frac{\xi C - Q}{\xi C} = e^{-\frac{t}{RC}}$$

or $$\xi C - Q = \xi C e^{-\frac{t}{RC}}$$

or $$Q = \xi C\left(1 - e^{-\frac{t}{RC}}\right)$$

or $$Q = Q_0\left(-e^{-\frac{t}{RC}}\right) \qquad \ldots(11.49)$$

where $Q_0 = \xi C$ is the final charge on the capacitor when the potential difference across it is equal to the applied emf. The equation (11.49) represents the growth of charge on the capacitor and shows that charge grows exponentially, as shown in Fig. 11.27.

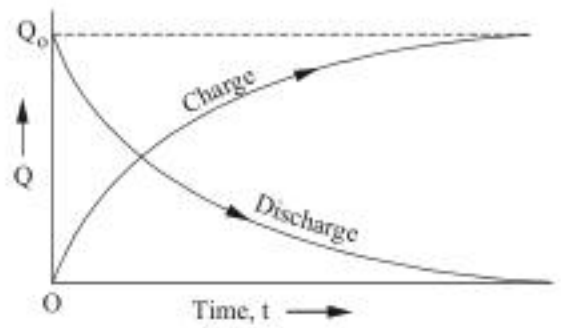


Fig. 11.27

The Current

Now, since, $I = \frac{dQ}{dt}$

The current during charge is obtained by differentiating equation (11.49),

$$I = \frac{dQ}{dt} = \frac{d}{dt}\left[Q_0\left(1 - c^{-\frac{t}{RC}}\right)\right]$$

$$= Q_0 \cdot \frac{1}{RC} e^{-\frac{t}{RC}}$$

$$= \frac{\xi C}{RC} e^{-\frac{t}{RC}}, \text{ since } Q_0 = \xi C$$

$$= \frac{\xi}{R} e^{-\frac{t}{RC}} \quad , \text{since } I_0 = \frac{\xi}{R}$$

$$= I_0 e^{-\frac{t}{RC}} \qquad \ldots(11.50)$$

The quantity RC in equations (11.49) and (11.50) has the dimensions of time and is called the "capacitative time constant" of the circuit. Let us verify this statement,

$$RC = \text{ohms. farad}$$

$$= \text{ohm} \cdot \frac{\text{Coul}}{\text{Volt}}, \text{ since} \left(C = \frac{q}{V}\right)$$

$$= \frac{\text{Coul}}{\text{amp.}}, \text{ since } \text{ohm} = \frac{\text{Volt}}{\text{amp.}}$$

$$= \text{Sec.} \qquad \text{amp} = \frac{\text{Coul}}{\text{Sec.}}$$

If we put $t = RC$ in equation (11.49), we have,

$$Q = Q_0 (1 - e^{-1})$$

$$= 0.63\, Q_0$$

Then the capacitative time constant RC is the time taken by the charge to rise $\frac{2}{3}$ *rd of its final equilibrium value.*

Discharge of a Condenser

If the external emf is withdrawn by releasing the key K, that is the capacitor which has been charged is now discharged through the resistance R. The equation of emf is given by,

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

or $$\frac{dQ}{Q} = -\frac{dt}{RC}$$

If charge reduces from Q_0 to Q in time interval t, then the Q after time t is obtained by integrating above equation,

$$\int_{Q_0}^{Q} \frac{dQ}{Q} = \int_{0}^{t} -\frac{dt}{RC}$$

$$\log_e \frac{Q}{Q_0} = -\frac{t}{RC} \qquad \ldots(11.51a)$$

taking antilog,

$$Q = Q_0\, e^{-\frac{t}{\lambda}} \qquad \ldots(11.51\ b)$$

where λ is the time constant is equal to RC. The equation (11.51b) represents the decrease of charge on capacitor and shows that charge decays exponentially. *The current during discharge* is obtained by differentiating equation (11.51b),

$$I = \frac{dQ}{dt} = \frac{d}{dt}\left[Q_0\, e^{-\frac{t}{RC}}\right]$$

$$= -\frac{Q_0}{RC} e^{-\frac{t}{RC}}$$

$$= -I_0\, e^{-\frac{t}{RC}} \qquad \ldots(11.52)$$

at $t = 0$ the current during charge is I_0 and during discharge $-I_0$. The variation of I is shown in Fig. 11.28, it falls exponentially with time.

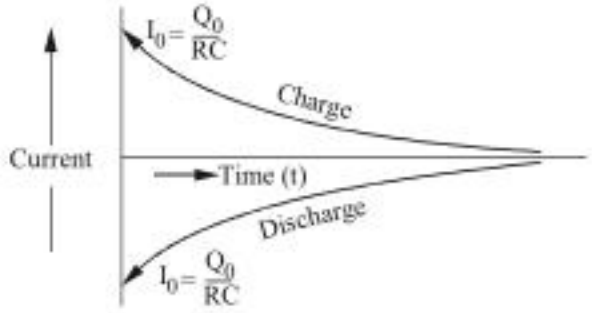


Fig. 11.28

Energy Stored in the Capacitor During Charging

To derive a qualitative expression for the storage of energy, consider an extra increase of charge *dq* on the capacitor, the small amount of additional work will be,

$$dU_E = \frac{q}{C}\, dq$$

This work done will increase the potential energy of the system. If the process is continued until a total charge Q has been transferred, the total work done will be,

$$U_E = \int dU_E = \int_0^Q \frac{q}{C}\, dq = \frac{1}{2}\frac{Q^2}{C}$$

or $$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} CV^2$$

is the total energy of the charged condenser.

Dissipation of energy during charging of the capacitor

Let us calculate the energy dissipated in the resistance R during the charging of a condenser, we have,

$$Q = Q_0\left(1 - e^{-\frac{t}{RC}}\right), \qquad \text{(see eq. 11.49)}$$

$$I = \frac{dQ}{dt} = \frac{Q_0}{RC} e^{-\frac{t}{RC}}$$

If *d*W is the energy dissipated in time *dt*, is given by,

$$dW = I^2 R\, dt$$

$$= \frac{Q_0^{\,2} R}{R^2 C^2} e^{-\frac{2t}{RC}}\, dt$$

The total energy dissipated during charging when the condenser has charged to its maximum value is obtained by integrating the above expression for *d*W, thus we get,

$$W = \int dW = \frac{Q^2}{C^2 R}\int_0^\infty e^{-\frac{2t}{RC}}\, dt$$

$$= \frac{1}{2}\frac{Q^2}{C}$$

$$= \frac{1}{2} Q\xi$$

This is independent of R.

The energy supplied by D.C. source is $Q\xi$, *that stored in the condenser is* $\frac{1}{2}Q\xi$ *and energy dissipated is* $\frac{1}{2}Q\xi$. *Thus energy supplied by the source is accounted for.*

11.26 MEASUREMENT OF HIGH RESISTANCE BY THE METHOD OF LEAKAGE

Equation (11.51 *a*) can be used to determine the value of high resistance of the order of 10^6 ohms. The condenser is first charged to value Q_0 by maintaining potential difference of V_0 across the condenser plates by C.K. then discharged through a ballistic galvanometer as shown in Fig. 11.29 and the deflection θ_0 is observed. The condenser is again charged to the same value and the charge is allowed to leak

for a known time *t* through the resistance R by pressing leakage key and its potential falls to V_1 *i.e.*, charge decreases. The remaining charge is instantaneously allowed to pass through the ballistic galvanometer and the deflection say θ_1 is observed. We have,

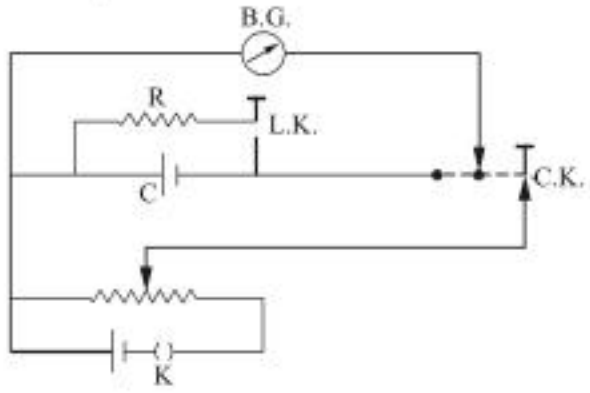


Fig. 11.29

$$Q_0 = CV_0 = K\theta_0\left(1+\frac{\lambda}{2}\right), \text{ see B.G. Theory}$$

and $$Q_1 = CV_1 = K\theta_1\left(1+\frac{\lambda}{2}\right)$$

Therefore,

$$\frac{Q_0}{Q_1} = \frac{V_0}{V_1} = \frac{\theta_0}{\theta_1} \qquad \ldots(1)$$

Now writing the relation (11.51b),

$$Q_1 = Q_0 e^{-\frac{t}{RC}}$$

where Q_1 is the charge after leakage through resistance R in time *t*.

$$\frac{Q_0}{Q_1} = e^{\frac{t}{RC}}$$

$$\frac{t}{RC} = \log_e \frac{Q_0}{Q_1}$$

$$\frac{t}{RC} = \log_e \frac{\theta_0}{\theta_1}, \text{ (from relation (1))}$$

$$R = \frac{t}{C \log_e \frac{\theta_0}{\theta_1}} = \frac{t}{2.3026\, C \log_{10} \frac{\theta_0}{\theta_1}} \qquad \ldots(11.53)$$

Hence R can be calculated by observing θ_0, θ_1 and leakage time *t*.

Now the capacitor is allowed to leak through resistance for different times *t* and the corresponding deflections θ_0 and θ_1 are observed in each case. If a graph is plotted between *t* and $\log_{10}\frac{\theta_0}{\theta_1}$ it will be a straight line. The slope of this line gives the value of R.

Why only High Resistance?

The method is suitable to determine the high resistance only, because if the resistance R is small, the time constant, RC, will be too small so the leakage time *t* during which a part of the charge is allowed to leak through the resistance is very small which cannot be measured accurately. The whole charge may leak through R as soon as the leakage key is closed.

Self-Leakage Through Condenser

The dielectric of the condenser may not be a perfect insulator and some charge may leak through it, so there may be leakage even without connecting the condenser to the high resistance to determine the self-leakage of the condenser. The condenser is immediately discharged through B.G. after charging it and deflection θ_0 is observed. Then the condenser is again charged and is kept isolated for self-leakage for a time t_1 and then discharge through B.G. and corresponding deflection θ_1 is observed. If R_L is resistance due to the self-leakage is given by,

$$R_L = \frac{t_1}{2.30\,C \log \frac{\theta_0}{\theta_1}} \qquad \ldots(11.54)$$

The resistance R_L of the leak is also called the capacitor insulation resistance and high resistance R are in parallel. Then the resultant resistance R′ across the condenser is given by,

$$\frac{1}{R'} = \frac{1}{R_L} + \frac{1}{R}$$

$$R' = \frac{R\,R_L}{R + R_L} \qquad \ldots(11.55)$$

R′ is determined by usual method *i.e.* charging the condenser and then discharge it through a leakage resistance for known time t_2 and finally pass the remaining charge to B.G. and note down the deflection θ_2

$$R' = \frac{R\,R_L}{R + R_L} = \frac{t_2}{2.303\,C \log_{10} \frac{\theta_0}{\theta_2}} \qquad \ldots(11.56)$$

R_L is known from equation (11.54) and R′ from equation (11.55), hence, we can calculate unknown resistance R.

11.27 COMPARISON OF CAPACITIES BY DE SAUTY'S METHOD

The Wheatstone's bridge principle can be used to compare the two capacitances as shown in Fig. 11.30. C_1 and C_2 are the two condensers and R_1 and R_2 are the two high resistance boxes. A ballistic galvanometer G is used for detecting the point of balance. When the Morse key K is pressed the circuit with battery B is closed. A difference of potential is set up between A and C and current will flow in the arms ABC and ADC. So the condensers get charged through the resistance R_1 and R_2. When the key K is released the condensers get discharged through the resistances. The resistances R_1 and R_2 are such adjusted that the deflection in the galvanometer is zero in both cases *i.e.* during charging and discharging. The deflection in the galvanometer is zero, means the potentials of B and D are the same when the key K is pressed or released.

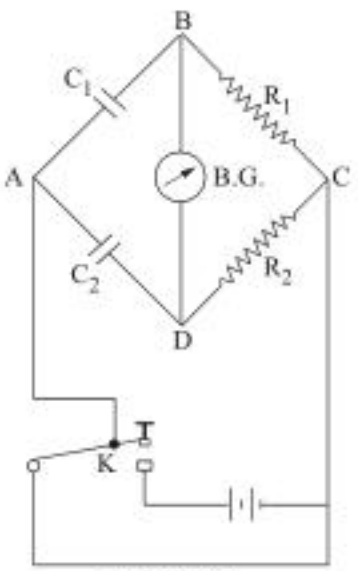


Fig. 11.30

The value of charge Q at any time t is given by eq. (11.49)

$$Q = Q_0\left(1 - e^{-\frac{t}{RC}}\right) \qquad \ldots(1)$$

when Q_0 is the maximum charge corresponding to the maximum potential, V_0 to which the condenser is charged. Then the potential V after a time can be calculated from above equation.

$$Q_0 = CV_0$$

therefore,

$$Q = CV_0\left(1 - e^{-\frac{t}{RC}}\right)$$

$$\frac{Q}{C} = V_0\left(1 - e^{-\frac{t}{RC}}\right)$$

$$V = V_0\left(1 - e^{-\frac{t}{RC}}\right) \qquad \ldots(2)$$

During the process of charging let V_1 and V_2 be the potentials across the condensers at any instant t then from equation (2),

$$V_1 = V_0\left(1 - e^{-\frac{t}{R_1C_1}}\right) \qquad \ldots(3)$$

$$V_2 = V_0\left(1 - e^{-\frac{t}{R_2C_2}}\right) \qquad \ldots(4)$$

In circuit one set of the plates of the condensers are connected to the point A and the second set of plates are connected to the points B and D. At the balance B and D are at the same potential. Therefore the condition for the balance of the bridge is,

$$V_1 = V_2$$

$$V_0\left(1 - e^{-\frac{t}{R_1C_1}}\right) = V_0\left(1 - e^{-\frac{t}{R_2C_2}}\right)$$

It is possible if,

$$R_1C_1 = R_2C_2 \qquad \ldots(5)$$

That is time constants of the two circuits ABC and ADC are the same or in other words the rate of growth of charge is same. Equation (5) can also be written as,

$$\frac{C_1}{C_2} = \frac{R_1}{R_2} \qquad \ldots(6)$$

Example 11.17. *A condenser of 1 micro farad is first charged and then discharged through a resistance of 1 megaohm. Calculate the time in which the charge of the condenser will fall to 63.2 per cent of its initial value.*

Solution. $\lambda = RC = 1\times10^{-6}\times1\times10^{-6} = 1\,\text{sec}$

$$e^{-t} = 0.368 = e^{-1}$$

$$t = 1\,\text{sec.}$$

Example 11.18. *A 4 microfarad condenser is connected to 240 volts d.c. source through a resistor of 0.25 megaohm. Calculate the charge on the condenser at 0.05 sec. Also calculate the potential difference across the condenser and the value of the current at the instant.*

Solution. The charge on the condenser is given by,

$$Q = Q_0\left(1-e^{-\frac{t}{RC}}\right)$$

$$= CV_0\left(1-e^{-\frac{t}{RC}}\right)$$

$$= 4\times10^{-6}\times240\left[1-e^{-\frac{0.05}{4\times10^{-6}\times0.25\times10^{6}}}\right]$$

$$= 960\times10^{-6}\left[1-2.718^{-0.05}\right]$$

$$= 960\times10^{-6}\left[1-0.9513\right]$$

$$= 46.75\times10^{-6}\text{ coul.}$$

The voltage across the condenser is given by,

$$V = \frac{Q}{C} = \frac{46.75\times10^{-6}}{4\times10^{-6}} = 11.68\text{ volts}$$

The current at the instant is given by,

$$I = \frac{dQ}{dt} = \frac{Q_0}{RC}e^{-\frac{t}{RC}} = \frac{V_0}{R}e^{-\frac{t}{RC}}$$

or $$I = \frac{240}{0.25\times10^{6}}\times0.9513 = 0.913\times10^{-3}\text{ ampere}$$

Example 11.19. *A condenser of 2 microfarad has been charged is shunted by high resistance. If half the charge leaks away in 1 min., find the value of the resistance.*

Solution. From equation (11.51b), we have,

$$Q = Q_0\,e^{-\frac{t}{RC}}$$

$$\frac{1}{2}Q_0 = Q_0\,e^{-\frac{t}{RC}}$$

$$\frac{t}{RC} = \log_e 2 = \frac{\log_{10}2}{\log_e 10}$$

$$= \frac{0.3030}{0.4343}$$

$$R = \frac{60\times0.4343}{0.3030\times2\times10^{-6}}$$

$$= 43\text{ megaohm.}$$

Example 11.20. *A condenser of capacity 0.1 microfarad and resistance 10 magaohms is charged to a certain potential and then insulated. Find the time the potential will take to fall to half its original value.*

Solution. Since $\frac{Q_0}{Q_1} = \frac{CV_0}{CV_1} = \frac{V_0}{V_1} = \frac{\theta_0}{\theta_1}$

hence from equation (11.53),

$$t = RC \log_e \frac{V_0}{V_1}, \text{ putting the given values, we get}$$

$$= 10^7 \times 10^{-7} \log_e 2$$

$$= 0.6391 \text{ sec.}$$

Example 11.21. *When a condenser is charged, a B.G. in series with the condenser gives a throw of 50 scale division. The condenser is allowed to discharge for 10 sec. through a resistance of 25 megaohms and discharged through the galvanometer. Calculate the capacitance of the condenser.*

Solution. From equation (11.53),

$$C = \frac{t}{2.3026\, R \log_{10} \frac{\theta_0}{\theta_1}}$$

$$= \frac{10}{2.3026 \times 25 \times 10^6 \times \log_{10} \frac{50}{45}}$$

$$= 8.83 \text{ microfarad.}$$

Example 11.22 *After how many time constants will the energy stored in the capacitor connected in series with resistance reach half of its maximum value or equilibrium value?*

Solution. $U = \frac{1}{2} \frac{Q^2}{C}$

Substituting Q from eqn. (11.49),

$$U = \frac{1}{2} \frac{Q_0^2}{C} \left(1 - e^{-\frac{t}{RC}}\right)^2$$

But $\frac{1}{2} \frac{Q_0^2}{C}$ is the maximum energy can be stored, hence

$$U = U_{max} \left(1 - e^{-\frac{t}{RC}}\right)^2$$

Putting $U = \frac{1}{2} U_{max}$, we get

$$\frac{1}{2} = \left(1 - e^{-\frac{t}{RC}}\right)^2$$

Solving for t

$$t = 1.22\, RC$$

$$= 1.2 \text{ time constant.}$$

11.28 IDEAL L.C. CIRCUIT

The circuits discussed so far have been non-oscillating in character. Perhaps the greatest interest in transient circuits lies in the possibility of their producing oscillating electric currents. The simplest case to consider is the circuit shown in Fig. 11.31 containing an ideal inductor and capacitor–by ideal we mean that there is no resistance in these components.

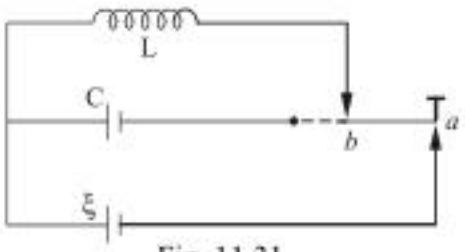


Fig. 11.31

When the key K is pressed the capacitor is charged, as the contact *a* is made and capacitor is connected with source of emf, ξ. When key K is released the capacitor C is connected with inductance L and capacitor discharges through it. We shall discuss only the *discharge case*.

The equation of emf is obtained by Kirchhoff's 2nd law,

$$L\frac{dI}{dt}+\frac{Q}{C}=0$$

using $I=\frac{dQ}{dt}$, we have

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0$$

$$\frac{d^2Q}{dt^2}+\frac{1}{LC}Q=0 \qquad \ldots(1)$$

This is the differential equation which describes the variation of Q with time and is identical to the equation of motion of a mass-spring executing the simple harmonic motion. The simple harmonic motion equation for a mass spring is given by,

$$\frac{d^2x}{dt^2}+\frac{K}{m}x=0$$

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

where ω is natural frequency of oscillation of an undamped mass spring system is,

$$\omega=2\pi f=\sqrt{\frac{K}{m}}$$

Therefore, the L.C. circuit is identical to a mass spring system executing simple harmonic motion.
To make it clear we have identity,

	Mechanical	Electromagnetic
Spring,	$U_P=\frac{1}{2}Kx^2$	Capacitor, $U_E=\frac{1}{2}\frac{q^2}{C}$
Mass,	$U_K=\frac{1}{2}mv^2$	Inductor, $U_B=\frac{1}{2}LI^2$
	$v=\frac{dx}{dt}$	$I=\frac{dq}{dt}$

The comparison suggests that a capacitor is behaving like a spring and an inductor like a mass and the electromagnetic quantities correspond to certain mechanical ones, namely,

q corresponds to x

I corresponds to v

C corresponds to $\frac{1}{K}$

L corresponds to m

The energy alternates in the mass-spring oscillation between two forms, kinetic and potential, and for L-C system between magnetic and electric. The solution of equation (1) is obtained just as solution of,

$\frac{d^2x}{dt^2} + \omega^2 x = 0$ and is given by,

$$Q = Q_m \cos(\omega t + \phi)$$

where Q_m which is constant is the maximum value of charge ω is angular frequency of oscillations and ϕ is arbitrary constant. We can test whether it is indeed a solution of equation (1) by putting Q and $\frac{d^2Q}{dt^2}$ in equation (1).

$$\frac{dQ}{dt} = I = -\omega Q_m \sin(\omega t + \phi)$$

$$\frac{d^2Q}{dt^2} = -\omega^2 Q_m \cos(\omega t + \phi)$$

Substituting in equation (1),

$$-L\omega^2 Q_m \cos(\omega t + \phi) + \frac{1}{C} Q_m \cos(\omega t + \phi) = 0$$

$$\omega = \frac{1}{\sqrt{LC}} \quad \ldots(11.57)$$

The phase angle ϕ is determined by the initial condition.

The equation (11.57) represents a sinusoidal variation of Q between limits $\pm Q_m$ with a period,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} \quad \ldots(11.58)$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \quad \ldots(11.59)$$

The current in the circuit is given by,

$$I = \frac{dQ}{dt} = -\omega Q_m \sin(\omega t + \phi) \quad \ldots(11.60)$$

The equation (11.60) indicates that the current is also oscillatory and has the same frequency as charge.

11.29 SERIES L.C.R. CIRCUIT (CHARGE CASE)

Consider the circuit containing an inductance, L, a capacitance C and a resistance R connected to a constant source of emf ξ as shown in Fig. 11.32. Let us investigate the transient condition here. Kirchhoff's second law gives the equation of emf at any time t when the current is I,

$$L\frac{dI}{dt} + \frac{Q}{C} + RI = \xi$$

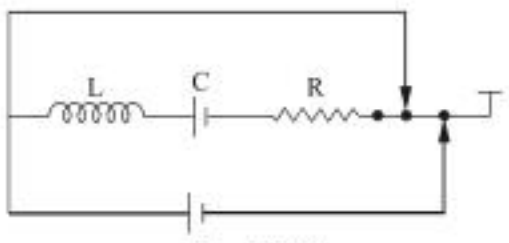


Fig. 11.32

putting $I = \frac{dQ}{dt}$

The equation reduces,

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{\xi}{L}$$

or $$\frac{d^2Q}{dt^2} + 2b\ \frac{dQ}{dt} + K^2 Q = \frac{\xi}{L}$$

where $2b = \frac{R}{L}$ and $K^2 = \frac{1}{LC}$

or $$\frac{d^2Q}{dt^2} + 2b\ \frac{dQ}{dt} + K^2\left(Q - \frac{\xi}{K^2 L}\right) = 0$$

Putting, $Q - \frac{\xi}{K^2 L} = x$,

we get after substitution,

$$\frac{dx}{dt} = \frac{dQ}{dt} \text{ and } \frac{d^2x}{dt^2} = \frac{d^2Q}{dt^2}$$

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + K^2 x = 0 \qquad \ldots(1)$$

This equation is identical to a damped harmonic oscillator. The differential equation, which is a homogeneous linear type of the second order, has at least one solution of the form,

$$x = A e^{\alpha t}$$

known as particular integral, where A and x are both arbitrary constants. We shall, therefore, use this as a trial solution.

Where A is constant after substitution, equation (1) reduces,

$$\alpha^2 + 2b\alpha + K^2 = 0$$

or $$\alpha = -b \pm \sqrt{b^2 - K^2}$$

Therefore α has to values and general solution of Eq. (1) is,

$$x = A\, e^{(-b+\sqrt{(b^2+K^2)})t} + B\, e^{(-b-\sqrt{(b^2-K^2)})t}$$

Where A and B are arbitrary constants which can be determined by the boundary conditions,

$$Q = x + \frac{\xi}{K^2 L} = \frac{\xi}{K^2 L} + A\, e^{(-b+\sqrt{(b^2-K^2)})t} + B\, e^{(-b-\sqrt{(b^2-K^2)})t}$$

For convenience, put $\sqrt{b^2 - K^2} = g$

and remembering $K^2 = \frac{1}{LC}$

Hence, $\frac{\xi}{K^2 L} = \frac{\xi LC}{L} = \xi C = Q_0$

where Q_0 is the final steady charge on the capacitor when the potential difference across it is equal to ξ the emf of the battery. Therefore,

$$Q = Q_0 + e^{-bt}(Ae^{gt} + Be^{-gt}) \qquad \ldots(2)$$

Now at $t = 0, Q = 0,$ or $A + B = -Q_0$ $\qquad \ldots(3)$

Differentiating equation (2), we get,

$$\frac{dQ}{dt} = -be^{-bt}(Ae^{gt} + Be^{-gt}) + ge^{-bt}(Ae^{gt} - Be^{-gt})$$

Again at $t = 0,\ 1 = \frac{dQ}{dt} = 0$

$$O = -b(A + B) + g(A - B)$$

or $$A - B = \frac{b}{g}(A + B)$$

or $$A - B = \frac{b}{g}Q_0 \qquad \text{(from equation (3))} \qquad \ldots(4)$$

Adding equations (3) and (4), we get,

$$A = -\frac{Q_0}{2}\left(1 + \frac{b}{g}\right)$$

and subtracting (3) from (4),

$$B = -\frac{Q_0}{2}\left(1 - \frac{b}{g}\right)$$

Putting the values of A and B in equation (2), we get,

$$Q = Q_0 - \frac{Q_0}{2}e^{-bt}\left\{\left(1 + \frac{b}{g}\right)e^{gt} + \left(1 - \frac{b}{g}\right)e^{-gt}\right\} \qquad \ldots(5)$$

The value of g depends on b and K.

Case 1:

When $b^2 > k^2$ *i.e.*, $\frac{R^2}{4L^2} > \frac{1}{LC}$, then $g = \sqrt{b^2 - k^2}$ is a real quantity and equation (5) shows that under this condition the charge on the condenser will increase in an exponential way till it finally acquires the steady value Q_0 as shown in Fig. 11.33 (a).

Case II:

When $b^2 = k^2$, under this condition the behaviour is critical one. Here the charge is neither dead beat nor oscillatory as shown in Fig. 11.33 (b).

Case III:

When $b^2 < k^2$ that is $\frac{R^2}{4L^2} < \frac{1}{LC}$

Then $g = \sqrt{b^2 - k^2}$ is an imaginary quantity

$$g = \sqrt{-(k^2 - b^2)} = j\omega \text{ where } \omega = \sqrt{k^2 - b^2}$$

Substituting $g = j\omega$ in eqn. (5) we get

$$Q = Q_0 - \frac{Q_0}{2} e^{-bt}\left[\left(1+\frac{b}{j\omega}\right)e^{j\omega t} + \left(1-\frac{b}{j\omega}\right)e^{-j\omega t}\right]$$

$$= Q_0 - \frac{Q_0}{2} e^{-bt}\left[\left(1+\frac{b}{j\omega}\right)(\cos\omega t + j\sin\omega t) + \left(1-\frac{b}{j\omega}\right)(\cos\omega t - j\sin\omega t)\right]$$

$$= Q_0 - \frac{Q_0}{2} e^{-bt}\left[2\cos\omega t + \frac{2b}{\omega}\sin\omega t\right]$$

$$= Q_0\left[1 - e^{-bt}\left(\cos\omega t + \frac{b}{\omega}\sin\omega t\right)\right]$$

$$= Q_0\left[1 - \frac{e^{-bt}}{\omega}\left(\omega\cos\omega t + b\sin\omega t\right)\right]$$

on putting the value of ω

$$Q = Q_0\left[1 - \frac{e^{-bt}}{\sqrt{K^2 - b^2}}\left(\sqrt{K^2 - b^2}\cos\omega t + b\sin\omega t\right)\right]$$

Now introducing, $\tan\alpha = \dfrac{\sqrt{K^2 - b^2}}{b} = \dfrac{2L}{R}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}$

i.e., $\sin\alpha = \dfrac{\sqrt{K^2 - b^2}}{K}$, $\tan\alpha = \sqrt{\dfrac{4L}{R^2 C} - 1}$

and $\cos\alpha = \dfrac{b}{k}$.

After substitution above equation reduces

$$Q = Q_0\left[1 - \frac{Ke^{-bt}}{\sqrt{K^2 - b^2}}\left(\sin\alpha\cos\omega t + \cos\alpha\sin\omega t\right)\right]$$

hence $$Q = Q_0\left[1 - \frac{Ke^{-bt}}{\sqrt{K^2 - b^2}}\sin(\omega t + \alpha)\right] \qquad \ldots(11.61)$$

The equation (11.61) represents a charge on the condenser which is oscillating with damped simple harmonic motion about its final steady value Q_0. The amplitude of the oscillating charge is given by,

$$\frac{Q_0 Ke^{-bt}}{\sqrt{K^2 - b^2}}$$

Thus the amplitude will die away exponentially due to the factor e^{-bt}. As $b = \dfrac{R}{2L}$, therefore, the circuit has low resistance, the value of b will be small and the amplitude will die slowly. *For, $R = 0$, the amplitude becomes Q_0 because $b = 0$ and damped oscillations become simple harmonic motion as damping factor resistance is removed.*

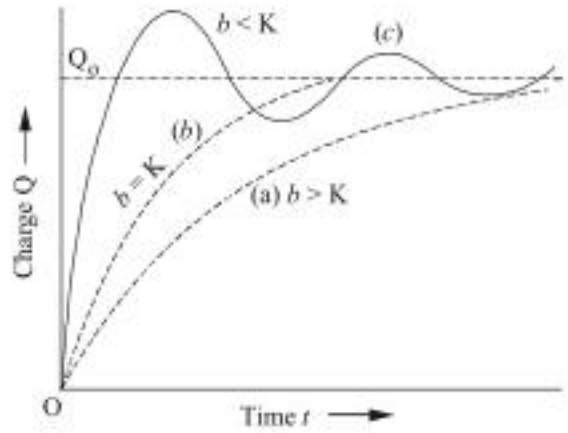


Fig. 11.33

The variation is shown in Fig. 11.33. It is to be noted that maximum charge is much larger than the steady value Q_0 and may raise the potential of the condenser plates to such an extent that the insulation may break down. To avoid this, a resistance is included in the circuit in the beginning which may be withdrawn later, but this will damp the oscillations.

The Current

By differentiating equation (11.61) with respect to time, we get $\frac{dQ}{dt}$ which is the current at a given instant,

$$I = \frac{dQ}{dt} = -\frac{Q_0 K}{\sqrt{K^2 + b^2}}\left[\omega e^{-bt}\cos(\omega t + x) - b\, e^{-bt}\sin(\omega t + \alpha)\right]$$

Since we have substituted,

$$\sin x = \frac{\sqrt{K^2 - b^2}}{K} = \frac{\omega}{K} \text{ and } \cos\alpha = \frac{b}{K}$$

Putting ω and b in the above equation,

$$I = -\frac{Q_0 K^2 e^{-bt}}{\sqrt{K^2 - b^2}}\left[\sin\alpha\cos(\omega t + \alpha) - \cos\alpha\sin(\omega t + \alpha)\right]$$

$$= \frac{Q_0 K^2 e^{-bt}}{\sqrt{K^2 - b^2}}\sin\left[(\omega t + \alpha) - \alpha\right]$$

$$= \frac{Q_0 K^2 e^{-bt}}{\sqrt{K^2 - b^2}}\sin wt$$

$$= \frac{Q_0 K^2 e^{-bt}}{\sqrt{K^2 - b^2}}\sin\sqrt{K^2 - b^2}\cdot t \qquad \ldots(11.62)$$

This expression indicates that the current in the circuit will also be oscillatory. The amplitude will die away exponentially in the same way as the charge.

Period of Oscillation

The period of oscillation is given by,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{K^2 - b^2}}$$

$$= \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \qquad \ldots(11.63)$$

The frequency of oscillation,

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad \ldots(11.64)$$

It can be seen from equations (11.61) and (11.62) that the current in the circuit has the same periodic time and frequency as the oscillating charge.

When R is very small *i.e.* R – 0 we get L.C. circuit. From the above calculation,

$$T = 2\pi\sqrt{LC}$$

and frequency $f = \frac{1}{2\pi\sqrt{LC}}$

The expression for current shows that the current in the circuit is alternating. Hence LC circuit is used for generating alternating currents of high frequency.

11.30 DISCHARGE OF A CONDENSER THROUGH INDUCTANCE AND RESISTANCE

After charging the condenser, the source of emf is withdrawn and the charge is allowed to pass through inductance and resistance. The equation of emf at any time t is given by,

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0$$

or $$L\frac{d^2Q}{dt^2} + \frac{R\,dQ}{dt} + \frac{Q}{C} = 0$$

or $$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + K^2Q = 0 \qquad \ldots(1)$$

with $2b = \frac{R}{L}$ and $K^2 = \frac{1}{LC}$

This is the same equation as equation (1) of article 11.29 except that x is replaced by Q here. Therefore, the solution of this equation is given by,

$$Q = A\,e^{\left(-b+\sqrt{(b^2-K^2)}\right)t} + B\,e^{\left(-b-\sqrt{(b^2-K^2)}\right)t}$$

The constants A and B are determined by boundary conditions for convenience put $g = \sqrt{b^2 - K^2}$,

$$Q = A\,e^{(-b+g)t} + B\,e^{(-b-g)t} = e^{-bt}\,(Ae^{gt} + Be^{-gt}) \qquad \ldots(2)$$

At $t = 0, Q = Q_0$

$$Q_0 = A + B \qquad \ldots(3)$$

Again the current $I = \frac{dQ}{dt} = 0$, at $t = 0$

The current I can be obtained by differentiating eq. (2),

$$I = ge^{-bt}(Ae^{gt} - Be^{-gt}) - b\,e^{-bt}(Ae^{gt} + Be^{-gt})$$

at $t = 0, I = 0$

$$0 = g\,(A - B) - b\,(A + B)$$

or $$A - B = \frac{b}{g}(A + B)$$

$$A - B = \frac{b}{g}Q_0, \quad \text{[from equation (3)]} \qquad \ldots(4)$$

Solving eqns. (3) and (4) for A and B

$$A = \frac{Q_0}{2}\left(1 + \frac{b}{g}\right)$$

$$B = \frac{Q_0}{2}\left(1 - \frac{b}{g}\right)$$

Putting the values of A and B in equation (2), we get,

$$Q = \frac{Q_0\,e^{-bt}}{2}\left[\left(1 + \frac{b}{g}\right)e^{gt} + \left(1 - \frac{b}{g}\right)e^{-gt}\right] \qquad \ldots(5)$$

Case I:

When $b^2 > K^2$ *i.e.*, $\frac{R^2}{4L^2} > \frac{1}{LC}$

Then g is a purely real quantity and charge on the condenser will decay exponentially as shown in Fig. 11.34.

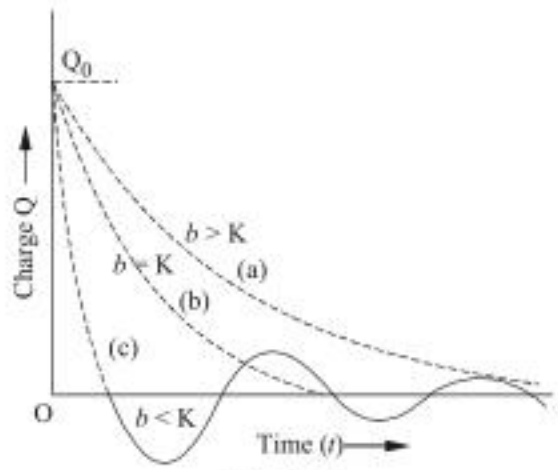


Fig. 11.34

Case 2:

When $b^2 = K^2$, under this condition the behaviour is critical one, and the discharge is neither dead beat nor oscillatory.

Case 3:

When $b^2 < \mathrm{K}^2$ *i.e.*, $\dfrac{\mathrm{R}^2}{4\mathrm{L}^2} < \dfrac{1}{\mathrm{LC}}$ g is imaginary because

$$g = \sqrt{b^2 - \mathrm{K}^2} = \sqrt{-(\mathrm{K}^2 - b^2)} = j\omega$$

Replacing g by $j\omega$ in equation (5) we get,

$$\mathrm{Q} = \frac{\mathrm{Q}_0}{2} e^{-bt} \left[\left(1 + \frac{b}{j\omega} \right) e^{j\omega t} + \left(1 - \frac{b}{j\omega} \right) e^{-j\omega t} \right]$$

Expanding $e^{j\omega t}$ and $e^{-j\omega}$ in above equation.

$$\mathrm{Q} = \frac{\mathrm{Q}_0}{2} e^{-bt} \left[\left(1 + \frac{b}{j\omega} \right) (\cos \omega t + j \sin \omega t) + \left(1 - \frac{b}{j\omega} \right) (\cos \omega t - j \sin \omega t) \right]$$

Collecting the coefficients of $\cos \omega t$ and $\sin \omega t$,

$$\mathrm{Q} = \frac{\mathrm{Q}_0}{2} e^{-bt} \left(2 \cos \omega t + \frac{2b}{\omega} \sin \omega t \right)$$

or $$\mathrm{Q} = \mathrm{Q}_0 e^{-bt} \left(\cos \omega t + \frac{b}{\omega} \sin t \right)$$

or $$\mathrm{Q} = \frac{\mathrm{Q}_0}{\omega} e^{-bt} (\omega \cos \omega t + b \sin \omega t)$$

$$= \frac{\mathrm{Q}_0 e^{-bt}}{\sqrt{\mathrm{K}^2 - b^2}} \left[\sqrt{\mathrm{K}^2 - b^2} \cos \omega t + b \sin \omega t \right]$$

Let, $\sin \alpha = \dfrac{\omega}{\mathrm{K}} = \dfrac{\sqrt{\mathrm{K}^2 - b^2}}{\mathrm{K}}$ and $\cos \alpha = \dfrac{b}{\mathrm{K}}$

$$\mathrm{Q} = \frac{\mathrm{Q}_0 e^{-bt} \mathrm{K}}{\sqrt{\mathrm{K}^2 - b^2}} [\sin \alpha \cos \omega t + \cos \alpha \sin \omega t]$$

$$= \frac{\mathrm{Q}_0 e^{-bt} \mathrm{K}}{\sqrt{\mathrm{K}^2 - b^2}} \sin (\omega t + \alpha) \qquad \ldots(11.65)$$

where α is given by $\tan \alpha = \dfrac{\sqrt{\mathrm{K}^2 - b^2}}{\mathrm{K}} = \sqrt{\dfrac{4\mathrm{L}}{\mathrm{R}^2 \mathrm{C}} - 1}$

This equation represents the general periodic oscillations. This shows that the discharge of condenser is oscillatory and the amplitude of oscillations is damped as shown in Fig. 11.34. The current is obtained by differentiating equation (11.65) with respect to time.

$$\mathrm{I} = \frac{d\mathrm{Q}}{dt} = \frac{\mathrm{Q}_0 \mathrm{K}}{\sqrt{\mathrm{K}^2 - b^2}} \left[e^{-bt} \omega \cos (\omega t + \alpha) - b e^{-bt} \sin (\omega t + \alpha) \right] \qquad \ldots(11.66)$$

Again substituting, $\sin \alpha = \dfrac{\omega}{\mathrm{K}}$ and $\cos \alpha = \dfrac{b}{\mathrm{K}}$

$$\mathrm{I} = \frac{\mathrm{Q}_0 \mathrm{K}^2 e^{-bt}}{\omega} \left[(\sin \alpha \cos (\omega t + \alpha) - \cos \alpha \sin (\omega t + \alpha) \right]$$

$$\mathrm{I} = \frac{\mathrm{Q}_0 \mathrm{K}^2 e^{-bt}}{\omega} \sin \omega t \qquad \ldots(11.67)$$

where $\omega = \sqrt{K^2 - b^2}$

Thus we *see that the current in the circuit during discharge is also oscillatory.* However, its amplitude dies exponentially due to damping term e^{-bt}.

Period of Oscillation

Since sine and cosine are periodic functions, hence the charge and current are also periodic. The periodic time T is given by,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{K^2 - b^2}}$$

$$= \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \qquad \ldots(11.68)$$

and the frequency,

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad \ldots(11.69)$$

when R = 0, the circuit reduces to an ideal L-C circuit.

Example 11.23 *A circuit has an inductance 10 mh, capacitance 0.1 microfarad and resistance 1000 ohms. Is the circuit periodic? If an emf 10^5 volts is in the circuit, what is the final charge on the capacitor?*

Solution. The circuit is periodic if $\frac{R^2}{4L^2} < \frac{1}{LC}$

From given values,

$$\frac{R^2}{4L^2} = \frac{10^6}{4\times(10-10^{-3})^2} = \frac{1}{4}10^{12}$$

and $$\frac{1}{LC} = \frac{1}{10\times10^{-3}\times0.1\times10^{-6}} = 10^9$$

Therefore $\frac{R^2}{4L^2} > \frac{1}{LC}$ *i.e.* the circuit is not oscillatory but is dead beat.

The final charge on the capacitor is given by,

$$Q = \xi C = 10^{-7}\times10^5 = 10^{-2} = 0.01 \text{ coul}.$$

Example 11.24 *If a battery of emf 100 volts is connected in series with an inductance of 10 mh, a capacitor of 0.05 microfarad and a resistance of 100 ohm, find (a) is the circuit oscillatory? (b) what is the frequency of the oscillatory current if such a current flows? (c) the final capacitor charge.*

Solution. The circuit is oscillatory if $\frac{R^2}{4L^2} < \frac{1}{LC}$

From given values,

$$\frac{R^2}{4L^2} = \frac{10^4}{4\times(10\times10^{-3})^2} = 2.5\times10^7$$

and $$\frac{1}{LC} = \frac{1}{10\times10^{-3}\times0.05\times10^{-6}} = 2\times10^9$$

Therefore, $\dfrac{R^2}{4L^2} < \dfrac{1}{LC}$

i.e. the discharge wil be oscillatory. The frequency of oscillations,

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi}\sqrt{2\times10^9 - 2.5\times10^7}$$

$$= 7078 \text{ cycles/sec.}$$

$$= 7.078 \text{ KC/sec.}$$

The final charge of the capacitor,

$$Q = C\xi = 0.05\times10^{-6}\times100$$

$$= 5\times10^{-6} \text{ volts.}$$

Example 11.25 *The condenser of an oscillatory circuit of low resistance is enclosed in a container. When the container is evaluated, the frequency of the circuit is 100 KC/sec. When it is filled with CO_2, the frequency changes by 50 cycles. Calculate the dielectric constant of the gas.*

Solution. Let C be the capacity of the condenser in vacuum and when filled with CO_2 the capacity becomes KC, where K is the dielectric constant. Now,

$$f = \frac{1}{2\pi\sqrt{LC}}$$

$$10^5 = \frac{1}{2\pi\sqrt{LC}}$$

and $$99,950 = \frac{1}{2\pi\sqrt{LKC}}$$

$$\frac{10^5}{99950} = \sqrt{K}$$

or $$K = \left(\frac{10^5}{99,950}\right)^2$$

$$= 1.001$$

11.31 COMPARISON OF TWO INDUCTANCES BY INDUCTANCE BRIDGE

Consider a bridge network made up of two uncoupled inductors L_1 and L_2 together with four resistances as shown in Fig. 11.35.

The resistance R_1 and R_2 are to include the resistances of the inductors, but may also include external resistances. The galvanometer is used to detect the balance. At balance, the current through galvanometer is zero, *i.e.* $I_g = 0$, which means potential difference between points B and D is zero.

Applying the second law of Kirchhoff to ABDA and BCDB, we get

$$R_1 I_1 + L_1\frac{dI_1}{dt} - L_2\frac{dI_2}{dt} - R_2 I_2 + GI_g = 0$$

and $$R_3(I_1 - I_g) - (I_2 + I_g)R_4 - GI_g = 0$$

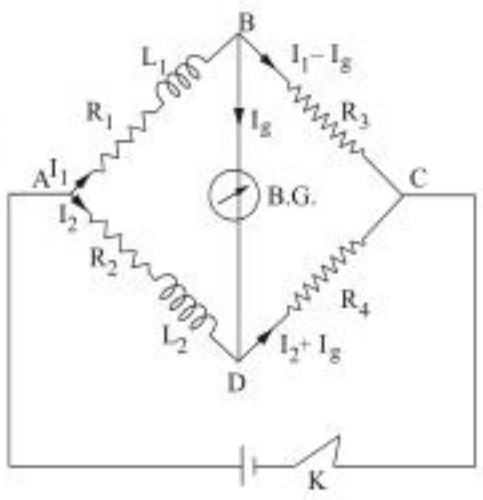


Fig. 11.35

At balance $I_g = 0$, so we have

$$R_1 I_1 + L_1 \frac{dI_1}{dt} - L_2 \frac{dI_2}{dt} - R_2 I_2 = 0 \quad \ldots(1)$$

$$R_3 I_1 - R_4 I_2 = 0 \quad \ldots(2)$$

From equation (2),

$$I_2 = \frac{R_3}{R_4} I_1 \quad \ldots(3)$$

Differentiating equation (3), we get,

$$\frac{dI_2}{dt} = \frac{R_3}{R_4} \frac{dI_1}{dt} \quad \ldots(4)$$

Substituting equations (3) and (4) into (1) we get,

$$\left(R_1 - \frac{R_2 R_3}{R_4}\right) I_1 + \left(L_1 - \frac{I_2 R_3}{R_4}\right) \frac{dI_1}{dt} = 0 \quad \ldots(5)$$

This condition of equation (5) must hold at all times. So that the coefficients of both I_1 and dI_1/dt must vanish. In this way we obtain two conditions for the transient balance of the network.

$$R_1 - \frac{R_2 R_3}{R_4} = 0 \quad \ldots(6)$$

$$L_1 - \frac{L_2 R_3}{R_4} = 0 \quad \ldots(7)$$

or

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \quad \ldots(8)$$

$$\frac{L_1}{L_2} = \frac{R_3}{R_4} \quad \ldots(9)$$

When these two balance conditions given in equations (8) and 9 are satisfied, the balance is obtained. Hence we can compare the unknown self-inductance with a standard. An external resistor may be used with one of the inductors satisfy equation (8).

REVIEW QUESTIONS

11.1 What is electromagnetic induction? Describe experiments to illustrate how this phenomenon occurs.

11.2 State and explain Faraday's laws of electromagnetic induction.

11.3 What is flux rule? Show that

$$\text{Curl } \vec{E} = -\frac{\delta \vec{B}}{\delta t}$$

11.4 (a) State Lenz's law of electromagnetic induction.

(b) Is there any difference between induced electric field and field due to static charges?

11.5 Show that

(1) $\vec{E} = -\frac{\delta \vec{A}}{\delta t}$

where $\vec{A}$ is vector potential

(2) $\nabla^2 I = \sigma \mu \frac{\delta I}{\delta t}$

Symbols have usual meaning.

11.6 What is self-inductance? Calculate the self-inductance of a solenoid of length L.

11.7 Give the physical significance of self-inductance. Calculate the self-inductance of

(1) Two coaxial cylinders.

(2) Toroidal coil of rectangular cross-section.

(3) Toroidal coil of circular cross-section.

11.8 Give a suitable method for determining the self-inductance. Show that the energy required to build up a current I in a circuit of self-inductance L is $\frac{1}{2} LI^2$.

11.9 Obtain an expression for energy density in a magnetic field.

11.10 Explain what is meant by coefficient of mutual inductances? Describe a method of finding it experimentally.

11.11 Show that if the two coils having coefficient of self-inductances L_1 and L_2 are mutally coupled together so that the whole of the flux from one coil links with other, then the mutual inductance between the two coils is given by, $M = \sqrt{L_1 L_2}$.

11.12 Derive Helmholtz equations for the growth and decay of currents in a circuit containing inductance, resistance and a constant emf. What is meant by time constant?

11.13 Obtain an expression for the growth and decay of charge in a condenser through a resistance. What is time constant of the circuit?

11.14 Describe with full theory the method of measuring high resistance by the method of leakage.

11.15 An electric circuit consists of a resistance, inductance and a condenser. Discuss the growth of current in the circuit when an emf is applied.

11.16 Give the theory of oscillatory discharge of a condenser through a circuit containing an inductance and a resistance.

PROBLEMS

11.1 A long solenoid of diameter 3.0 cm has 200 turns/cm, and carries a current of 1.5 amp. A close packed coil of diameter 2.0 cm having 100 turns is placed at the centre of the solenoid is parallel to the axis of the coil. There current through the solenoid is reduced to zero and then increased to 1.5 amp in the reverse direction at a steady rate over a period of 0.05 sec. Calculate the emf induced in the coil while the current is being changed.

11.2 A rectangular coil of 100 turns and area 400 sq. cm. is rotated about a horizontal axis at a constant rate of 50 revolutions per sec. in a horizontal magnetic field of 2000 oersted perpendicular to the axis. Calculate (1) the maximum value of the induced emf in the coil, (2) the induced emf when the plane of the coil is 30° to the horizontal and (3) the induced emf when the place of the coil is vertical.

11.3 When a copper disc. of radius 10 cm makes 20 rotations per sec with its plane perpendicular to a uniform magnetic field an induced emf of 3.14 millivolts is generated between the centre and the edge of the disc. Calculate B.

11.4 A coil has an inductance of 0.03 henry. Calculate the emf induced when a current in the coil changes at a rate of 200 amperes per second.

11.5 An air core solenoid of 80 cm length has 500 turns and its circular cross-section has a diameter of 2.00 cm. Calculate

(a) The self-inductance of the solenoid.

(b) The self-linked flux when the current in the solenoid is 2 amp.

(c) The rate of change of current in the solenoid that will produce a self-induced emf of 0.3 volt.

11.6 A solenoid having a core of 10^{-3} sq. metre its cross-section, half air and half iron is 1 metre long. If the number of turns on it is 5000, what will be its coefficient of self-inductance?

11.7 An emf of 25 millivolts is induced in a coil when the current in the neighbouring circuit changes by 10 amperes in 0.1 sec. What is mutual inductance of the coil?

11.8 A long solenoid of length 1 metre, cross-section 10 cm^2 having 100 turns, has wound over its centre a small coil of 20 turns. Find the mutual inductance of this system.

11.9 Two coils, a primary of 600 turns and a secondary of 20 turns, are wound on an iron ring of mean radius 10 cm and cross-section 4 cm. diameter. Find their mutual inductance for iron $\mu = 800$.

11.10 What energy is stored in a magnetic field of solenoid of inductance 5 millihenry when maximum current of 3 amp flows through it?

11.11 A coil of resistance 11 ohms and inductance 0.1 henry is connected to a 110 volts D.C. mains. Find (*a*) the current finally established in the coil, (*b*) the voltage used in overcoming the resistance when the rising current is 3 amp. (*c*) the rate at which the current is rising at that instant.

11.12 A coil having resistance of 150 ohms and an inductance of 10 henries is connected to a 90 volts supply. Determine the value of current after 2 sec.

11.13 An inductor of self-inductance 500 millihenry and resistance of 5 ohms is connected to a battery of negligible internal resistance. Calculate the time in which the current will attain half of its final steady value.

11.14 A coil of resistance 20 ohms and inductance 0.5 henry is switched in direct current 200 volts supply. Calculate the rate of increase of current

(1) at the instant of closing the switch and

(2) at $t = L/R$ sec. after the switch is closed; also

Find the steady value of the current in the circuit.

11.15 An inductor of 3 henries and resistance 6 ohms is connected to the terminals of a battery of emf 12 volts and of negligible internal resistance. Calculate:

(*i*) the initial rate of increase of current in the circuit.

(*ii*) the rate of increase of current at the instant when the current in the circuit is 1 amp.

(*iii*) the instantaneous value of current at 0.2 sec. after the circuit is closed.

(*iv*) the final steady-state current.

(*v*) the power input to the inductor at the instant when the current is 0.5 amp.

(*vi*) the rate of development of heat at this instant.

(*vii*) the rate at which the energy of magnetic field is increasing at this instant.

(*viii*) the energy stored in the magnetic circuit when the current has attained its steady value.

11.16 A capacitor of capacity 1 microfarad is discharged through a high resistance. The time taken for half the charge in the capacitor to leak is found to be 10 sec. Compute the value of resistance.

11.17 In an oscillatory circuit L = 0.2 henry and C = 0.0012 microfarad. What is the maximum value of resistance for the circuit to be oscillatory?

11.18 Given a coil of self-inductance 5 mH and resistance 0.5 ohm, calculate the value of the capacity of the condenser required to generate the frequency of 1 kc/sec.

11.19 Find whether the discharge of a condenser through the following inductive circuit is oscillatory. C = 0.1 microfarad, L = 10 mH, R = 200 ohms. If so, calculate frequency.

11.20 If a condenser of 10 microfarad be charged to a potential difference of 50 volts and then discharged through a coil of negligible resistance and inductance 10 mH, find the frequency and maximum amplitude of the resultant current oscillations.

ANSWERS

1. -4.74×10^{-2} volts **2.** 251.4 volts, 217.7 volts, zero

3. 5×10^{-3} weber/metre^2 **4.** – 6 volts

5. 1.23×10^{-4} henries, 2.464 webers, 2.43×10^{2} amp/sec.

6. 7.87 henries **7.** 0.24 millihenry

8. 25.12 microhenries **9.** 3.45×10^{-2} henry

11. 22.5×10^{-3} joules **11.** 10 amp. 33 volts, 770 amp./sec.

12. 5.70 amp. **13.** 0.0693 sec.

14. 400 amp/sec. 147.04 amp/sec. 10 amp.

15. 4 amp/sec. 2 amp/sec., 0.65 amp, 2 amp, 6 watt, 0.357 cal/sec. 4.5 watt, 6 joules

16. 2.171×10^{6} ohms **17.** 2.58×10^{4} ohms

18. 5.066 microfarad **19.** yes, 4775 hertz

20. 1592 hertz, 0.5 amp.

SELF-TEST OBJECTIVE QUESTIONS

11.1 The law of electromagnetic induction was given by
(*a*) Faraday (*b*) Lenz
(*c*) Fleming (*d*) Neumann

11.2 The direction of induced emf in a circuit is given by
(*a*) Lenz (*b*) Fleming's left hand rule
(*c*) Faraday's law (*d*) None of the above

11.3 A coil does not consume any power. It should be
(*a*) Resistive (*b*) Inductive
(*c*) Capacitive (*d*) None of the above

11.4 The inductance of a coil depends upon
(*a*) Number of turns of coil
(*b*) Type of core
(*c*) Spacing between the turns
(*d*) All of the above

11.5 Inductance has the dimensions of
(*a*) Flux/length (*b*) Flux/current
(*c*) Ohms/Hz (*d*) none of the above

11.6 RC has the dimensions of
(*a*) Sec (*b*) 1/Sec
(*c*) H/Sec. (*d*) Sec/H

11.7 Current through an inductor follows (when circuit is closed)
(*a*) A linear growth (*b*) A linear decay
(*c*) An exponential growth (*d*) An exponential decay

11.8 In a R.L. circuit voltage across R is a
(*a*) Linar decay (*b*) Linear rise
(*c*) An exponential decay (*d*) An exponential rise

11.9 The time constant is the time it takes the current in an inductive circuit to rise to
(*a*) 50 per cent of its final steady state value
(*b*) 70 per cent of its final steady state value
(*c*) 66 per cent of its final steady state value
(*d*) 63 per cent of its final steady state value

11.10 The mutual inductance of two coils is maximum when the coils are
(*a*) inclined at an angle of 45° (*b*) at right angle to each other
(*c*) facing each other (*d*) touching each other

11.11 Coupling coefficient denotes
(*a*) The degree of magnetic leakage
(*b*) Whether the reluctance remains constant
(*c*) The variation of inductance between the two coils
(*d*) Whether the flux linkage is constant

11.12 In LCR circuit if $\frac{1}{LC} < \frac{R^2}{4L^2}$ the circuit will be
(*a*) oscillatory
(*b*) dead beat
(*c*) critically damped
(*d*) none of the above

11.13 In question 12, if $\frac{1}{LC} = \frac{R^2}{4L^2}$ the circuit is
(*a*) dead beat
(*b*) critically damped
(*c*) oscillatory
(*d*) none of the above

11.14 If a LCR circuit is to be oscillatory, then it should satisfy
(*a*) $\frac{1}{LC} < \frac{R^2}{4L^2}$
(*b*) $\frac{1}{4L^2} = \frac{R^2}{4L^2}$
(*c*) $\frac{1}{LC} > \frac{R^2}{4L^2}$
(*d*) $\frac{L}{C} = \frac{R^2}{4}$

11.15 For a constant inductance of L henrys the energy stored in a magnetic field is given by
(*a*) $\frac{1}{2}mv^2$
(*b*) $\frac{1}{2}Lv^2$
(*c*) $\frac{1}{2}LI^2$
(*d*) $\frac{1}{2}LV^2$

11.16 Faraday's law of voltage induction relates the voltage induced in the coil linking the flux
(*a*) The coil stationary and the flux changing with time
(*b*) The coil moving and the flux changing with time
(*c*) both
(*d*) None of these

11.17 $\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}$ is representing
(*a*) Ampere's law
(*b*) Gauss's law
(*c*) Ohm's law
(*d*) Faraday's law

ANSWERS

1. (a)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (b)	**6.** (a)
7. (c)	**8.** (b)	**9.** (d)	**10.** (d)	**11.** (a)	**12.** (b)
13. (b)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (d)	

CHAPTER 12

PRACTICAL APPLICATIONS OF ELECTROMAGNETIC INDUCTION

12.1 EARTH INDUCTOR

It is an important device based on electromagnetic induction. It consists of a circular coil C (Fig. 12.1) of 1000 to 5000 turns of insulated copper wire wound round a wooden frame of about 25 cm diameter. The circular frame is capable of rotation about an axis passing through two diametrically opposite points. There is a spring key which allows the coil to rotate through 180° only. The two ends of coil are connected to two terminals T_1 and T_2. They are sometimes also joined to a ballistic galvanometer through a commutator. The handle is used to bring the coil to its position so that on pressing the release key the coil is set in rotation.

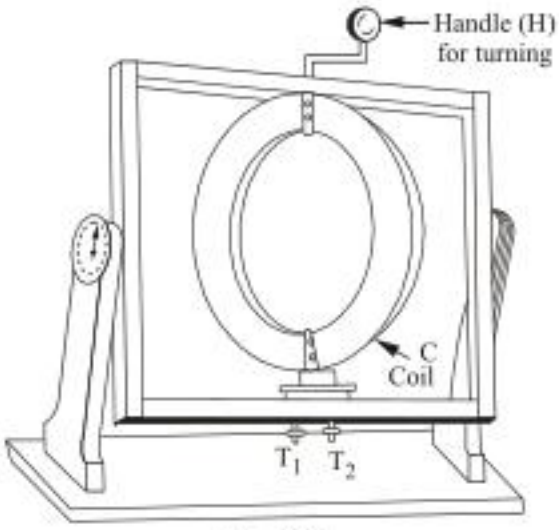


Fig. 12.1

Theory

Consider, a coil of n turns and area A is placed in a magnetic field B. When this coil is rotated in this uniform magnetic field, the magnetic flux associated with it changes and according to Faraday's law an induced emf is set up in it. Suppose at any instant the plane of coil is at right angles to the field, then flux linked with coil = n BA.

Suppose at any instant it makes an angle θ with a direction at right angles to that of the coil (Fig. 12.2), then the flux linked with it nBA cos θ.

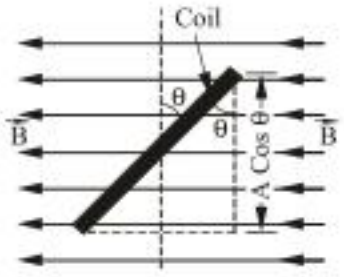


Fig. 12.2

Since flux = normal component × area

The induced emf,

$$\xi = -\frac{d\phi}{dt} = -\frac{d}{dt}(n\,\mathrm{BA}\cos\theta) = n\,\mathrm{BA}\sin\theta\frac{d\theta}{dt}$$

If R is the resistance of a circuit, induced current is given by,

$$\mathrm{I} = \frac{\xi}{\mathrm{R}} = \frac{n\,\mathrm{BA}}{\mathrm{R}}\sin\theta\frac{d\theta}{dt}$$

Therefore, the charge that flows through B.G during a small interval of time dt is,

$$dq = \mathrm{I}dt$$

Substituting I,

$$dq = \frac{n\,\mathrm{BA}}{\mathrm{R}} \sin\theta\, d\theta$$

When the coil is rotated through half the complete rotation *i.e.* 0 to 180°, then the total charge passing through the ballistic galvanometer is given by,

$$\int dq = \int_0^{\pi} \frac{n\,\mathrm{BA}}{\mathrm{R}} \sin\theta\, d\theta$$

$$q = \frac{n\,\mathrm{BA}}{\mathrm{R}} \left[-\cos\theta \right]_0^{\pi}$$

$$= -\frac{n\,\mathrm{BA}}{\mathrm{R}} \left[\cos\pi - \cos\theta \right]$$

$$= \frac{2n\,\mathrm{BA}}{\mathrm{R}} \qquad \ldots(12.1)$$

If θ_1 is the corresponding first throw in the ballistic galvanometer,

$$\frac{2n\,\mathrm{BA}}{\mathrm{R}} = \mathrm{K}\theta_1 \left(1 + \frac{\lambda}{2} \right) \qquad \ldots(12.2)$$

where K is the ballistic constant and λ is the logarithmic constant.

12.2 STANDARD MAGNETIC FLUX

Standard magnetic fluxes are used for calibrating ballistic galvanometer. The earth inductor also works as a source standard flux is discussed below.

Standard Solenoid or Flux Meter

It consists of a long solenoid of large number of turns as primary coil PP. The secondary coil SS of few turns wound over primary coil as shown in Fig. 12.3. Suppose the number of turns per unit length of the primary be n_p. When a steady current I is sent through the primary. The magnetic field inside the solenoid is given by,

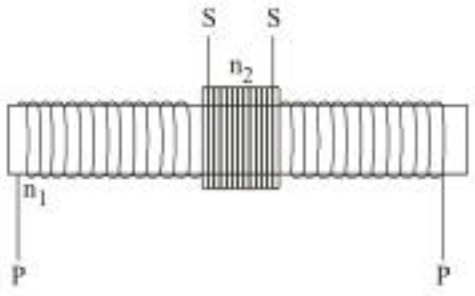


Fig. 12.3

$$\mathrm{B} = \mu_0\, n_p\, \mathrm{I}$$

Now if A_s be the mean area of the secondary coil and N_s the total number of turns in it. As the lines of magnetic field $\overline{\mathrm{B}}$ inside the solenoid also pass through the secondary, producing the magnetic flux through the secondary. The total flux through the secondary is,

$$\phi = \mathrm{BA}_s\, \mathrm{N}_s$$

The magnitude of the induced emf

$$\xi = \frac{d\phi}{dt} = \mu_0\, n_{\mathrm{P}}\, \mathrm{N_S}\, \mathrm{A_S} \frac{d\mathrm{I}}{dt}$$

$$\text{Induced current} = \frac{\xi}{R} = \frac{\mu_0 n_p N_S A_S}{R} \frac{dI}{dt}$$

where R is the total resistance of the secondary.

And the charge q is given by,

$$q = \int_0^I \frac{\mu_0 n_P N_S A_S}{R} \frac{dI}{dt} dt$$

$$= \frac{\mu_0 n_P N_S A_S I}{R}$$

If θ is the observed throw in B.G, when this charge passes through it, then

$$\frac{\mu_0 n_P N_S A_S I}{R} = K \theta_H \left(1 + \frac{\lambda}{2}\right) \quad \ldots(12.3)$$

This equation is used for determining the ballistic constant K. Thus, the galvanometer is calibrated.

12.3 USES OF EARTH INDUCTOR

1. Measurement of Horizontal Component H of the Earth Magnetic Field

To measure the horizontal component of the earth's magnetic field at any place, the coil is set with its axis of rotation vertical and the plane of the coil east-west, *i.e.* perpendicular to the magnetic meridian. The coil is now rapidly rotated through 180° by spring key, consequently induced charge due to change in flux flows through the B.G which gives a deflection θ_H. From equation (12.2), we get,

$$\frac{2NAH}{R} = K\theta_H \left(1 + \frac{\lambda}{2}\right) \quad \ldots(12.4)$$

where R is the total resistance and the coil rotates in field H, *i.e.*, the horizontal component of earth's magnetic field. If the value of K is known, we can determine H. K is calculated by calibrating B.G

K can be calculated by sending a known current through the primary of standard solenoid and noting the observed ballistic throw in the galvanometer. The circuit used for this purpose is shown in Fig. 12.4. Let the current in the primary be adjusted to I and the direction of the current is suddenly reversed.

The flux linked with the secondary will change by an amount, given by,

$$\mu_0 N_P N_S A_S I - (-\mu_0 N_P N_S A_S I) = 2\mu_0 N_P N_S A_S I$$

charge due to this change influx,

$$q = \frac{2\mu_0 N_P N_S A_S I}{R}$$

If θ is the ballistic throw when this charge passes through ballistic galvanometer,

$$\frac{2\mu_0 N_P N_S A_S I}{R} = K\theta \left(1 + \frac{\lambda}{2}\right) \quad \ldots(12.5)$$

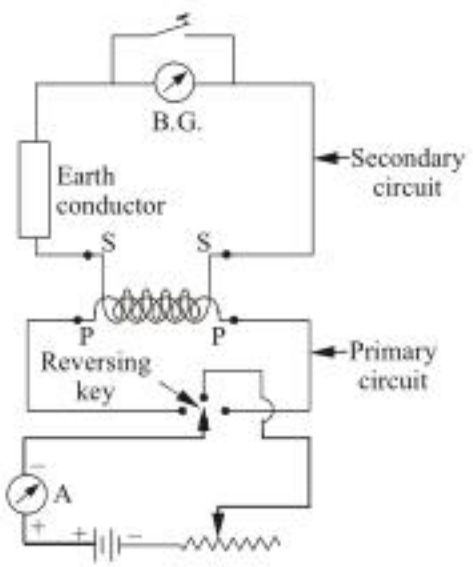


Fig. 12.4

Dividing equation (12.4) by (12.5),

$$\frac{NAH}{\mu_0 N_P N_S A_S I} = \frac{\theta_H}{\theta}$$

or $$H = \frac{\mu_0 N_P N_S A_S I}{NA} \frac{\theta_H}{\theta} \text{ weber/meter}^2 \quad \ldots(12.6)$$

Thus H the horizontal component of the earth magnetic field is known in terms of known quantities.

2. Measurement of Vertical Component V of the Earth Magnetic Field

Here, the coil is adjusted so that its plane is horizontal and it can rotate about a horizontal axis in the magnetic meridian. The coil now rotates in the vertical component V of the earth's magnetic field. The coil is now rapidly rotated through 180° and the deflection θ in the ballistic galvanometer is observed. Proceeding exactly as in previous case, a relation for V is obtained *i.e.*,

$$V = \frac{\mu_0 N_P N_S A_S}{NA} I \frac{\theta_v}{\theta} \text{ weber/meter}^2 \quad \ldots(12.7)$$

The intensity of the earth's magnetic field

$$= \sqrt{H^2 + V^2}$$

$$= \frac{\mu_0 N_P N_S A_S I}{NA} \frac{1}{\theta} \sqrt{\theta_H^{\,2} + \theta_v^{\,2}} \quad \ldots(12.8)$$

3. Measurement of the Angle of Dip

The above two experiments are performed and the angle of dip is given by

$$\tan\delta = \frac{V}{H}$$

Dividing equation (12.7) by (12.6), we get

$$\tan\delta = \frac{V}{H} = \frac{\theta_v}{\theta_H} \quad \ldots(12.9)$$

The calibration of galvanometer is not needed in this case.

4. Calibration of Ballistic Galvanometer Using Earth Inductor

Equations (12.4) can be used for the calibration of the ballistic galvanometer, because if H is known, we can determine the constant K as given below:

$$\frac{2NAH}{R} = K\theta_H \left(1 + \frac{\lambda}{2}\right)$$

or $$K = \frac{2NAH}{R\theta_H \left(1 + \frac{\lambda}{2}\right)} \quad \ldots(12.10)$$

12.4 MEASUREMENT OF A STRONG MAGNETIC FIELD USING A SEARCH COIL

A search coil is a solenoid of small diameter having few (about 50) turns of insulated copper wire and is suitable for measuring strong magnetic field.

Suppose we have to find the magnetic field between the pole pieces of an electromagnet. The search coil is placed in the gap between the pole pieces and is connected in series with a ballistic galvanometer through a high resistance box R to control the throw in B.G. and the secondary coil SS of standard solenoid as shown in Fig. 12.5. We can change the magnetic flux in two ways.

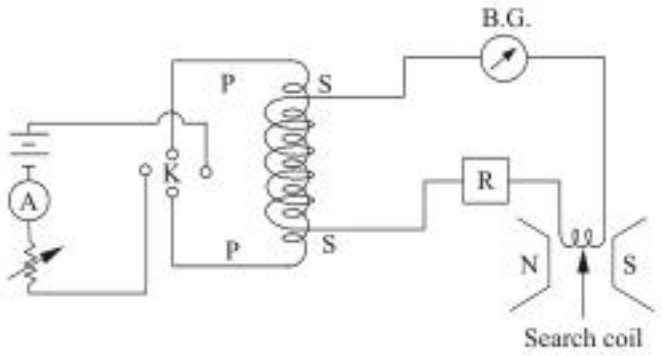


Fig. 12.5

(1) Put on the magnetic field of the electromagnet and the search coil is either suddenly inserted in or withdrawn from the field, *i.e.* the region between pole pieces NS, with its plane at right angles to the magnetic field.

(2) The search coil is held between the pole pieces and the magnetic field is suddenly put on.

In both cases the flux linked with the search coil changes and produces an induced emf in the circuit which gives throw in ballistic galvanometer.

Suppose the throw $= \theta$

Number of turns of the search coil $= N$

The area of each turn of the coil $= A$

The magnetic field (or flux density) $= B$

Then flux linked with the search coil $= NAB$

But this represents change in flux in both cases mentioned above, since in the first case initially the flux was zero whether the coil was outside or inside, as the case may be. In the second case mentioned above the flux becomes zero when the field is put off. Thus in both cases,

The change in flux $d\phi = NAB$

Induced emf in time $dt = \dfrac{NAB}{dt}$

Current $I = \dfrac{NAB}{R dt}$

The charge flowing in time dt

$$q = I dt = \frac{NAB}{R}$$

This charge flows through B.G. then,

$$\frac{NAB}{R} = K\theta\left(1 + \frac{\lambda}{2}\right) \qquad \ldots(12.11)$$

In order to determine K, *i.e.* for calibrating the galvanometer a series circuit containing a battery, an ammeter and a rheostat is connected to the primary of the standard solenoid. A known current in the primary circuit is passed. It produces the field inside the solenoid $\mu_0 n_P I$. This causes a magnetic flux to pass through the secondary. The flux linked with secondary coil $= \mu_0 n_P N_S A_S I$,

where n_P are the number of turns per unit length in the primary coil,

N_s the total number of turns in the secondary coil,

A_s the area of each secondary coil and I is the current through the primary.

Now, the direction of current in the primary is suddenly reversed with reversing key, the total change in flux,

$$= \mu_0 n_P N_s A_s I - (-\mu_0 n_P N_s A_s I)$$

$$= 2\mu_0 n_P N_s A_s I$$

As above, the charge due to this change in flux

$$= \frac{2\mu_0 n_P N_s A_s I}{R}$$

Let θ_1 be the throw observed in ballistic galvanometer when this charge passes through it then,

$$\frac{2\mu_0 n_P N_s A_s I}{R} = K\theta_1 \left(1 + \frac{\lambda}{2}\right) \qquad \ldots(12.12)$$

Dividing eqn. (12.11) by (12.12), we get,

$$\frac{NAB}{2\mu_0 n_P N_s A_s I} = \frac{\theta}{\theta_1}$$

or $$B = \frac{2\mu_0 n_P N_s A_s I}{NA} \frac{\theta}{\theta_1} \text{ weber/metre}^2 \qquad \ldots(12.13)$$

Hence, B can be determined.

While calibrating the galvanometer, if instead of reversing the direction of current in the primary, the current is suddenly started or stopped, the change in flux linked with secondary will be $\mu_0 n_P N_s A_s I$ and the throw θ_2 in B.G will now be half of the θ_1, thus,

$$B = \frac{\mu_0 n_P N_s A_s I}{NA} \frac{\theta}{\theta_2} \qquad \ldots(12.14)$$

Hence, B can be determined.

12.5 GRASSOT'S FLUX METER

It is essentially a moving coil galvanometer in which mechanical damping and restoring torque are very much reduced but electromagnetic damping is relatively increased. Hence it is a modified moving coil ballistic galvanometer and is used to measure magnetic flux. The advantage of a Grassot's flux meter over a ballistic galvanometer is that while measuring the flux, the change in flux did not take place rapidly, since the fluxmeter coil is at rest before and after the change. Thus in Grassot's fluxmeter, the deflection depends only upon the change of flux and not upon the time taken for the flux to change, so the deflection is independent of the rate at which flux changes. The moving coil is suspended by a single silk fibre, the upper end of the fibre being attached to a flat spiral spring A. The coil is connected through two spirals B, B of thin silver foil and is suspended in the magnetic field of a permanent magnet NS. The search coil is connected to the terminals T_1 and T_2 of the moving coil of the fluxmeter as shown in Fig. 12.6.

The deflection of the coil is indicated by the usual lamp and scale arrangement. The search coil is inserted in the space where the magnetic flux is to be measured and then withdrawn, an emf is induced in the circuit which gives rise to a current. The corresponding deflection produced in the fluxmeter is noted.

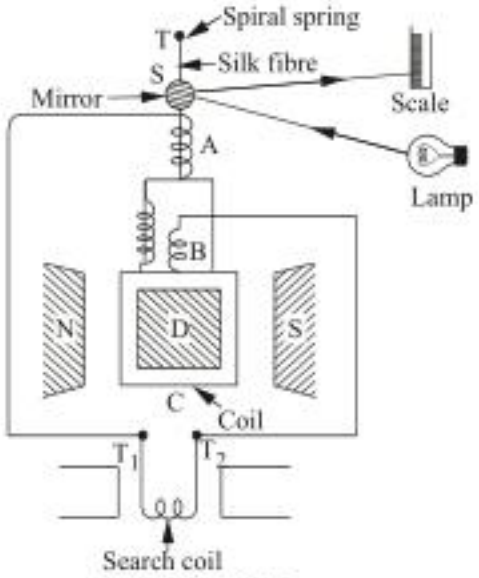


Fig. 12.6

$$\text{The induced current} = \frac{\xi}{R}$$

where ξ is induced emf and R is the resistance of the circuit.

Suppose area of the fluxmeter coil = A

Number of turns = N

Magnetic field = B

The magnitude of the deflecting couple = iANB

(Since $\tau = \vec{m} \times \vec{B}$ and $m = iA$)

This couple acts on the fluxmeter coil. If I is the moment of inertia of the fluxmeter coil and $\frac{d\omega}{dt}$ is its angular acceleration then, we have,

$$NBiA = I\frac{d\omega}{dt}$$

$$\text{or} \qquad ABN\frac{\xi}{R} = I\frac{d\omega}{dt} \qquad \ldots(12.15)$$

The emf ξ is the resultant of two emfs' one due to the rate of change of flux *i.e.* $\frac{d\phi}{dt}$ in the search coil and the other due to the rotation of fluxmeter coil with angular velocity ω i.e., $NAB\omega$

$$\xi = \frac{d\phi}{dt} - NAB\omega$$

Substituting ξ in equation (12.15), we get,

$$\frac{ABN}{R}\left(\frac{d\phi}{dt} - NAB\omega\right) = I\frac{d\omega}{dt}$$

Integrating and putting $\omega = \frac{d\theta}{dt}$

$$\frac{NAB}{R}\left(\int_0^t \frac{d\phi}{dt}\,dt - NAB\int_0^t \frac{d\theta}{dt}\,dt\right) = I\int_0^t \frac{d\omega}{dt}\,dt$$

The integral of the right-hand side *i.e.*,

$$\int_0^t d\omega = |\omega|_{t=0}^{t=t} = 0$$

because the coil is at rest before and after the change of flux *i.e.* angular velocity is zero when $t = 0$ and $t = t$.

$$\frac{NAB}{R}\left[\int_0^t d\phi - NAB\int_0^t d\theta\right] = 0$$

or $$\int_0^t d\phi - \text{NAB}\int_0^t d\theta = 0$$

or $$\phi - \text{NAB}\theta = 0$$

or $$\phi = \text{NAB}\theta$$

or $$\phi = \text{K}\theta \qquad \ldots(12.16)$$

where $$\text{K} = \text{NAB}.$$

Thus, *the total change in flux in the search coil is independent of time in which the change of flux takes place.* The deflection θ in the fluxmeter is proportional to the flux linkage in the search coil. Therefore, the displacement of the fluxmeter coil, when the search coil is inserted in or withdrawn from the unknown field, measures the flux linked with the search coil.

12.6 A.C. GENERATOR OR DYNAMO

Principle

It is based on the principle of electromagnetic induction. When a coil rotates in a uniform magnetic field, the associated flux changes and an induced emf is generated according to Faraday's law of electromagnetic induction. *Thus the rotation of coil (Mechanical Processes) produces induced emf (electrical effects) if the process is made continuous, then we can convert continuously mechanical energy into electrical energy.*

Suppose a coil of wire is rotated about an axis OO' in a uniform magnetic field $\vec{B}$. The flux passing through the coil, as shown in Fig. 12.7.

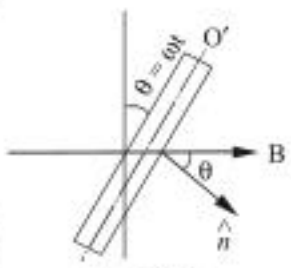


Fig. 12.7

$$\phi = \int_s \vec{B}\cdot\hat{n}\,dS$$

where $\hat{n}$ is a vector along the outward normal to the plane of the coil and dS is an element of the area enclosed by the coil. Let at any instant θ be the angle between normal to the plane of the coil and $\vec{B}$. Further, let N be the total number of turns in the coil and A be the area of each turn, then, total flux

$$\phi = \int_s \text{B}\cos\theta\, dS$$

$$= \text{B}\cos\theta\int_s dS$$

$$= \text{NBA}\cos\theta, \text{ since } \int_s dS = \text{NA}$$

The induced emf in the coil at any instant t is given by,

$$\text{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}(\text{BAN}\cos\theta)$$

$$= \text{BAN}\sin\theta\frac{d\theta}{dt}$$

If ω is the angular velocity, then $\omega t = \theta$, therefore,

$$\text{E} = \text{BAN}\,\omega\sin\omega t \qquad \ldots(12.17)$$

The value of induced emf is maximum when $\omega t = \frac{\pi}{2}$, $\sin\omega t = 1$, *i.e.*, when the plane of the coil is parallel to the magnetic field, this maximum value of E is denoted by E_0.

$$\text{E}_0 = \text{BAN}_\omega$$

Equation (12.17) becomes,

$$E = E_0 \sin \omega t \qquad \ldots(12.18)$$

This relation (12.18) shows the variation of emf with time.

1. When $\omega t = 0$, *i.e.* the plane of the coil is perpendicular to the field (coil is vertical), $E = 0$.
2. When $\omega t = \frac{\pi}{2}$, *i.e.* the plane of the coil is parallel to B (coil is horizontal) we have $E = E_0$.
3. When $\omega t = \pi$, *i.e.*, the plane of the coil is again perpendicular to the field, $E = 0$.
4. When $\omega t = \frac{3\pi}{2}$, *i.e.* the plane of the coil is again parallel to the field $\sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$, so that $E = -E_0$. It shows that the direction of the induced emf is again maximum but in the reverse direction.
5. When $\omega t = 2\pi$, the plane of the coil is again perpendicular to the field and $E = 0$.

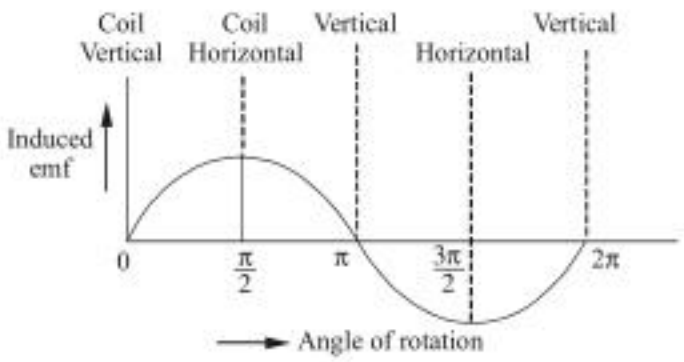


Fig. 12.8

Thus the emf induced in the coil goes through a cycle. Such emf is called sinusoidal just like sine wave. It is also referred to as alternating emf as shown in Fig. 12.8. The corresponding current in the external circuit,

$$I = \frac{E}{R} = \frac{E_0}{R} \sin \omega t$$

$$= I_0 \sin \omega t \qquad \ldots(12.19)$$

where I_0 is maximum current.

Construction

A simple electric generator consists of a coil which can rotate in magnetic field. The coil of the wire is called the *armature*. It is held between the pole pieces of strong magnet called the *field magnet*. When the armature rotates, the magnetic flux linked with it changes and the current is induced in the coil. This current is taken out by means of an arrangement called *slip rings and brushes*. The two rings are insulated from each other, the brush lightly touches each ring and connects the armature with the external circuit, as shown in Fig. 12.9.

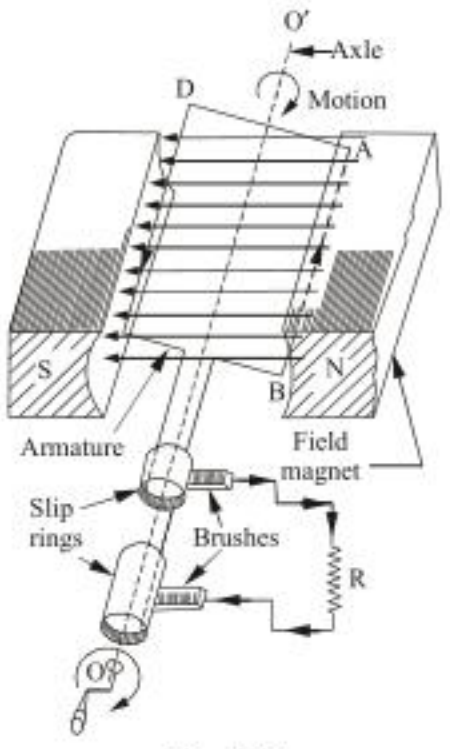


Fig. 12.9

Source of energy

The emf is generated when the armature rotates, hence, there must be some device for rotating the armature. Such as in a cycle dynamo the armature is rotated by the rotation of the wheel. For other purposes, steam engines, gas turbines etc. are used to rotate the armature. Thus the mechanical energy is converted into electrical energy. *In every generator, some form of mechanical energy is put into the generator, and*

transformed into electrical energy. Therefore, the generator basically does not produce energy but merely it transforms mechanical to electrical forms.

12.7 D.C. GENERATOR OR DYNAMO

In d.c. dynamo the current and emf are unidirectional. The principle of working is same as that of a.c. generator except that an arrangement is provided which reverses the emf after half cycle in a.c. generator. This is achieved by reversing the connections to the ends of the coil of the generator, when the direction of emf changes every time. This is done by using a *split ring commutator* instead of the slip rings. The split ring consists of two segments S_1 and S_2 of a brass cylinder which are insulated from each other. Each end of the armature coil is connected to one segment of the split ring. *The brushes and the split rings are so mounted that the contact of one brush with the first segment changes over to that with the second segment as soon as the direction of emf in the coil reverses.* This is done by mounting the commutator on the same axis as the coil and rotating along with it. The emf reverses the sign when the plane of the coil is at right angles to the magnetic field $\vec{B}$ at this instant emf is zero. Such a split ring cummutator is shown in Fig. 12.10. The emf time graph is shown in Fig. 12.11.

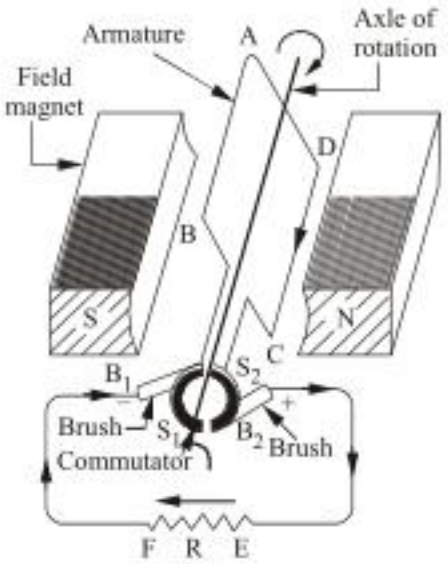


Fig. 12.10

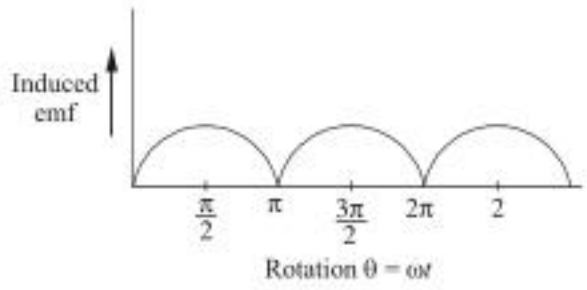


Fig. 12.11

12.8 ENERGY LOSSES IN GENERATOR AND THEIR REDUCTION

The energy losses in a dynamo are due to:

1. **Heating or copper losses.** Part of energy is wasted in overcoming the ohmic resistance of the armature coils, field coils and due to sparking at the brushes. This can be reduced by using thick copper wire of low resistance.
2. **Hysteresis Loss.** This is the energy used up in magnetising the core. This is reduced by choosing the material of the core having very thin B.H. loop such as soft iron.
3. **Eddy currents.** This is also known as iron loss and is due to eddy currents in the iron core. This is reduced by using laminated core.
4. **Flux leakage.** Leakage of flux is reduced by making the pole-shoe large so as to enclose completely the armature and also by reducing the air gap.
5. **Mechanical loss.** This is due to friction at the bearings, air resistance, etc. It is reduced by using roller or ball bearing and proper lubrication.

12.9 EFFICIENCY OF A GENERATOR

Efficiency is defined as

$$\eta = \frac{\text{Electrical energy supplied by the generator}}{\text{Mechanical energy supplied to the generator}}$$

If V is terminal voltage developed and I is current in external circuit, then power developed = VI. If the total power loss is W watts, then power used

$$= (VI + W) \text{ watts}$$

and $$\%\,\eta = \frac{VI}{VI + W} \times 100 \qquad \ldots(12.20)$$

12.10 D.C. MOTORS

It is a device by which electrical energy is converted into mechanical energy. Hence, it is a reverse of the generator or dynamo.

Principle. The principle of a d.c. motor is the same as that of moving coil galvanometer which is the reverse of the d.c. generator. When a current carrying conductor capable of a free movement is placed in a magnetic field as shown in Fig. 12.12, it experiences a mechanical force and begins to move. When a coil carrying a current I is placed in a magnetic field $\vec{B}$, then it experiences a force $(\vec{F} = I\vec{l} \times \vec{B})$ which turns it about in a direction perpendicular to both the field and the current. Thus the coil rotates and the rotating torque

$\tau = \vec{m} \times \vec{B}$ (see eq. 9.13)

where m is magnetic moment given by m = NIA, here A is area of the each coil and N are the number of the turns. When the plane of the coil is vertical, no torque acts on the armature (or coil). As the coil has sufficient inertia and frictional forces small it will cross the zero torque position. In order to make the motion of the coil continuous, the direction of the current is reversed after the vertical position (*i.e.* after each half revolution) with the help of split ring commutator so the torque acts in the same direction. Thus, we see that after each half cycle, there are positions of no torque and the principle used is reverse of d.c. generator.

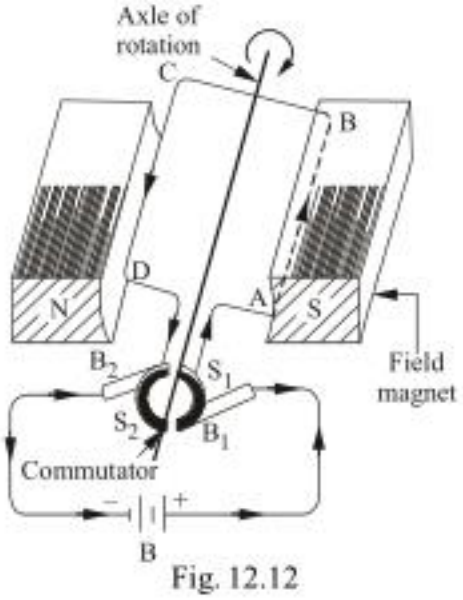


Fig. 12.12

For smooth motion, the single coil is replaced by several coils set such as drum winding called *armature*.

The magnetic field is generally provided by electromagnets called *field magnet*. From the direct current sources, the armature is connected in series with electromagnet . Then motor is said to *series-wound*. If the armature is connected in parallel with the field magnet, the motor is then called *parallel to shunt wound*. Sometimes the windings over the armature and the field magnet are partly in series or partly in parallel, the motor is then called *compound wound*.

Back emf in a Motor

When the armature coil rotates an induced emf is set up in the coils of the winding and the direction of the induced emf opposes the current that rotates the armature. This emf is called back emf of the motor. Thus, every motor acts as a generator also.

The value of back emf goes on increasing steadily as the motor picks up speed. An equilibrium is established when this back emf plus the voltage drop due to the resistance of the armature coil is equal and opposite to potential difference applied to the coil.

Suppose applied potential difference = V
Back emf developed = E
Resistance of armature circuit = R
Current in armature = I

then,

$$V = E + IR \qquad \ldots(12.21)$$

or $$I = \frac{V - E}{R} \qquad \ldots(12.22)$$

Thus, at the start when the speed is low, the back emf is small so that armature current I is large. At full speed the back emf is large.

Starter

At start, sometimes athe armature current is sufficiently large and it can damage the armature windings. To prevent this, a variable resistance called starter is connected in series with the armature when the speed increases, the starter resistance is reduced.

Efficiency of Motor

The input power used in the following ways,

1. Produces heat in the armature *i.e.* heat loss I^2R.
2. In magnetising the core of the armature *i.e.* magnetic loss.
3. Mechanical power available.

If V is potential difference and I is current in armature, then,

Input power = VI

$VI = I^2 R$ + magnetic loss + machanical power

If magnetic losses are negligible, we have,

$VI = I^2 R$ + Mechanical power

or mechanical power available,

$= VI - I^2 R$

$= I(V - IR)$, from equation (12.21)

$= IE$, $\quad E = V - IR$ = back emf

Efficiency is defined as the ratio of the output mechanical power to the electrical power

$$= \frac{\text{available mechanical power}}{\text{Input electrical power}}$$

$$= \frac{IE}{IV} = \frac{E}{V}$$

$$= \frac{\text{Back emf}}{\text{Applied emf}} \qquad \ldots(12.23)$$

This shows that efficiency increases as back emf E increases. Efficiency is maximum *i.e.* 100 per cent.

When E = V, but from the relation E = V – IR, under this condition, I = 0, and motor will do no work because the current is zero, hence this cannot be realised in practice.

However, the efficiency is maximum when,

$$EI = \text{maximum}$$

or $$E\left(\frac{V-E}{R}\right) = \text{maximum,} \qquad \text{(from eq. 12.22)}$$

or $$\frac{EV}{R} - \frac{E^2}{R} = \text{maximum}$$

Differentiating with respect to E, we get,

$$\frac{V}{R} - \frac{2E}{R} = 0$$

or $$V = 2E \quad \text{or} \quad E = \frac{V}{2} \qquad \ldots(12.24)$$

Hence, for maximum efficiency, the back emf should be half the applied voltage. In actual practice this condition cannot be satisfied as the armature resistance being small, the armature current will become high and burn the coil of the armature. Therefore, the back emf in actual practice is allowed to go up to 90 to 95 per cent of the applied emf to keep armature current within the safe limit.

12.11 FOR ROTATING MAGNETIC FIELD (OR A.C. INDUCTION MOTOR) AND TRANSFORMER, See Chapter 17 on A.C.

REVIEW QUESTIONS

12.1 Describe with necessary theory the use of an earth inductor to find the angle of dip.

12.2 Describe with relevant theory the method for measuring strong magnetic field.

12.3 Discuss the theory and construction of a Grassot's fluxmeter. How it is used in actual practice?

12.4 Describe the principle, construction and working of an AC generator or dynamo.

12.5 What do you mean by a D.C. generator? Describe its working.

12.6 What are energy losses in a generator? How these are reduced?

12.7 Describe the principle and construction of a D.C. electric motor. Define the efficiency of an electric motor. Prove that for maximum output, the back emf should be half the applied voltage.

SELF-TEST OBJECTIVE QUESTIONS

12.1 The earth inductor is an instrument for measuring the
- (*a*) magnetic elements
- (*b*) strong magnetic field
- (*c*) only horizontal component of earth's field
- (*d*) H,V and dip

12.2 The fluxmeter is special type of ballistic galvanometer in which
- (*a*) Controlling torque and damping ratio are large
- (*b*) Controlling torque is small and damping is very high
- (*c*) Controlling torque and damping are small
- (*d*) Controlling torque is large but damping is very high

12.3 Strong magnetic field is measured by special type of instrument known as
- (*a*) Fluxmeter
- (*b*) Search coil
- (*c*) Ballistic galvanometer
- (*d*) None of the above

12.4 A coil rotating in a magnetic field develops emf. This is the base of

(*a*) dynamo (*b*) motor
(*c*) both (a) and (b) (*d*) none of (a) and (b)

12.5 A.C. dynamo differs from D.C. one in that

(*a*) A.C. dynamo has electromagnet but D.C. one has permanent magnet
(*b*) A.C. dynamo has slip rings but D.C. one has commutators
(*c*) A.C. dynamo generates lower voltage
(*d*) A.C. dynamo has coil wound on iron but in D.C. one it is wound on copper

12.6 An electric motor is an instrument for converting

(*a*) mechanical energy into electrical
(*b*) electrical energy into mechanical
(*c*) electrical energy to any other form
(*d*) all the above

12.7 The brushes of a d.c. machine are made of

(*a*) carbon (*b*) steel
(*c*) soft-iron (d) copper

12.8 The e.m.f induced in a signal loop of a d.c. machine is picked up by means of

(*a*) slip rings (*b*) slip rings and brushes
(*c*) an armature (*d*) slip rings and an armature

12.9 Which of the following motors will not operate on D.C.?

(*a*) Series motor (*b*) Shunt motor
(*c*) Induction motor (*d*) Compound motor

12.10 The brushes of a generator carry the

(*a*) Excitation voltage to the permanent field magnets
(*b*) Generated voltage to the load, to field magnets
(*c*) Excitation voltage to the armature coil
(*d*) Commutator end of the armature

12.11 The armature cores of motors and dynamos should be made of materials having

(*a*) large $\bar{B} - \bar{H}$ loop area (*b*) small $\bar{B} - \bar{H}$ loop area
(*c*) None of the above

ANSWERS

1. (*d*)	**2.** (*b*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)	**6.** (*b*)
7. (*a*)	**8.** (*b*)	**9.** (*c*)	**10.** (*b*)	**11.** (*b*)	

CHAPTER 13

DYNAMICS OF CHARGED PARTICLES

This chapter briefly reviews the forces experienced by the charged particles in electric and magnetic fields. The systems that shall be discussed here relate to the motion of charged particles in uniform (static) electric and magnetic fields and time-changing magnetic fields are important in experimental applications. We shall assume that velocity of the particles is small compared with the velocity of light, *i.e.* we shall treat only non-relativistic problems. Also the effect of the particles on each other is neglected.

13.1 CHARGED PARTICLE IN UNIFORM (STATIC) ELECTRIC FIELD APPLIED ALONG THE DIRECTION OF THE PARTICLE MOTION

Let a particle of charge q be placed in an electric field $\vec{E}$. Since $\vec{E}$ is the force per unit charge (newtons per unit coulomb), the force $\vec{F}$ on the particle is given by,

$$\vec{F} = q\vec{E} \qquad \ldots(13.1)$$

The force is in the same direction as the field $\vec{E}$ if the charge is positive and opposite to the field if the charge is negative.

When the field is applied, the particle is accelerated uniformly in the direction of field. According to Newton's second law the force on a particle of mass m is given by,

$$\vec{F} = m\vec{a} = q\vec{E}$$

where a is acceleration and given by,

$$\vec{a} = \frac{q}{m}\vec{E} \qquad \ldots(13.2)$$

or
$$\frac{d^2\vec{r}}{dt^2} = \frac{q}{m}\vec{E} \qquad \ldots(13.3)$$

As $\vec{E}$ is constant on integrating eqn. (13.3), we will get velocity of the particle,

$$\frac{d\vec{r}}{dt} = \vec{v} = \frac{q\vec{E}}{m}t + C \qquad \ldots(13.4)$$

where C is the constant of integration when $t = 0$, $\vec{v} = \vec{u}$ (initial velocity of the particle).

From equation (13.4), $C = u$

Substituting C in equation (13.4), it reduces,

$$\vec{v} = \frac{q\vec{E}}{m}t + \vec{u} \qquad \ldots(13.5)$$

Again integrating (13.5), we have

$$\vec{r} = \frac{q\vec{E}}{2m}t^2 + \vec{u}t + C_1 \qquad \ldots(13.6)$$

where $t = 0$. Suppose the position of particle is given by $\vec{r}_0$, $C_1 = r_0$.

Substituting C_1 in equation (13.6), we get

$$\vec{r} = \frac{q\vec{E}}{2m}t^2 + \vec{u}t + \vec{r}_0 \qquad \ldots(13.7)$$

The relation (13.7) is applicable in general whatever may be direction of $\vec{r}_0$, $\vec{u}$ and $\vec{E}$. The relation (13.7) can be represented by vector diagram shown in Fig. 13.1. $\overrightarrow{AD}$ is the displacement $\vec{r}$ at any time *t*.

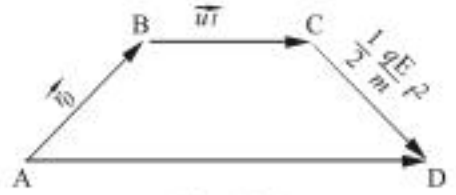


Fig. 13.1

If the charge is placed at rest on the origin of the coordinates, then,

$$\vec{u} = 0, r_0 = 0$$

We have from equation (13.7),

$$\vec{r} = \frac{q\vec{E}}{2m}t^2 \qquad \ldots(13.8)$$

and from equation (13.5),

$$\vec{v} = \frac{q\vec{E}}{m}t \qquad \ldots(13.9)$$

Energy Acquired

The field imparts energy to the charged particle. The force is given by,

$$F = m\vec{a} = q\vec{E}$$

The work done by this force in moving the particle as shown in Fig. 13.2 gives the energy W acquired by the particle. Thus,

$$W = m\int_1^2 \vec{a}\cdot d\vec{l} \qquad \text{(since } W = \vec{F}\cdot d\vec{l}\text{)}$$

Substituting $\vec{a} = \frac{d\vec{v}}{dt}$

and $d\vec{l} = \vec{v}\cdot dt$

$$W = m\int_1^2 \frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = m\int_1^2 \vec{v}\cdot d\vec{v}$$

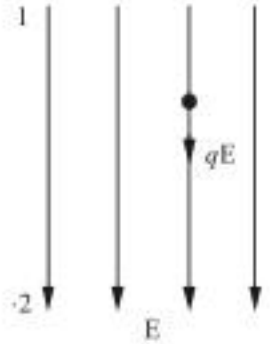


Fig. 13.2

If V is the potential difference between the points, then,

$$W = qV$$

Then, $$W = \frac{1}{2}m\left(v_2^2 - v_1^2\right) = qV \qquad \ldots(13.10)$$

where v_1 and v_2 are the initial and final velocities of the particle.

If the particle starts from rest, the initial velocity is zero ($v_1 = 0$) so,

$$W = qV = \frac{1}{2}mv^2 \qquad \ldots(13.11)$$

where v = final velocity. Solving for v

$$v = \sqrt{\frac{2qV}{m}} \text{ metre/sec} \qquad \ldots(13.12)$$

On squaring equation (13.9), we have,

$$v^2 = \frac{q^2 E^2}{m^2}t^2$$

$$\text{Kinetic energy} = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot \frac{q^2 E^2}{m^2}t^2$$

But from equation (13.8),

$$\frac{qE}{2m}t^2 = r$$

Hence, Kinetic energy $= qEr$ $\qquad \ldots(13.13)$

Electron Volt

If $q = e$, *i.e.* particle is electron, then the energy acquired by an electron,

$$W = eV$$

Further, if V = 1 volt,

$$W = e = 1.6 \times 10^{-19} \text{ joule}$$

This amount of energy is called one electron-volt.

$$1\,eV = 1.6 \times 10^{-19} \text{ joule} \qquad \ldots(13.14)$$

This is a convenient unit in designating the energies of particles.

Example 13.1 *Calculate the value of the electric field intensity which will give a proton an acceleration equal to the acceleration due to gravity.*

Solution. Force $= eE = mg$

or $$E = \frac{mg}{e}$$

$$E = \frac{1.67 \times 10^{-27} \times 9.8}{1.6 \times 10^{-19}} \text{ volt/cm}$$

$$= 10.2 \times 10^{-10} \text{ volt/cm}$$

Example 13.2 *Two parallel plates are kept in air 1 cm apart and potential difference of 100 volts is applied. If there be an electron resting on the plate at lower potential, calculate (1) the force on the electron, (2) the kinetic energy of this electron, when it reaches the other plate.*

Solution. $\mathrm{E} = \dfrac{\mathrm{V}}{d} = \dfrac{100}{0.01} = 10^4$ volts/m

Force $\mathrm{F} = e\mathrm{E} = 1.6\times10^{-19}\times10^4 = 1.6\times10^{-15}$ Newton

Acceleration $a = \dfrac{\mathrm{F}}{m} = \dfrac{1.6\times10^{-15}}{9.1\times10^{-31}} = 1.75\times10^{15}$ metres/sec^2

velocity when it reaches the other plate,

$$v = \sqrt{2ad}$$

Since $v^2 = u^2 + 2a\mathrm{S}$ and

$$u = 0, \mathrm{S} = d$$

or $$v = \sqrt{2\times1.75\times10^{15}\times0.01}$$

$$= 5.9\times10^6 \text{ m/sec.}$$

Therefore, the kinetic energy

$$= \frac{1}{2}mv^2 = \frac{1}{2}\times9.1\times10^{-31}\times(5.9\times10^6)^2$$

$$= 1.6\times10^{17} \text{ joules}$$

Example 13.3 *What must be the initial velocity of electrons which can just strike a conductor at –1200 volts potential?*

Solution. The electrons, having negative charge, are repelled by a conductor at negative potential. In order that an electron just reaches the conductor, it should have the kinetic energy given by,

$$\frac{1}{2}mv^2 = e\mathrm{V}$$

$$v = \sqrt{\frac{2e\mathrm{V}}{m}} = \sqrt{\frac{2\times1.6\times10^{-19}\times1200}{9.1\times10^{-31}}}$$

$$= 2.06\times10^6 \text{ metre/sec.}$$

Example 13.4 *A positive ion of charge q and mass m is accelerated by a potential difference V, passes with a constant velocity through the space between two plates separated by distance d. Show that transit time between plates is* $\dfrac{d}{(2q\mathrm{V}/m)^{1/2}}$

Solution. From equation (13.12),

$$v = \sqrt{\frac{2q\mathrm{V}}{m}}$$

If t be the transit line between the plates,

$$t = \frac{d}{v} = \frac{d}{\left(\dfrac{2q\mathrm{V}}{m}\right)^{1/2}}$$

13.2 MOTION OF A CHARGED PARTICLE IN UNIFORM TRANSVERSE ELECTRIC FIELD

In this case we shall examine the motion of charged particles when the uniform electric field is perpendicular to the direction of its initial velocity.

Fig. 13.3 shows a beam of charged particles (*e.g.* electrons beam) moving along x-axis through a pair of parallel plates of length l and separated by distance d. If V be the potential difference between the plates, then $E_y = \frac{V}{d}$. The force on the charged particle = qE_y.

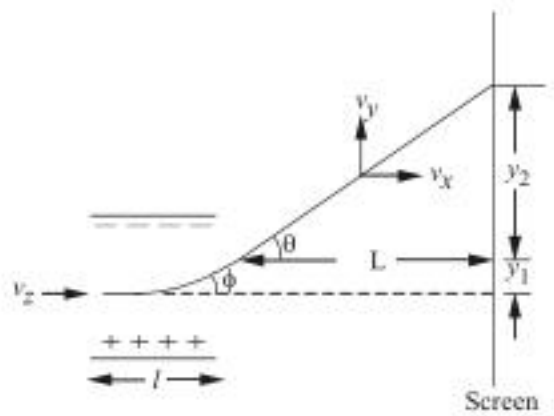


Fig. 13.3

The force is along the direction of the field *i.e.* Y-axis, hence the initial velocity of the particle v_x is not affected by this force. So v_x remains constant. If the electron travels a distance x along x-axis in the field, the time t required to cover this distance is given by,

$$t = \frac{x}{v_x} \qquad \ldots(13.15)$$

In this time, the transverse velocity gained by the particle is,

$$\frac{dy}{dt} = v_y = 0 + at$$

$$= \frac{qE_y}{m}\frac{x}{v_x} \qquad \ldots(13.16)$$

where $a = \frac{qE_y}{m}$ is acceleration of the particle and initial velocity along y-axis is zero.

The displacement along y-axis is,

$$y = \frac{1}{2}at^2 = \frac{1}{2}\frac{qE_y}{m}\left(\frac{x}{v_x}\right)^2$$

$$= \frac{qE_y}{2mv_x^2}x^2 \qquad \ldots(13.17)$$

The quantity $\frac{qE_y}{2mv_x^2}$ is constant, say A, then,

$$y = Ax^2 \qquad \ldots(13.18)$$

which represents a parabola. Thus the path of a charged particle in a transverse uniform electric field is parabola. The direction of motion is changed due to field. At any instant t if the inclination of the direction of motion to x-axis be ϕ, then from equation (13.17),

$$\tan\phi = \frac{dy}{dx} = \frac{qE_y}{2mv_x^2} \cdot 2x$$

$$= \frac{qE_y}{mv_x^2} x \qquad \ldots(13.19)$$

But the particle remains in the field for a distance l the length of the plates. Hence the displacement (y_1) along the y-axis due to entire field will be obtained by substituting $x = l$ in equation (13.17),

$$y_1 = \frac{qE_y}{2mv_x^2} l^2 \qquad \ldots(13.20)$$

After traversing the field, the direction of the particle is changed through the angle θ

$$\tan\theta = \frac{qE_y}{mv_x^2} l \qquad \ldots(13.21)$$

(by putting $\phi = 0, x = l$ in equation (13.19))

It is to be noted that as soon as the particle emerges from the field, no force acts on it. The particle travels (neglecting gravity) toward screen in a straight line tangent to the parabola at the exit point. If screen is placed at a distance L from the end A of the plate, then the displacement y_2 along y-axis is given by,

$$y_2 = L\tan\theta \qquad \ldots(13.22)$$

Now, the total displacement on the screen is,

$$y = y_1 + y_2 \qquad \text{[Substituting from eqns. (13.20) and (13.22)]}$$

$$y = \frac{qE_y l^2}{2mv_x^2} + L\tan\theta$$

Substituting $\tan\theta$ from equation (13.21),

$$y = \frac{qE_y l^2}{2mv_x^2} + L\frac{qE_y}{mv_x^2} l \qquad \ldots(13.23)$$

Such a deflected electron beam when falls on the screen makes a "visible small luminous spot". This is the principle of the "*electrostatics cathode-ray oscilloscope*" which we shall discuss later on.

Example 13.5 *The electric field between the plates of a cathode-ray oscilloscope is 1.2×10^4 Newton/coul. What deflection will an electron experience if it enters at right angles to the field with a kinetic energy of 2000 eV? The plates are 1.5 cm long.*

Solution. Using equation (13.17),

$$y = \frac{qE_y}{2mv_x^2} x^2$$

As given $\frac{1}{2}mv_x^2 = 2000\ eV$

$$y = \frac{qE_y x^2}{4\times(2000\times1.6\times10^{-19})}$$

$$= \frac{1.6\times10^{-19}\text{ coul}\times1.2\times10^{1}\text{ Nt/coul}\times(1.5\times10^{-2}\text{ metre})^2}{4\times3.2\times10^{-16}\text{ joul}}$$

$$= 3.4\times10^{-4}\text{ metre}$$

$$= 0.34\text{ mm}$$

Example 13.6 *An electron beam is accelerated through a potential difference of 1000 V. An electric field is applied at right angles to the path of the beam by means of rectangular parallel plates. The distance between the plates is 2.5 cm and potential difference 50 V. The edges of the plates are at a distance of 25 cm and 30 cm from the fluorescent screen. What is the displacement of the fluorescent spot?*

Solution. $\mathrm{E} = \dfrac{50}{2.5\times10^{-2}}\dfrac{\text{Volts}}{\text{metre}}$

$$\text{Kinetic energy} = \frac{1}{2}mv^2 = e\mathrm{V}$$

$$\frac{1}{2}\times9.1\times10^{-31}\times v^2 = 1000\times1.6\times10^{-19}$$

On solving this for v, we get

$$v = 5.6\times10^{7}\text{ metre/sec}$$

Using equation (13.23),

$$y = \frac{q\mathrm{E}}{m}\left(\frac{l}{2}+\mathrm{L}\right)\frac{l}{v^2}$$

We have omitted the subscript and assumed that E is along y-axis and particle is moving along x-axis.

Substituting $\mathrm{L} = 25\times10^{-2}$ metre

$$l = 5\times10^{-2}\text{ metre}$$

and v and E calculated above, we get,

$$y = \frac{50\times1.6\times10^{-19}}{2.5\times10^{-2}\times9.1\times10^{-31}}\left[0.25+\frac{0.05}{2}\right]\frac{5\times10^{-2}}{[5.65\times10^{7}]^2}$$

$$= 0.153\times10^{-2}\text{ metre}$$

$$= 0.153\text{ cm.}$$

Example 13.7 *If a stream of electrons, each of mass m and charge e and velocity 3 × 10⁶ metre/sec., is deflected by 2 mm in traversing a distance of 10 cm through an electrostatics field of 18 V/cm perpendicular to their path, find e/m in couls/kg.*

Solution. $\mathrm{E} = 18\dfrac{\text{Volts}}{\text{cm}} = \dfrac{18}{10^{-2}}\dfrac{\text{Volts}}{\text{metre}}$

$$e\mathrm{E} = ma$$

$$y = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{eE}{m}\right)\left(\frac{1}{v}\right)^2$$

$$2\times10^{-3} = \frac{1}{2}\left(\frac{18}{10^{-2}}\cdot\frac{e}{m}\right)\left(\frac{10\times10^{-2}}{3\times10^{6}}\right)^2$$

or $$\frac{e}{m} = \frac{2\times10^{-3}\times2\times10^{-2}\times9\times10^{12}}{18\times100\times10^{-4}}$$

$$= 2\times10^{11} \text{ couls/kg.}$$

13.3 CHARGED PARTICLE IN A CONSTANT UNIFORM MAGNETIC FIELD

When a charged particle moves through a magnetic field, the Lorentz force acts on it. If the charge q is moving with velocity v in a magnetic field $\vec{B}$, the force acting on the particle is given by,

$$\vec{F} = q\,(\vec{v}\times\vec{B}) \qquad \ldots(13.24)$$

From Newton's second law, F = ma. Thus,

$$m\vec{a} = q\,(\vec{v}\times\vec{B}) \qquad \ldots(13.25)$$

where $\vec{a}$ is acceleration and given by,

$$\vec{a} = \frac{q\,(\vec{v}\times\vec{B})}{m} \qquad \ldots(13.26)$$

According to eqn. (13.26), the acceleration is normal to the plane containing the particle path (i.e. the velocity $\vec{v}$) and $\vec{B}$. *We can draw the following important conclusions.*

(1) From eq. (13.26), the acceleration is maximum when angle between $\vec{v}$ and $\vec{B}$ is 90° *i.e.*, if the direction of particle path (indicated by v) is normal to $\vec{B}$.

(2) If $\vec{v} = 0$, *i.e.*, if the particle is at rest, the acceleration $\vec{a}$ (as well as force $\vec{F}$) is zero. It means field has no effect.

(3) If angle between $\vec{v}$ and $\vec{B}$ is zero, then acceleration $\vec{a} = 0$.

Thus, if the particle is moving in the same direction as $\vec{B}$, field $\vec{B}$ has no effect as $\vec{a}$ is zero. It means the particle moves with the same velocity *i.e.* $\vec{v}$.

Thus, *the field $\vec{B}$ affects the motion of the charged particle only when the path or the velocity $\vec{v}$ has a component normal to $\vec{B}$.* Further if a magnetic field of large extent is at right angles to the direction of motion of a charged particle, the particle is deflected into a circular path.

If, however, the velocity v of the particle be inclined to the magnetic field at an angle other than 90° *i.e.* not normal to the field, it can be resolved into two components. $\vec{v}_1$ parallel to the field, $\vec{B}$ and $\vec{v}_2$ perpendicular to $\vec{B}$. Suppose $\vec{B}$ is towards the right as shown in Fig. 13.4 and v_2 is normally outward from the page. According to the cross product of $\vec{v}_2 \times \vec{B}$, the acceleration $\vec{a}$ is upward so that positively charged particle describes a curved path in the clockwise direction in the plane of the page.

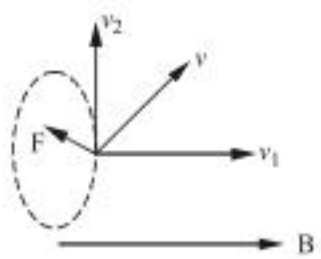


Fig. 13.4

The magnitude of the force on the particle

$$= qv_2 B$$

If particle moves in a curved path of radius R, this force is balanced by centrifugal force,

$$\frac{mv_2^2}{m} = qv_2 \mathrm{B}$$

or $$\mathrm{R} = \frac{mv_2}{q\mathrm{B}} \qquad \ldots(13.27)$$

It is the radius of the curvature of the path of the particle moving in the magnetic field. This is called *gyro radius or cyclotron radius*. Thus, the larger the velocity of the particle or the larger its mass, the greater the radius. On the other hand, the larger the charge or $\overline{\mathrm{B}}$, the smaller the radius.

Since the force acts perpendicular to v_2 hence it changes its direction but not magnitude. And, as the component v_1 of the velocity, parallel to the field $\overline{\mathrm{B}}$ remains constant, so that particle moves in the direction of $\overline{\mathrm{B}}$. Therefore, the combined effect of the circular and linear horizontal motion is to produce a helical motion, as shown in Fig. 13.5, with the axis of the helix parallel to $\overline{\mathrm{B}}$.

Fig. 13.5

Suppose the initial velocity is perpendicular to the magnetic field *i.e.* $v_1 = 0$, in other words particle is moving in a direction perpendicular to $\overline{\mathrm{B}}$. Then the curved path is exactly circular.

The angular velocity,

$$\omega = \frac{v_2}{\mathrm{R}}$$

$$\omega = \frac{q\mathrm{B}}{m} \qquad \text{(by eq. 13.27)} \qquad \ldots(13.28)$$

The time required to complete one circle is period of *revolution*, is given by,

$$\mathrm{T} = \frac{2\pi}{\omega} = \frac{2\pi m}{q\mathrm{B}} \qquad \ldots(13.29)$$

The number of revolutions per second by a charged particle in a magnetic field is termed *gyro frequency or cyclotron frequency* is,

$$n = \frac{1}{\mathrm{T}} = \frac{q\mathrm{B}}{2\pi m} \text{ r.p.s} \qquad \ldots(13.30)$$

This relation suggests that cyclotron frequency of charged particle is independent of its velocity and hence also of the energy of the particle so long as the velocity lies in the non-relativistic region *i.e.* so long as $v \ll \mathrm{C}$. But if particle velocity is comparable to the velocity of the light, we cannot overlook the fact, that mass changes with velocity according to the relation,

$$m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

Pitch of the Helix

It is defined as the distance travelled by the charged particle along $\overline{\mathrm{B}}$ in the time in which it completes one revolution, *i.e.*,

$$\text{Pitch of helix} = v_1 \mathrm{T}$$

$$= \frac{2\pi m v_1}{q\mathrm{B}} \qquad \ldots(13.31)$$

Example 13.8 *For a proton initially travelling with velocity $v_x = 5 \times 10^8$ cm/sec, calculate the following:*

(1) Force experienced in a magnetic field $\vec{B} = 2000\,\hat{j} + 4000\,\hat{k}$ gauss

(2) Acceleration experienced in an electric field E, $\vec{E} = 200\hat{i} + 100\,\hat{j}$ volts/cm

(3) Transverse deflection in travelling a length $L_x = 10$ cm in an electric field $E_y = 200$ volts/cm.

Solution. (1) Force in magnetic field

$$\vec{F} = q\,(\vec{v} \times \vec{B})$$

$$\vec{v}_x = 5 \times 10^8 \text{ cm/sec} = 5 \times 10^6 \text{ metre/sec}$$

$$\vec{B} = 2000\,\hat{j} + 4000\,\hat{k}$$

$$\vec{B} = 0.2\,\hat{j} + 0.4\,\hat{k} \text{ weber/m}^2$$

Since, $\dfrac{1 \text{ Weber}}{\text{metre}^2} = 10^4$ gauss

hence, $\vec{F} = 1.6 \times 10^{-19}\left[(5 \times 10^6\,\hat{i}) \times (0.2\,\hat{j} + 0.4\hat{k})\right]$

$$= 1.6 \times 10^{-19} \times 5 \times 10^6 \times 2\,[0.1\hat{k} - 0.2\,\hat{j}]$$

$$= 1.6 \times 10^{-13}\,[\hat{k} - 2\hat{j}] \text{ Newtons}$$

Magnitude of the force $F = 1.6 \times 10^{-13}\sqrt{5} = 3.57 \times 10^{-13}$ Nt.

Its direction cosines $\left(0, -\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$

(2) In electric field,

$$ma = qE$$

$$a = \frac{qE}{m} = \frac{1.6 \times 10^{-19}}{1.67 \times 10^{-27}}\left(\frac{200\hat{i} + 100\,\hat{j}}{10^{-2}}\right)$$

$$= \frac{1.6}{1.67}10^{12}\,(2\bar{i} + \bar{j})$$

$$= \frac{1.6}{1.67}10^{12}\sqrt{5}$$

$$= 2.14 \times 10^{12} \text{ met/sec}^2$$

(3) Transverse deflection,

$$Y = \frac{1}{2}at^2$$

$$= \frac{1}{2}\left(\frac{gE_y}{m}\right)\left(\frac{l}{v_x}\right)^2$$

Since $E_y = 200\,\dfrac{\text{volt}}{\text{cm}} = \dfrac{200}{10^{-2}}\,\dfrac{\text{volt}}{\text{metre}}$

$$Y = \frac{1}{2} \times \frac{1.6 \times 10^{-19} \times 200}{1.67 \times 10^{-27} \times 10^{-2}} \times \left(\frac{10 \times 10^{-2}}{5 \times 10^{-6}}\right)^2 \text{ metre}$$

$$= 0.0383 \times 10^{-2} \text{ metre}$$

$$= 0.0383 \text{ cm}$$

Example 13.9 *An electron of energy 20 eV moving in a direction perpendicular to a magnetic field $B = 10^2$ weber/m^2 describes a circle of radius R. Calculate the value of R.*

Solution. Kinetic energy $\frac{1}{2}mv^2 = 20\ eV$

$$= 20 \times 1.6 \times 10^{-19} \text{ joules}$$

hence, $$v = \left(\frac{2 \times 20 \times 1.6 \times 10^{-19}}{m} \right)^{1/2} \text{ m/sec}$$

The radius of the circular path is given by equation (13.27),

$$\mathrm{R} = \frac{mv}{q\mathrm{B}}$$

$$= \frac{m \times (64 \times 10^{-19})^{1/2}}{\sqrt{m} \times q \times \mathrm{B}}$$

$$= \frac{(9 \times 10^{-31} \times 64 \times 10^{-19})^{1/2}}{1.6 \times 10^{-19} \times 10^2}$$

$$= 15.0 \times 10^{-8} \text{ metre}$$

Example 13.10 *An electron of velocity $\vec{v} = (2\hat{i} + 3\hat{j})\,10^8$ cm/sec. enters a region of uniform magnetic field, $\mathrm{B} = 500\hat{i}$ gauss. Show that its path becomes helical.*

(1) In what direction does the axis of the helix lie?

(2) Calculate the radius of helix.

(3) Calculate the number of rotations as the electron advances 10 cm along the axis of the helix.

Solution. $\vec{v} = (2\hat{i} + 3\hat{j})\,10^8$ cm/sec. and

$$\vec{\mathrm{B}} = 500\hat{i} \text{ gauss}$$

The component of velocity $\vec{v}_1 = 2\vec{i} \times 10^8$ cm/sec is along $\vec{\mathrm{B}}$ and electron moves along x-axis and due to the component $\vec{v}_2 = 3\hat{j} \times 10^8$ cm/sec which is perpendicular to $\vec{\mathrm{B}}$ the motion is circular so that it moves in a circular path, [see sec (13.3)].

(1) The axis of helix lines along B *i.e.* x-axis.

(2) Radius of helix R by equation (13.27)

$$\mathrm{R} = \frac{mv_2}{q\mathrm{B}} + \frac{9 \times 10^{-31} \times 3 \times 10^6}{1.6 \times 10^{-19} \times 500 \times 10^{-4}}$$

(1 Weber/metre^2 = 10^4 gauss)

$$\mathrm{R} = 3.37 \times 10^{-4} \text{ metre}$$

$$= 0.0337 \times 10^{-2} \text{ metre}$$

$$= 0.0337 \text{ cm}$$

(2) Distance travelled in one rotation = pitch of the helix by equation (13.31),

$$\frac{2\pi m v_1}{q\mathrm{B}} = \frac{2\pi \times 9 \times 10^{-31} \times 2 \times 10^6}{1.6 \times 10^{-19} \times 500 \times 10^{-4}}$$

Number of rotations made by the electron as it advances through a distance of 10 cm

$$= \frac{10\times10^{-2}}{\text{pitch of helix}}$$

$$= \frac{10\times10^{-2}\times1.6\times10^{-19}\times500\times10^{-4}}{2\pi\times9\times10^{-31}\times2\times10^{6}}$$

$$= 70.74$$

$$= 71$$

Example 13.11 *An electron has a velocity of 10^4 meters/sec normal to a magnetic field of 0.1 weber/metre^2. Find the radius of the electron path and also its frequency.*

Solution. From equation (13.27), the radius,

$$R = \frac{9.1\times10^{-31}\times10^{4}}{1.6\times10^{-19}\times0.1} = 5.7\times10^{-7} \text{ metre}$$

This is a very small circle. The frequency,

$$n = \frac{v}{2\pi R} = \frac{10^4}{2\pi\times5.7\times10^{7}}$$

$$= 2.8\times10^{9} \text{ rps}$$

13.4 CYCLOTRON

The cyclotron is a heavy particle accelerator, invented in 1929 by E.O. Lawrence for accelerating charged particles to high velocities such as protons, deutrons, or alpha particles.

Construction

It essentially consists of a flat cylindrical vacuum chamber C inside which there are two semi-cylindrical hollow electrodes DD, called the dees on account of their shape like the letter D. These dees are arranged with small gap between them. The chamber C is placed horizontally between the pole pieces of a large electromagnet capable of producing a field of order of 1.5 weber/metre^2. The source F arranged near the centre of the dees supplies the positive ions to be accelerated as shown in Fig. 13.6 (*a*). The positive ions are produced by ionising the gas at low pressure. For example, when electrons ionise the hydrogen gas the protons are produced, deutrons from deutrium gas and α-particles from helium gas. An alternating potential of the order of 10,000 volts and high frequency is connected to the terminals of dees as shown in Fig. 13.6 (*b*).

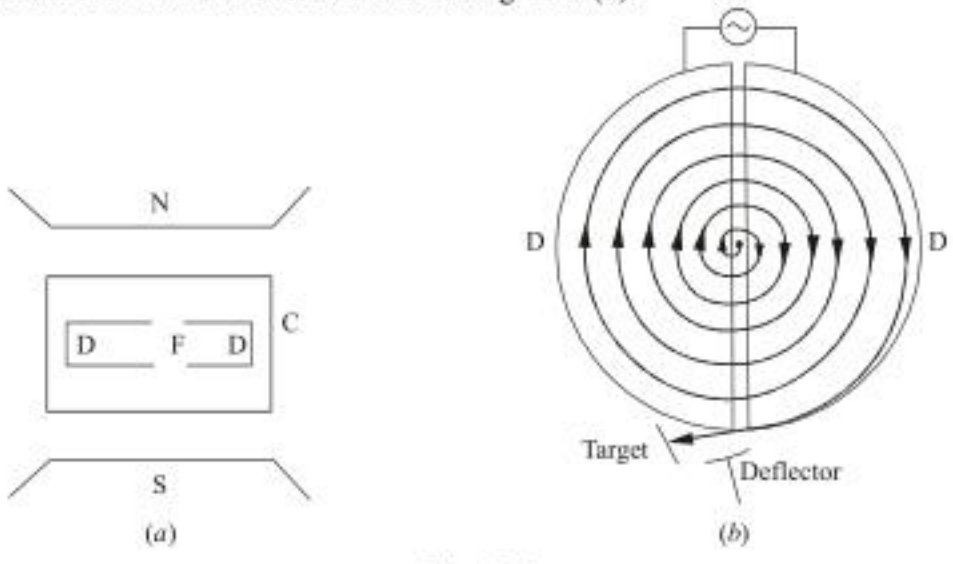


Fig. 13.6

Principle of Action

The positive ion introduced at F will be attracted towards the dee which happens to be negative at that moment.

Entering the dee the ion is in a region free of electric field and will move with a constant speed. Suppose the magnetic field $\overline{B}$ is acting vertically upward *i.e.* perpendicular to the path of the ion, the ion will describe a small semi-circular path in clockwise direction, before arriving at the gap between the dees. Now, if the frequency of the applied alternating potential is so adjusted that the ion completes a semi-circle in the half cycle of the alternating potential difference then it will reach in the gap at the instant, when the second D is negative. Thus the particle will be further accelerated by the new reversed electric field across the gap towards the opposite dee. The ion will acquire additional energy, hence now it again moves in a circle of larger radius. The process of acceleration and motion in circular path is repeated after each cycle. However, the ion still completes the semi-circle inside the dee in the half cycle of alternating potential, because the period of revolution of an ion circulating in a magnetic field does not depend on the speed of the ion [see equation (13.29)]. By repetition of this process, the energy of the ions is increased in steps until the radius of the path acquires the value equal to the radius of the dees R. From here the ion is pulled out of the system by a negatively charged deflecting electrode through an opening in the dee so that it can impinge on a specimen placed outside the dee.

It is to be noted that the *operation* of the cyclotron requires the condition given in equation (13.30), *i.e. the frequency with which ion circulates in the field must be equal to frequency of the electric field applied between the dees.* Therefore, from equation (13.30), we get,

$$n = \frac{qB}{2\pi m} \qquad \ldots(13.32)$$

As q/m is constant for a given ion and usually the oscillator is designed to work at constant frequency n. So we tune the cyclotron by varying B until equation (13.32) is satisfied and an accelerated beam appears. We can also satisfy equation (13.32) by varying n.

The final energy of a particle accelerated by the cyclotron is determined by the radius R and magnetic field B. From equation (13.27),

$$R = \frac{mv_{max}}{qB}$$

or $$v_{max} = \frac{RqB}{m} \qquad \ldots(13.33)$$

The maximum kinetic energy acquired by the particle

$$W = \frac{1}{2} m v_{max}^2$$

Substituting v_{max}, we get,

$$\text{Energy of the particle } W = \frac{1}{2} \frac{(RqB)^2}{m} \qquad \ldots(13.34)$$

Clearly to get the particles of high energy R and B should be large. As energy is inversely proportional to mass, the lighter particles will gain larger value of energy for the same R and B. *Equation (13.34) shows that the energy of the particle is independent of the potential difference applied to the dees.*

If the voltage V applied between the gap is small, the particle will acquire the same energy through a large number of revolutions. But when it is large the particle will acquire this energy in few reolutions. However, to reduce the dispersion of the beam it is desirable to make the number of revolutions small

and hence, V is as large as possible. Because when ion crosses the gap between the dees it gains energy qV. It crosses the gap twice in each revolution hence gain in energy in one revolution is $2q$V joules or 2V eV. Hence, the total energy W acquired in N revolutions is,

$$W = 2Nq V \qquad \ldots(1.3.35)$$

Thus for a given energy W as determined by (13.34), and for a particular voltage, the number of revolutions are

$$\text{Number of revolutions } N = \frac{W}{2qV} \qquad \ldots(13.36)$$

where W is in joules. If W is expressed in electron volt, We_v

$$N = \frac{W e_v}{2qV} \times 1.6 \times 10^{-19} \qquad \ldots(13.37)$$

Limit to Maximum Energy

However, to get the particles of very high energy the velocity becomes comparable to the velocity of light and so the mass of the particle changes,

$$\left[m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \right]$$

and the angular velocity of the particle will not remain constant because $\omega = \frac{qB}{m}$, from equation (13.28).

The cyclotron frequency ω decreases so that it lags in phase behind the accelerating alternating voltage. Therefore, the particle will not reach the gap at the right time when alternating potential completes half cycle, *i.e.* there is no resonance between the potential difference and circular motion. This sets a limit to the energy which a particle can have inside a cyclotron.

The electron cannot be accelerated to high energy by cyclotron because due to its small mass it will have a large velocity with a comparatively small kinetic energy and then its mass will increase according to relativity theory.

The Synchro-Cyclotron or the Relativistics Ion Cyclotron

Lawrence in 1946 developed synchro-cyclotron to compensate for the relativistics increase of mass either by modifying the magnetic field or the frequency of the accelerating potential difference as the particle gains energy.

Example 13.12 *Find the frequency of the oscillator that is to be connected between the dees of a cyclotron with a magnetic field* $\overline{B} = 1.5$ *weber/metre*^{2} *for accelerating deutron. How the result is modified if we choose* α *-particle or proton?*

Solution. $q = 1.6 \times 10^{-19}$ couls.

Mass of deutron = 2 × mass of proton

$$m = 3.34 \times 10^{-27} \text{ kg}.$$

From equation (13.30),

$$n = \frac{qB}{2\pi m}$$

$$= \frac{1.6\times10^{-19}\times1.5}{2\pi\times3.34\times10^{-27}} = 1.14\times10^{7} \text{ rps}$$

$$= 11.4 \text{ MC/sec.} \qquad [1 \text{ MC} = 10^{6} \text{ cycles}]$$

Accordingly, the oscillator frequency must be 11.4 MC/sec. If instead of deutron proton is accelerated, then,

$n = 22.8$ MC/sec. since q/m for proton = twice of deutron

and for α -particle n = 11.4 M/C sec. as q/m for a particle = q/m of deutron (see Table 13.1.)

Example 13.13 *In the above example, if the radius of the dees is 50 cm, find the energy of the emergent deutrons. Also calculate the energy of α = particle or proton if they were accelerated, under the same condition.*

Solution. From equation (13.34),

$$W = \frac{1}{2}\frac{(RqB)^2}{m}$$

$$= \frac{(0.5\times1.6\times10^{-19}\times1.5)^2}{2\times3.34\times10^{-27}}$$

$$= 2.15\times10^{-12} \text{ joule}$$

$$= \frac{2.15\times10^{-12}}{1.6\times10^{-19}} eV$$

$$= 1.33\times10^{7} eV$$

$$= 1.33 \text{ MeV} \qquad (1 \text{ million electron volt} = 10^{6} eV)$$

If either α-particles or protons have been used the energy would be double, *i.e.* 26.6 MeV because for α-particle q in the above example will be $2q$ and m will be 2m.

Hence,

$$W = \frac{4}{2}\left(\frac{1}{2}\frac{(RqB)^2}{m}\right) = 26.6 \text{ MeV}$$

Similarly for proton q is same. But mass is half of the deutron, *i.e.* $q = q$, $m = m/2$.

$$W = 2\left[\frac{1}{2}\frac{(RqB)^2}{m}\right] = 26.6 \text{ MeV}$$

Example 13.14 *A stream of protons and deutrons in a vacuum chamber enters a uniform magnetic field. Both are subjected to the same accelerating potential, hence kinetic energies are the same. If ion stream is perpendicular to the magnetic field and protons move in a circular path of radius 15 cm, find the radius of the path traversed by the deutrons. Given that mass of a deutron is twice that of a proton.*

Solution. The radius is given by equation (13.27),

$$R = \frac{mv}{qB}$$

So that, radius of the path of proton

$$R_p = \frac{m_p v_p}{qB}$$

and radius of the path of deutron

$$R_d = \frac{m_d v_d}{qB}$$

Charge on deutron is same as on proton and moving in the same field. Thus,

$$\frac{R_d}{R_P} = \frac{m_d v_d}{m_P v_P}$$

Since they have same kinetic energies, *i.e.*,

$$\frac{1}{2} m_d v_d^2 = \frac{1}{2} m_P v_P^2$$

or $$\frac{v_d}{v_P} = \sqrt{\frac{m_P}{m_d}}$$

Since $m_d = 2 m_p$, so

$$\frac{v_d}{v_P} = \sqrt{\frac{m_P}{2m_P}} = \frac{1}{\sqrt{2}}$$

Hence,

$$\frac{R_d}{R_P} = \frac{2m_P}{m_P} \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$$

or $$R_d = \sqrt{2}\, R_P \quad \text{since } R_P = 15 \text{ cm (given)}$$

hence $$R_d = 0.15 \times \sqrt{2} = 0.212 \text{ metre}$$
$$= 21.2 \text{ cm}.$$

Example 13.15 *A cyclotron accelerating protons has dees of 40 cm. radius and a radio frequency supply of 12 magacycles per second at 10000 volts maximum value. Calculate:*

(1) The magnetic field B

(2) The speed and kinetic energy acquired by protons

(3) Maximum number of revolutions and time spent by a proton within dees.

Solution. (1) From equation (13.30),

$$\text{Cyclotron frequency } n = \frac{qB}{2\pi m}$$

or $$B = \frac{2\pi m n}{q}$$

$$= \frac{2 \times 3.14 \times 1.67 \times 10^{-27} \times 12 \times 10^6}{1.6 \times 10^{-19}}$$

$$= 0.79 \text{ weber/metre}^2$$

(2) From equation (13.27),
Velocity acquired by proton,

$$v = \frac{qBR}{m}$$

$$= \frac{1.6 \times 10^{-19} \times 0.79 \times 0.40}{1.67 \times 10^{-27}}$$

$$= 3\times10^{7} \text{ metre/sec.}$$

$$\text{Energy} \quad = \frac{1}{2}mv^2 = \frac{1}{2}\times1.6\times10^{-27}\times(3\times10^{7})^2 \text{ joules}$$

$$= 7.5\times10^{-13} \text{ joules}$$

$$= \frac{7.5\times10^{-13}}{1.6\times10^{-19}} \text{eV}$$

$$= 4.7\times10^{6} \text{ eV}$$

(3) From equation (13.37),

Number of revolutions inside dees

$$N = \frac{\text{WeV}}{2q\text{V}}\times1.6\times10^{-19}$$

$$N = \frac{4.7\times10^{6}\times1.6\times10^{-19}}{2\times1.6\times10^{-19}\times10,000}$$

$$= 235$$

$$\text{Time spent within dees} = \frac{235}{n}$$

$$= \frac{235}{12\times10^{6}}$$

$$= 195\times10^{-5} \text{ sec.}$$

13.5 CATHODE RAY OSCILLOGRAPH

Principle. The cathode ray oscillograph is based on the facts that an electron beam is deflected by an electric and magnetic field and a bright spot is produced when electron beam strikes the fluorescent screen. A cathode-ray tube is a device for observing rapid voltage variation.

Construction. In a cathode-ray tube (Fig. 13.7) a beam of electrons is emitted from a cathode C. G is a control grid which is made negative with respect to filament so that electrons are concentrated into a thin line by repulsion along the axis. The beam is minutely accelerated by anode A_1 which consists of metal disc with a central hole so that electrons can pass through it. The anode A_1 is kept at higher potential. The beam can be further accelerated by anode A_2 maintained at higher protential than A_1. This device is called "electron gun". The hole tube is evacuated at low pressure.

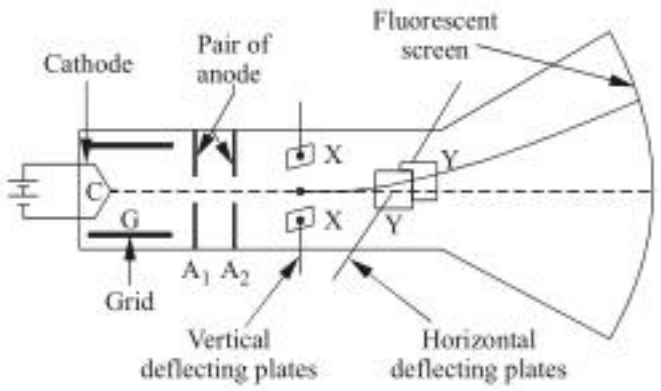


Fig. 13.7

By means of either a transverse electric or a transverse magnetic field the beam may be deflected. This is done by two pairs of parallel plates XX and YY at right angles to each other. These are called deflector plates and are connected to external electric or magnetic sources. *Depending on the nature of electric or magnetic field the cathode ray oscillograph is called electrostatics or magnetostatics C.R.O.* In the case of electric field, as the electron beam passes between the plate XX it will be attracted by the positive plate and repelled by the negative plate and the spot will move on the screen. Now by applying potential difference between YY which is right angles to XX, the spot will move in a direction right angles to first pair XX. Thus, if XX cause the vertical deflection of the spot then YY will cause the horizontal deflection of the spot on the screen.

13.6 DEFLECTION SENSITIVITY IN ELECTRIC FIELD

The electric deflection sensitivity is defined as the amount of deflection of the electron spot produced when a voltage of one volt d.c. is applied between the corresponding deflection plate. This is different for X and Y plates. Fig. 13.9 shows the electron deflection system, l is the length of deflecting plates and L is the distance from edge of the plate to the screen, then the distance of the mid-point O of the field from the screen is L + l/2 and displacement on screen is Y. From Fig. 13.8

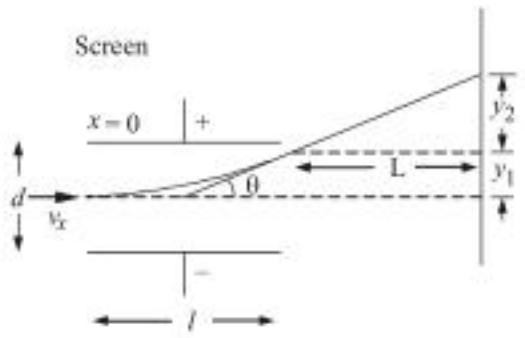


Fig. 13.8

$$\text{Tan } \theta = \frac{Y}{(L + l/2)}$$

by equation (13.19) i.e.

$$\text{Tan } \theta = \frac{q E_y l}{m v_x^2}$$

Here charge on electron is e and say deflecting field is E_d and electron enters the deflecting plates with velocity v_x

$$\text{Tan } \theta = \frac{Y}{(L + l/2)} = \frac{e E_d l}{m v_x^2}$$

If the potential difference between the plates is V_d and these are separated by distance, d then

$$E_d = \frac{V_d}{d}$$

therefore,

$$\frac{Y}{L + l/2} = \frac{e V_d l}{d m v_x^2}$$

$$\text{deflection sensitivity} = \frac{\text{deflection}}{\text{deflecting voltage}}$$

$$\frac{Y}{V_d} = \frac{e(L + l/2) l}{m v_x^2 d}$$

If the electron beam before entering the deflecting plates was accelerated to achieve this velocity v_x by accelerating voltage V_a

$$\frac{1}{2} m v_x^2 = e V_a, \text{ substituting this we get}$$

$$\frac{Y}{V_d} = \frac{e\left(L + \frac{l}{2}\right) l}{2\, eV_a \cdot d}$$

$$\text{deflection sensitivity} = \frac{l}{2d \cdot V_a}\left(L + \frac{l}{2}\right) \qquad \ldots(13.38)$$

Thus the deflection sensitivity is

(1) directly proportional to the length of the deflection plates,
(2) directly proportional to $(L + l/2)$,
(3) inversely proportional to the final accelerating voltage,
(4) inversely proportional to the spacing d between the plates.

The sensitivity is inversely proportional to the accelerating voltage V_a. This implies that by increasing the v_x which is done for increasing the brightness of the spot, the deflection sensitivity decreases.

13.7 MAGNETIC DEFLECTION SENSITIVITY

The magnetic deflection sensitivity is defined as deflection in metres per weber per square metre.

The magnetic field deflects the beam perpendicular to its direction of motion and the electron follows the circular path in the field. The magnitude of the deflecting force eBv from relations $\vec{F} = e(\vec{v} \times \vec{B})$ where B is the magnitude of the magnetic field and v is the velocity of electron, say radius of the circular path is R given by centrifugal force,

$$\frac{mv^2}{R} = eBv$$

$$R = \frac{mv}{eB} \qquad \ldots(13.39)$$

Let the magnetic field B be uniform in a region of length l in the direction of beam and zero outside this region. The electron after leaving the field travels in a straight line tangential to the circular path as shown in Fig. 13.9.

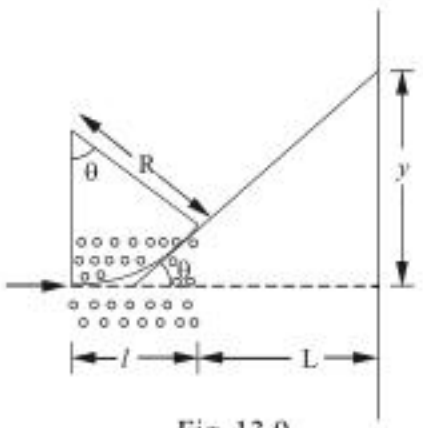


Fig. 13.9

If θ is the angular deflection suffered by beam due to the magnetic field and considering that θ is small

$$\theta = \frac{l}{R} = \frac{Y}{(L + l/2)} \qquad \ldots(13.40)$$

where L is the distance from edge of l to the screen. By eqn. (13.39),

$$\frac{Y}{(L + l/2)} = \frac{l}{R} = \frac{leB}{mv} \qquad \ldots(13.41)$$

$$\text{Deflection sensitivity} = \frac{Y}{B} = \frac{el\,(L + l/2)}{mv}$$

If electron is accelerated to the velocity v by accelerating potential V_a

$$\frac{1}{2} mv^2 = eV_a$$

$$m^2v^2 = 2meV_a$$

$$mv = \sqrt{2meV_a}$$

Hence deflection sensitivity

$$\frac{Y}{B} = \frac{el\,(L + l/2)}{\sqrt{2me\,V_a}}$$

or
$$\frac{Y}{B} = (L + l/2)\sqrt{\frac{e}{m}}\,\frac{l}{\sqrt{2\,V_a}} \qquad \ldots(13.42)$$

Thus deflection sensitivity is proportional to l, $L + l/2$ and $\sqrt{e/m}$ inversely proportional to the square root of the final anode voltage.

The comparison of equations (13.38) and (13.42) shows that the effect of changing the accelerating potential V_a for changing the brightness of the spot will be less, when magnetic field is used for deflection than when electric field is used. *So, the brightness of the spot can be increased with smaller decrease in sensitivity for magnetic C.R.O. than for electric C.R.O.*

13.8 TIME BASE

In many applications of the C.R.O., the movement of the spot in one direction 'generally the horizontal' is required to be proportional to time. The arrangement used to achieve this is called time base. By using time base the base line of any pattern on the screen can be regarded as time axis and the pattern will record the variation of the quantity under study with time.

Time Base Circuit

The simplest time base circuit is shown in Fig. 13.10, a neon lamp is connected to the terminals of condenser C. The condenser is charged through R by D.C. mains and discharged through neon lamp. The condenser is charged until the voltage across it reaches the critical value needed to glow the neon lamp. When the neon lamp glows, it short-circuits the condenser which thus discharges through the lamp causing the rapid fall of the potential and the spot on screen returns to the initial position suddenly. The process is repeated, the new cycle starts again, and the spot traces the same line several times per second and looks like a continuous line. The neon lamp will flash at regular interval depending upon the product RC. The frequency of time base can be calculated assuming that the charging is linear. If V is the voltage of the D.C. mains,

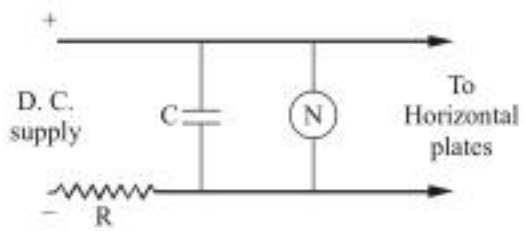


Fig. 13.10

$$i = \frac{V}{R}$$

charge on the condenser $Q = it$, where t is time of the charging. The voltage on the condenser V' which is the same as the striking voltage for the neon lamp

$$V' = \frac{Q}{C}$$

But $$t = \frac{Q}{i} = \frac{V'CR}{V}$$

and the frequency $$f = \frac{1}{t} = \frac{V}{V'CR}$$

f can be made to vary by changing C and R within certain limits.

13.9 USES OF CATHODE RAY OSCILLOSCOPE

Cathode ray oscilloscope is widely used in a number of measurements and few of them are given below:

1. **In television.** In television receiver C.R.O. is used for recreation of image.

2. **In radar.** In radar C.R.O. is used to detect the position of the target such as aircraft and then to give the visual indication.

3. **In industry.** C.R.O. is used for study of mechanical pressures and indicator diagrams of internal combustion engines and moisture contents in a factory.

4. **In medicine and surgery.** C.R.O. is used for electro-cardiography *i.e.* the study of the heart's action.

5. **In engineering.** It is used to detect faults in generators, motors and in locating the faults in radio receivers, transmitters and other radio equipment.

6. **Measurement of Direct or Alternating Voltage.** As we know, the deflection on C.R.O. screen is directly proportional to the voltage applied to the deflecting plates. First the deflection sensitivity is calculated *i.e.* the deflection on the screen is directly calibrated in terms of known voltage. The direct voltage to be measured is applied between a pair of deflecting plates. Measure the deflection produced on the screen when this deflection is multiplied by the deflection sensitivity gives the values of direct voltage.

The alternating voltage to be measured is applied between a pair of deflecting plates and the length of the straight line produced on the screen is measured knowing the deflection sensitivity, peak to peak value of the applied voltage can be calculated.

7. **Measurement of Direct or Alternating Current.** As we know that C.R.O. is a device working on the voltage applied to the deflecting plates, therefore, the current to be measured is first passed through a known resistance and then voltage developed is applied to one pair of the deflection plates. The voltage is calculated as in the previous case 6 from the known deflection on the screen. The current can easily be calculated by Ohm's law.

8. **Study of Waveforms.** The periodic waveforms such as alternating voltage, sound etc. can be traced with C.R.O. Suppose the waveform of voltage oscillations is to be traced, this voltage is applied to the vertical *i.e.* XX plates and a "time base" voltage of suitable frequency and amplitude is applied to the horizontal *i.e.* YY plates. If voltage under study is applied alone, then spot traces along the vertical direction. Similarly, if the time base circuit alone is connected, then it makes the spot sweep across the screen in the horizontal direction at a uniform speed and then come back quickly to the starting point. But under the combined influence of the two, the frequency of time base is adjusted to some exact sub-multiple of frequency of applied a.c. voltage. In other words, two frequencies have been "synchronized". Then a stationary trace is obtained which will give the required waveform as shown in Fig. 13.11.

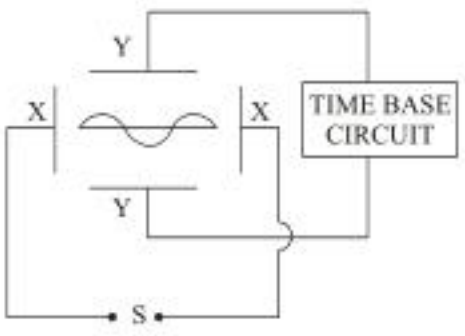


Fig. 13.11

The number of complete a.c. voltage waves will be equal to the ratio of the frequency of a.c. voltage to the frequency of time base voltage.

9. **Measurement of Phase Difference**. For measuring phase difference between two given voltages, one sinusoidal voltage is applied to the XX plates and other to YY plates, a Lissajous figure pattern is obtained on the screen. This pattern is measure of the phase difference between the two voltages. Generally an ellipse is obtained when the two voltages are of equal amplitude and of the

same frequency. The orientation of ellipse with respect to coordinate axes depends upon the phase difference between the two voltages. The phase difference can be calculated with the help of the pattern on the C.R.O. by noting A and B as shown in Fig. 13.12. The phase difference ϕ is given by

$$\phi = \theta_1 - \theta_2 = \sin^{-1}\frac{B}{A}$$

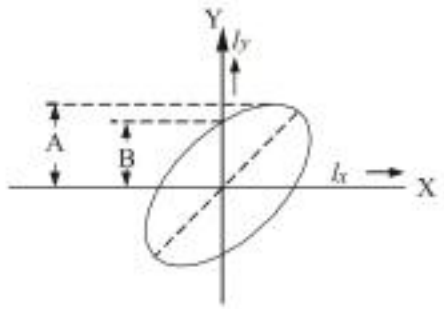


Fig. 13.12

Various cases are shown in Fig. 13.13.

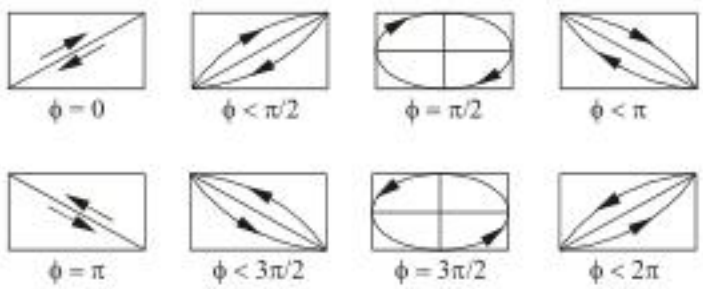


Fig. 13.13

Similarly, the phase difference between the voltage and current in an oscillatory circuit can be studied by applying the current to one (XX) pair and the voltage to other pair (YY) of the deflecting plates. Traces obtained for inductive circuit indicate that current lags behind the voltage by $90°$.

10. Measurement of Frequency. The unknown frequency can be determined by comparison with a known frequency. Suppose we want to determine the frequency of an alternating current. This unknown frequency (current) is applied to one pair of deflecting plates and a known frequency from some variable frequency, standard oscillator is applied to the other deflecting plates. The frequency of standard oscillator is varied until a single loop stationary pattern is obtained. Then the frequency of sinusoidal current is the same as that of the oscillator.

11. **Hysteresis Curve on CRO Screen.** To trace the hysteresis curve, consider the specimen in the form of a ring. A smple circuit is shown in Fig. 13.14.

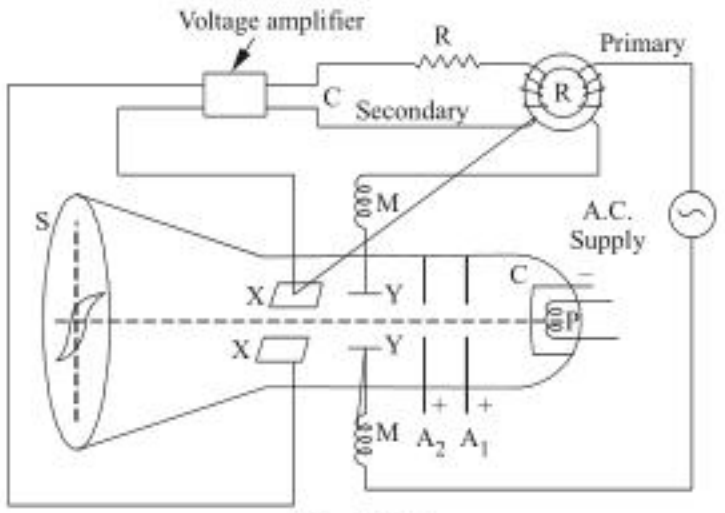


Fig. 13.14

A.C. is passed through the primary of the coil wound over the ring of the specimen. A part of this is fed to the YY plates as the magnetising field. Due to magnetic induction an alternating voltage is produced in the secondary which is connected to XX through a condenser C and a high resistance R. The condenser C helps to keep the induced emf in phase with the magnetising current. The voltage across the condenser is amplified with a value amplifier and fed to the plates XX. The deflection along the X-axis corresponds to the magnetising field H and the deflection along the Y-axis corresponds to the magnetic induction B. A $\vec{B}\cdot\vec{H}$ curve is traced on the screen.

Example 13.16 *A cathode-ray tube with electrostatics deflection has an accelerating voltage, V_a = 1500 volts, length of deflecting plate l = 1 cm. Spacing between the plates (d) = 1 cm and screen is placed at a distance L = 29.5 cm from the end of the deflecting plates. Find the voltage required to deflect the spot by 1 cm on the screen. Assuming that there is no fringing of the field.*

Solution. From eq. (13.38),

$$\frac{V_d}{Y} = \frac{2d\,V_a}{l\,(L + l/2)} = \frac{2\times10^{-2}\times1500}{10^{-2}\times(29.5+0.5)\,10^{-2}}$$

or $$\frac{V_d}{Y} = 10{,}000 \text{ volts/metre}$$

or $$\frac{V_d}{Y} = 100 \text{ volts/cm}$$

The required voltage is 100 volts.

Example 13.17 *A cathode-ray tube with magnetic deflection has an accelerating voltage V = 1500 volts, a magnetic deflecting field is uniform over length l = 2 cm and screen is placed at a distance L = 29 cm from the end of the deflecting field. Find the magnetic field B required to deflect the spot of an electron beam 1 cm on the screen.*

Solution. From eq. (13.42),

$$\frac{B}{Y} = \frac{1}{l\,(L + l/2)}\sqrt{\frac{2m\,V_a}{e}}$$

For an electron on substituting e and m, we get,

$$\frac{B}{Y} = \frac{1}{l\,(L + l/2)}\,3.38\times10^{-6}\sqrt{V_a}$$

or $$\frac{B}{Y} = \frac{3.38\times10^{-6}\times1500^{1/2}}{2\times10^{-2}(29+1)\,10^{-2}}$$

or $$\frac{B}{Y} = 2.18\times10^{-2} \text{ weber/metre}^2\text{/metre}$$

or $$\frac{B}{Y} = 2.18\times10^{-4} \text{ weber/metre}^2\text{/cm}$$

Therefore the required magnetic field is 2.18×10^{-4} weber/metre^2.

13.10 CHARGED PARTICLE IN PARALLEL MAGNETIC AND ELECTRIC FIELDS

Suppose the magnetic field $\vec{B}$ and electric field $\vec{E}$ are along Y-axis and the particle of charge q is moving along z-axis with velocity v as shown in Fig. 13.15, therefore,

$$\vec{B} = B\hat{J},\ \vec{E} = E\hat{J} \text{ and } \vec{v} = -v\hat{k}$$

The force due to magnetic field on charged particle is directed in a direction perpendicular to B and v *i.e.* along x-axis,

the magnitude = qBv

If a' is acceleration then equation of motion is

$$ma' = \frac{md^2x}{dt^2} = qBv$$

or $$a' = \frac{d^2x}{dt^2} = \frac{qBv}{m} \qquad \ldots(13.43)$$

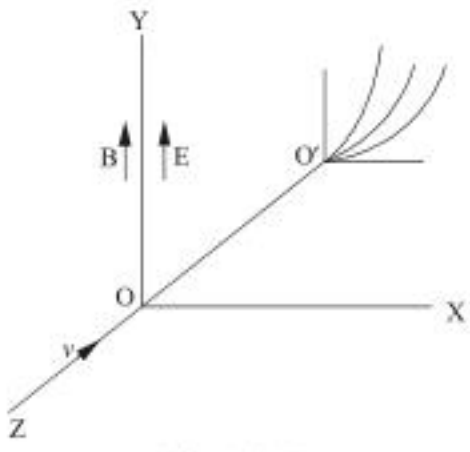


Fig. 13.15

The acceleration may be assumed to be constant if the change in magnitude and direction of the velocity due to the two fields is small. Initially particle did not have any velocity along the x-axis *i.e.* $\frac{dx}{dt} = 0$. Hence velocity attained after time t is given by

$$v_x = \frac{dx}{dt} = a't \qquad \ldots(13.44)$$

and distance travelled in time t, if represented by

$$x = \frac{1}{2}a't^2 \qquad \ldots(13.45)$$

If the magnetic field is applied in a region of length l', then the time of travel in the magnetic field is $t = \frac{l'}{v}$.

Therefore, the velocity of the particle after the emergence from the magnetic field by Eq. (13.44) is

$$v_x = \frac{a'l'}{v}$$

and its displacement x_1 by eq. (13.45),

$$x_1 = \frac{1}{2}a'\left(\frac{l'}{v}\right)^2 \qquad \ldots(13.46)$$

Now putting the value of a' from (13.43) in (13.46)

$$x_1 = \frac{1}{2}\frac{q\mathrm{B}}{mv}l'^2 \qquad \ldots(13.47)$$

If the screen is placed at a distance L' from the point where the magnetic field ends. The time required by the electron to reach the screen is L'/v.

So the electron is further displaced through

$$x_2 = v_x\frac{\mathrm{L}'}{v}$$

$$= a'\frac{l'}{v}\cdot\frac{\mathrm{L}'}{v}$$

$$= \frac{q\mathrm{B}v}{m}\cdot\frac{l'}{v}\cdot\frac{\mathrm{L}'}{v}$$

$$= \frac{q\mathrm{B}}{m}l'\,\mathrm{L}'$$

Total displacement on the screen

$$x = x_1 + x_2$$

$$x = \frac{q\mathrm{B}}{mv}\left(\frac{l'^2}{2} + l'\mathrm{L}'\right) \qquad \ldots(13.48)$$

The electric field is along the Y-axis, acceleration is given by

$$a = \frac{d^2y}{dt^2} = \frac{q\mathrm{E}}{m}$$

(force on a charge q due to field intensity is $q\mathrm{E}$)

We have solved it in article (13.2). And the total displacement is given by equation (13.23), *i.e.*

$$y = y_1 + y_2 = \frac{q\mathrm{E}}{mv^2}\left(\frac{l^2}{2} + l\,\mathrm{L}\right) \qquad \ldots(13.49)$$

where l is the length of the electric field region and L the distance of the screen from the end of the electric field eliminating v from eqns. (13.48) and (13.49),

$$\frac{x^2}{y} = \frac{q}{m}\cdot\frac{\mathrm{B}^2}{\mathrm{E}}\frac{\left(\frac{l'^2}{2} + l'\,\mathrm{L}\right)^2}{\left(\frac{l^2}{2} + l\mathrm{L}\right)}$$

$$= \mathrm{A}$$

$$x^2 = \mathrm{A}y \qquad \ldots(13.50)$$

This represents a parabola. The particles of same q/m but of different velocities will lie on different points of a parabola. Thomson used these parabolae to determine e/m for positive ions.

Now the ratio y/x

$$\frac{y}{x} = \frac{\mathrm{E}}{v\mathrm{B}}\left[\frac{\frac{l^2}{2} + l\,\mathrm{L}}{\frac{l'^2}{2} + l'\,\mathrm{L}}\right]$$

and q/m is eliminated.

$$y = mx$$

This equation represents a straight line. Therefore all the particles with same velocity fall on a straight line passing through the centre of the screen regardless of the different values of q/m.

13.11 POSITIVE RAYS (OR CANAL RAYS) AND CATHODE RAYS

The detailed analysis of the phenomena in the discharge tube at low pressure has brought to light the existence of two important radiations called the cathode and positive rays. Under the influence of the electric field some of the gas molecules in the tube will be ionised, forming positive and negative ions. The process of ionisation expelled the electrons from the neutral molecules. The molecules which have been deprived of their electrons will become positively charged and constitute the positive ions. The electrons are repelled away from the cathode and positive ions accumulated at near the cathode which produces the intense field gradient and large potential gradient there. The accelerated positive ions strike the cathode and cause it to liberate a great number of electrons. These electrons travel some distance accelerated through the strong electric field near the cathode before they acquire sufficient energy to ionize gas molecules by collision. As moving electrons do not emit light by themselves, the space they travel before colliding with molecules will be dark, and this is called the "Crookes dark space". When the pressure is reduced to the order of 0.01 mm the Crookes dark space practically fills the tube producing fluorescence on the walls of the tube. These results appear to be produced by some radiations emitted from the cathode surface and are called "cathode rays". If the cathode of discharge tube is perforated, some rays also pass through the cathode to the opposite side. These are called positive rays (canal rays) because they travel from anode towards cathode *i.e.* opposite to cathode rays as shown in Fig. 13.16. In fact,

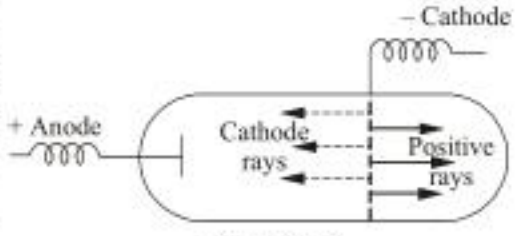


Fig. 13.16

these rays are positive ions in motion produced by the collision of cathode rays with gas atoms between anode and cathode.

The positive ions are attracted by the cathode and pass through it with velocity forming the positive rays. The positive rays are deflected in electric and magnetic field but the deflection is being opposite to that of cathode rays.

13.12 DETERMINATION OF E/M OF POSITIVE RAYS BY J.J. THOMSON METHOD (THOMSON MASS SPECTROGRAPH)

Theory. It consists essentially in subjecting a beam of positive ions simultaneously to parallel electric and magnetic fields of uniform intensities and registering the resultant effect on a photographic plate, *i.e.* one has to examine the motion of positive ions in parallel E and B. We have discussed in the previous article and obtain a relation between the deflection produced in two fields for a charge q of mass m, here we shall assume that charge on ion is e.

$$\frac{x^2}{y} = \frac{e}{m}\frac{\mathrm{B}^2}{m}\frac{\left(\frac{l'^2}{2} + l'\mathrm{L}\right)^2}{\left(\frac{l^2}{2} + l\mathrm{L}\right)} \qquad \ldots(1)$$

From Eq. (13.50) replacing q by charge of ion e.

If the length of the magnetic field region is equal to the length of the electric field region, we have,

$$l' = l$$

Eq. (1) reduces,

$$\frac{x^2}{y} = \frac{e}{m}\frac{\mathrm{B}^2}{m} l\left(\frac{l}{2} + \mathrm{L}\right)$$

or $$\frac{x^2}{y} \propto \frac{e}{m}$$

which represents a parabola when e/m is constant. These parabolae obtained on the photographic plate are shown in Fig. 13.18. From the knowledge of these parabolic traces and constants of apparatus e/m can be calculated.

Apparatus. Apparatus used by Thomson is shown in Fig. 13.17. It consists of a bulb fitted with an anode A (an aluminium disc) and a cathode B. The cathode B is placed in the neck of the flask.

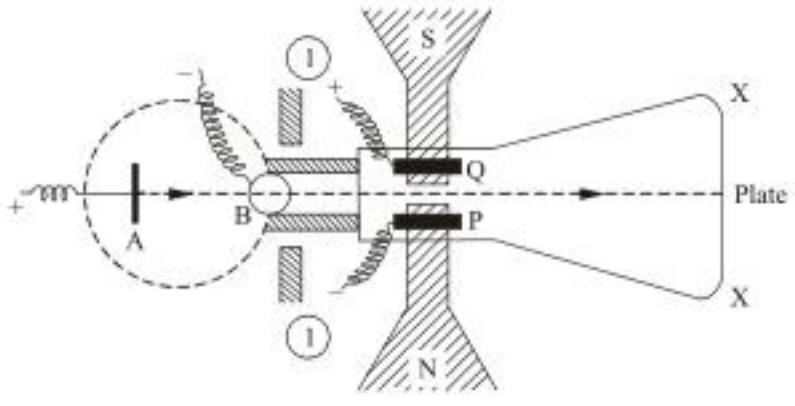


Fig. 13.17

Its face is made of aluminium and is pierced at the centre by a brass tube of narrow bore of about 0.1 mm. A strong electric field is maintained between P and Q and magnetic field by the strong electromagnets of pole pieces N and S. The $\overline{E}$ and $\overline{B}$ are parallel to each other and perpendicular to the direction of the motion of the positive ion. 1, 1 are protecting iron shield. The chamber where deflection takes place is evacuated to prevent the neutralisation of ions by collisions with gas molecules. XX is a photographic plate.

Suppose the particle enters the deflecting system after it has acquired the velocity v_m corresponding to voltage V.

$$\frac{1}{2}mv^2m = \mathrm{V}e$$

The corresponding minimum electrostatics deflection, by eqn. (13.49).

$$y_0 = \frac{\mathrm{E}e}{mv^2m} l\left(\frac{l}{2}+\mathrm{L}\right)$$

$$y_0 = \frac{\mathrm{E}e}{2e\mathrm{V}} l\left(\frac{l}{2}+\mathrm{L}\right)$$

$$y_0 = \frac{1}{2} l\left(\frac{l}{2}+\mathrm{L}\right)\frac{\mathrm{E}}{\mathrm{V}}$$

which is independent of e/m. The all parabolic arcs terminate at a minimum distance y_0 from apex O as shown in Fig. 13.18 and the y_0 is called "cut-off value".

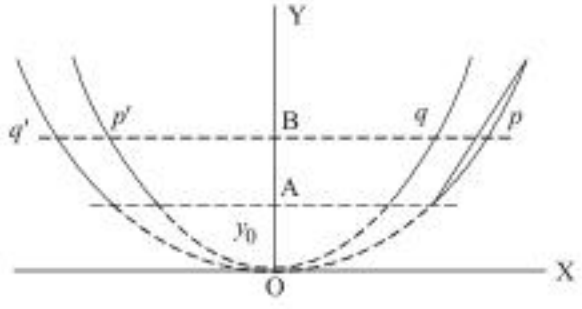


Fig. 13.18

Comparison of e/m by Analysis of Traces

Let the parabola corresponding to ions of mass m_1 be pp' while the ions of mass m_2 but of the same charge be qq', the distances pp' and qq' are corresponding to same value of y (by cutting the parabola for the same value of y).

$$\frac{\frac{e}{m_1}}{\frac{e}{m_2}} = \frac{\frac{x_1^2}{y}}{\frac{x_2^2}{y}}$$

$$\frac{m_2}{m_1} = \frac{x_1^2}{x_2^2} = \frac{(2x_1)^2}{(2x_2)^2} = \frac{(pp')^2}{(qq')^2}$$

$$m_2 = m_1 \frac{(pp')^2}{(qq')^2} \qquad \ldots(13.51)$$

m_2 can be calculated in terms of m_1. Eq. (13.51) represents that masses of the ions are inversely proportional to the square of the distance from the axis of the parabola. If there are number of isotopes present, they will be deposited into various parabolae. The distance from the axis of the parabola is greater for ion of larger mass.

Limitation of the parabola method

It is not an accurate method because of the following reasons:

1. Since positive ions are lost by collisions while passing through the narrow tube in the cathode, so the intensity available for photography is very much reduced. This intensity is further reduced as particles spread out into a parabolic arc.
2. The traces on the photographic plate are blurred with no definite edges and hence not suited for accurate measurements because the wider slits are used to increase the intensity of parabolic arc.
3. The influence of secondary rays makes analysis diffcuilt. Though the pressure inside the tube is as low as possible, is never entirely negligible, the positive ions under test may make collisions and so gain or loss of electrons while passing through the deflecting field. This results in "secondary rays" which may be of different types. Some appear on the plate as a general fog, others as straight beams, others even as parabolae which may be mistaken for the genuine ones.
4. Resolving power is much less than other mass spectrographs discussed below.

13.13 CHARGED PARTICLE IN CROSSED MAGNETIC AND ELECTRIC FIELD (E/M OF ELECTRON BY J.J. THOMSON METHOD)

The force acting on the particle is given by,

$$\vec{F} = q\vec{E} + q\,(\vec{V} \times \vec{B})$$

$$= q\,(\vec{E} + \vec{V} \times \vec{B})$$

The electric and magnetic fields simultaneously applied in a mutually perpendicular direction and are adjusted such that the beam remains undeflected. The arrangement of the apparatus is shown in Fig. 13.19.

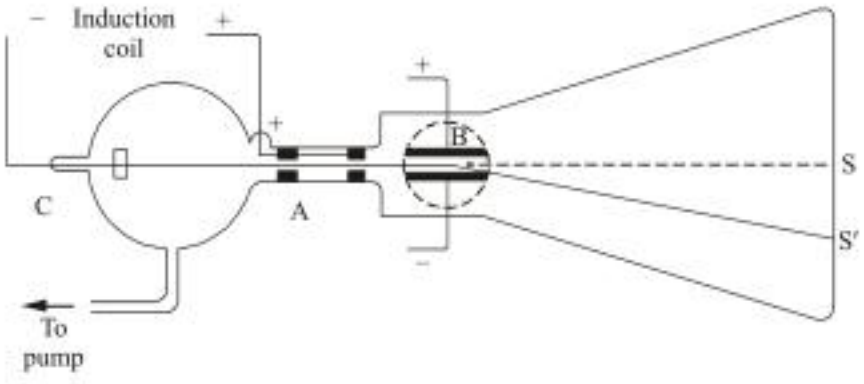


Fig. 13.19

Deflection in the Magnetic Field

Due to magnetic fields, the particle will experience a force B*ev* at right angles to $\vec{v}$ and electron moves in a circular path of radius R as shown in Fig. 13.20. Thus,

$$\mathrm{B}ev = \frac{mv^2}{R}, \text{ or } \mathrm{R} = \frac{mv}{\mathrm{B}e}$$

Here from geometry of the apparatus,

$$\theta = \frac{l}{R} = \frac{y_1}{(L + l/2)}$$

$$y_1 = \frac{l\,(L + l/2)}{R}$$

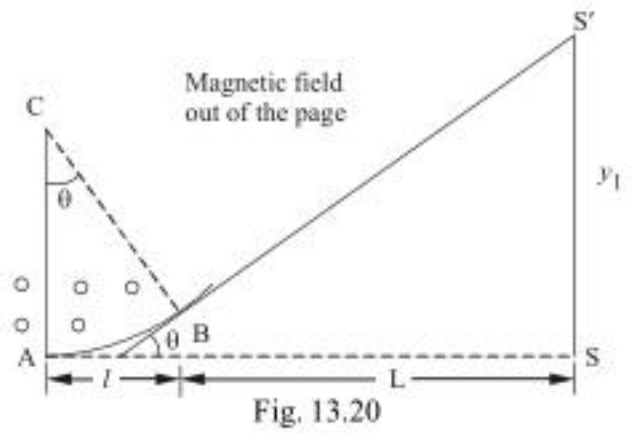


Fig. 13.20

Putting the value of R

$$y_1 = \frac{Be}{mv} l \left(L + \frac{l}{2} \right) \qquad \ldots(13.52)$$

It is a deflection produced by magnetic field in vertical direction.

Deflection produced in Electric Field

The electron on entering the electric field of the intensity E experiences a force F = *e*E. The deflection produced at distance L from the end of deflector plates as shown in Fig. 13.21 has been determined in article 13.2 and is given by eq. (13.21) *i.e.*,

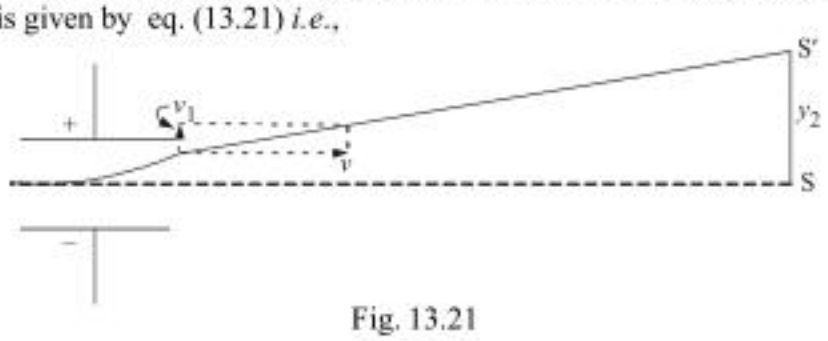


Fig. 13.21

$$y_2 = \frac{e E l^2}{2mv^2} + \frac{e E L l}{mv^2}$$

$$= \frac{e E l}{mv^2} \left(\frac{l}{2} + L \right) \qquad \ldots(13.53)$$

When E and B are perpendicular to each other, *their magnitudes are so adjusted that the deflection produced by the magnetic field is annulled by that produced by electric field i.e.*

$$y_1 = y_2$$

$$\frac{Be}{mv} l \left(\frac{l}{2} + L \right) = \frac{eEl}{mv^2} \left(\frac{l}{2} + L \right)$$

or

$$v = \frac{E}{B} \qquad \ldots(13.54)$$

E is known by dividing the potential difference between the plates by the distance between them (E = V/*d*). The value of B is measured by search coil.

Determination of e/m = substituting the value of *v* from eqn. (13.54) in (13.52)

$$y_1 = \frac{Be}{m}\frac{B}{E}l\left(L+\frac{l}{2}\right)$$

or
$$\frac{e}{m} = \frac{E}{B^2}\frac{1}{l\left(L+\frac{l}{2}\right)}y \qquad \ldots(13.55)$$

In eq. (13.55) all the quantities are known, hence *e/m* can be calculated. The value obtained by J.J. Thomson was,

$$\frac{e}{m} = 1.75\times10^{11}\ \text{coul/kg}$$

Mass of the Electron

The mass of electron has not been determined directly but it can be estimated from the value of *e/m*

$$\frac{e}{m} = 1.75\times10^{11}\ \text{coul/kg}$$

$$e = 1.6\times10^{-19}\ \text{coulomb}$$

$$m = 9.1023\times10^{-31}\ \text{kg}.$$

13.14 ASTON'S MASS SPECTROGRAPH

It is an improvement on Thomson method to increase the accuracy by modifying the principle of operation.

Principle

The main modification introduced is that the electric and magnetic fields are not applied simultaneously and in parallel direction as Thomson did, but successively, first the electric field and then the magnetic field, their directions being at right angles to each other. Therefore, the electric field produces a dispersion of the positive rays with respect to the velocity, while the magnetic field brings to a focus the dispersed rays.

Apparatus and its working

A narrow beam of positive ions is made to pass through two parallel narrow slits A and B (0.01 mm in radius and 20 cm. apart) and a very fine beam is obtained. This beam is subjected to a uniform electric field intensity maintained between the plates C and D. The beam as a whole is deflected towards the negative electrode D as shown in Fig. 13.22. The beam is also dispersed through a certain angle because the beam contains ions of different velocities. The fast moving ions remain for a shorter time under the influence of electric field and will be deflected less than the slower ones taking a longer time to traverse the field. Thus the beam is spread out into an electric spectrum according to different velocities. The divergent beam limited by slit S next enters the magnetic field B ⊥ E and the beam is bent in a direction

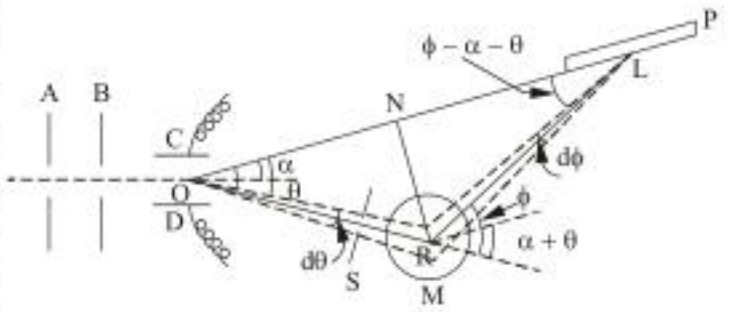


Fig. 13.22

opposite to that caused by E. Again the slowly moving ions are deflected more than the faster ions. Thus the beam can be focused at some point on photographic plate.
If S is the linear displacement of the ions and l the length of their path in the electric field, then from eq. (13.6),

$$S = \frac{eEl^2}{2mv^2}$$

(Force $eE = ma$, $a = eE/m$, initial velocity is zero

$$S = \frac{1}{2}at^2,\ t = \frac{l}{v},\ S = \frac{1}{2}\frac{eE}{m}\frac{l^2}{v^2}\Bigg)$$

Similarly if S' is linear displacement and l' the length of path of the ion in the magnetic field

$$S' = \frac{1}{2}\frac{eB}{mv}l'^2$$

$$\Bigg(\text{Force} = evB = ma',\ a' = \frac{evB}{m},\ S' = \frac{1}{2}a't'^2,\ t' = \frac{l'}{v}$$

$$S' = \frac{1}{2}\frac{evB}{m}\frac{l'^2}{v^2} = \frac{1}{2}\frac{eB}{mv}l'^2\Bigg)$$

Let θ and ϕ be the mean angles of deviation in the electric and magnetic fields respectively,

$$\theta = \frac{S}{l} \qquad \text{and} \qquad \phi = \frac{S'}{l'}$$

$$\theta = \frac{eEl}{2mv^2} \qquad \phi = \frac{eBl'}{2mv}$$

or $$\theta = K_e \frac{e}{mv^2} \qquad \ldots(13.56)$$

and, $$\phi = K_m \frac{e}{mv} \qquad \ldots(13.57)$$

where K_e and K_m are constants, depending on the strength and distribution of electric and magnetic fields.

Let $d\theta$ be the dispersion angle due to electric field and $d\phi$ the convergence angle due to the magnetic field. Differentiating equations (13.56) and (13.57), we get,

$$d\theta = -2K_e \frac{e}{mv^3}dv = -2\theta\frac{dv}{v}$$

or $$\frac{d\theta}{\theta} = -2\frac{dv}{v} \qquad \ldots(13.58)$$

Similarly,

$$d\phi = -K_m \frac{e}{mv^2}dv$$

$$= -\phi\frac{dv}{v}$$

or $$\frac{d\phi}{\phi} = -\frac{dv}{v} \qquad \ldots(13.59)$$

Comparing Eqs. (13.58) and (13.59), we get

$$\frac{d\theta}{\theta} = 2\frac{d\phi}{\phi} \qquad \ldots(13.60)$$

Let OR = a, and RL = b.

The dispersion of the selected group at R is $ad\theta$. In the absence of magnetic field the dispersion produced in the beam for a distance $a + b$

$$= (a+b)\, d\theta$$

As this dispersion (or divergence) is annulled by the magnetic field acting at R so that the group comes to focus at L, distance b from R the condition of focusing is given by,

$$(a+b)\, d\theta = b d\phi$$

$$\frac{a+b}{b} = \frac{d\phi}{d\theta} \qquad \ldots(13.61)$$

$$\frac{a+b}{b} = \frac{\phi}{2\theta}, \text{ from eqn. (13.60)}$$

$$(a+b)\, 2\theta = d\phi$$

$$b(\phi - 2\theta) = 2a\theta \qquad \ldots(13.62)$$

This is the condition for focussing. If plate P is inclined at an angle α to the incident beam AB and RN be the perpendicular on the line joining O and plate P, from triangle RNO

$$RN = a \sin(\alpha + \theta) \qquad \ldots(13.63)$$

and from triangle RNL,

$$RN = b \sin RLN$$

$$RN = b \sin(\phi - \alpha - \theta) \qquad \ldots(13.64)$$

Equating eqns. (13.63) and (13.64),

$$a \sin(\alpha + \theta) = b \sin(\phi - \alpha - \theta)$$

for small angle,

$$a(\alpha + \theta) = b(\phi - \alpha - \theta) \qquad \ldots(13.65)$$

Comparison of eqns. (13.62) and (13.65) shows that two equations are identical if $\alpha = \theta$. *Thus the condition for focusing is that photographic plate must be placed at an angle* θ *with the direction of incident positive ray.*

Separation of Isotopes by Analysis of Traces

Substances of known atomic masses are mixed up with elements whose isotopes are to be determined and traces are obtained on photographic plate. The traces of the known atomic mass serve as reference, thus the traces of unknown isotopes are known by measuring the distance from the reference point.

Advantages of Aston's Mass Spectrograph over Thomson Spectrograph

The modified technique of Aston has the following advantages over Thomson method:

1. The dispersion of the ionic beam according to different velocities of ions is greater than in Thomson mass spectrograph.
2. All the ions having the same e/m are focussed at one point, instead of spreading them into a parabola thus securing greater intensity for photography. Hence a fine slit can be used to obtain sharp lines to be obtained on the photographic plate, thereby securing greater sensitivity. The accuracy was 1 in 1,000.

13.15 180° MAGNETIC FOCUSSING

Let a beam of charged particles enters a region through a slit in which there be a uniform magnetic field $\vec{B}$ perpendicular to the beam. The particles will be deflected with a radius of curvature R given by the relation

$$\frac{mv^2}{R} = qBv$$

$$R = \frac{mv}{qB}$$

As shown in Fig. 13.23, where v is the component of velocity in the plane of the paper *i.e.* normal to B. If we examine the beam after 180° of the motion, we find that it is spread out in the plane of motion because the different particles with different masses and velocities describe the circular path of different radii. Thus we can obtain a beam of particles having closely equal momentum after 180° deflection, provided that the particles have the same charge. Therefore, such device can be used as "*momentum selector*". One advantage of using a deflection of 180° is that particles of equal momenta, but moving through an entrance slit at slightly different angles are brought to an approximately common focus after 180°.

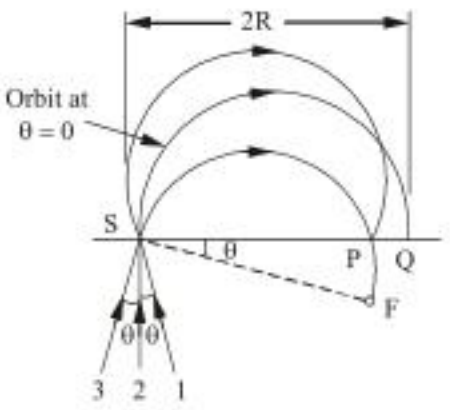


Fig. 13.23

The particle 2 after describing circular path reaches at Q and SQ = 2R, the diameter of the circle. If the particle is making an angle θ to 2, say 1 or 3, then it will reach at P after describing an arc of circle whose diameter is inclined to SQ at an angle θ. The width of the focussed image is,

$$PQ = SQ - SP$$
$$= 2R - 2R\cos\theta$$
$$= 2R(1 - \cos\theta)$$

and $$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4}$$

for small value of θ, then θ^2 be very small hence width PQ is small, 180° magnetic focussing is also a method used in mass spectrographs.

13.16 DEMPSTER MASS SPECTROGRAPH

Dempster in 1918 devised this spectrograph suitable for analysis of substances available in solid form.

Principle. It is based on 180° magnetic focussing method. Therefore, in this method there is only one deflecting agency, namely, the magnetic field, although an electric field is used to accelerate the ions.

Suppose the ions are produced with negligible velocity and these ions are accelerated by the same potential difference and so possessed the same velocity and energy. When a magnetic field $\overline{B}$ is applied at right angles to the direction of motion of these accelerated ions, the ions will describe a circular path.

Let ions acquire velocity v by accelerating potential V,

$$\frac{1}{2}mv^2 = Ve \qquad \ldots(1)$$

R is radius of curvature of circular path is given by,

$$\frac{mv^2}{R} = Bev \qquad \ldots(2)$$

$$\frac{e}{m} = \frac{v}{BR}$$

$$\frac{e}{m} = \frac{1}{BR}\sqrt{\frac{2V_e}{m}}$$

On squaring,

$$\frac{e^2}{m^2} = \frac{1}{B^2 R^2} \cdot \frac{2V_e}{m}$$

$$\frac{e}{m} = \frac{2V}{B^2 R^2} \qquad \ldots(13.66)$$

By keeping B constant and varying V the ions of different atomic masses are made to describe the same circular orbit governed by eqn. (2).

Apparatus

Apparatus is shown in Fig. 13.24. The front surface C of a metal cylinder A is coated with a salt of the element under test. An electrically heated filament *f* emits electrons, by maintaining the filament at a P.D. of 30 to 60 volts with respect to A by means of B_2, the electrons bombard the heated salt and the anode emits the positive ions of the element. These ions are collimated into a fine line by slit S_1 and then accelerated toward slit S_2 by means of variable potential maintained between S_1 and S_2. Then the ions enter the magnetic field region and travel along a semi-circular path towards S_4. Finally the ions fall upon the metal plate P which is connected to a quadrant electrometer. The ions collected at P give rise to an ionic current I.

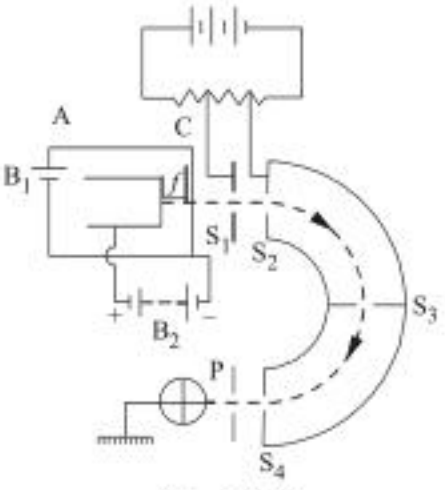


Fig. 13.24

Results

Fig. 13.25 shows the curve for lithium isotopes of masses 39 and 41. The ionic current I reaching the plate P as measured by the electrometer is plotted against V, a curve is obtained showing sharp maxima each of which corresponds to a definite value of *e/m*. Using the value of V corresponding to *peaks* and using the relation (13.66) masses of the ions can be determined. The curve shown above is for lithium. The resolving power is improved by Neir modification is 1 part in 10^5 parts.

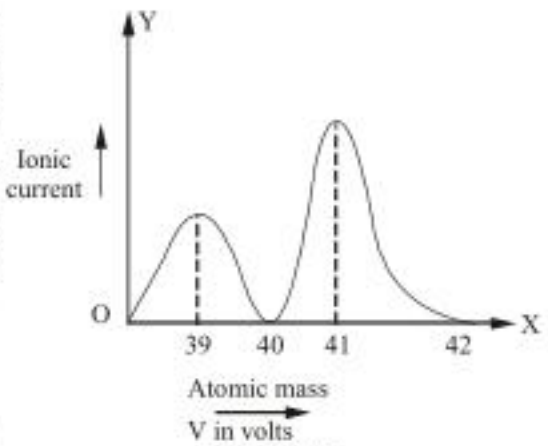


Fig. 13.25

13.17 VELOCITY SELECTOR

Let us consider that a charged particle is moving in crossed electric and magnetic field of intensities $\overline{E}$ and $\overline{B}$ respectively. If $\overline{E}$ and $\overline{B}$ are acting along Y and Z axes respectively, if at any instant, the velocity of the particle, subjected to the two fields simultaneously, be $\vec{v}$, we have $\vec{v} = i\hat{v}_x + \hat{j}v_y + \hat{k}v_z$.

The velocity of charged particles along the *z*-axis remains unaffected in the crossed magnetic and electric fields because neither the magnetic nor the electric field exerts any force on it. The magnetic field can only exert a force on a charged particle if it be moving in a direction perpendicular to the field, whereas the velocity of the particle, v_z, here is in the direction of the field. And the electric field produces no effect on it because it is moving along the *z*-axis *i.e.* in a direction perpendicular to that

of the field which is along the y-axis. The force acting on the particle due to the magnetic field will be because of its velocity v_x and will be equal to qv_x B along the negative direction of the y-axis *i.e.* downward direction and the force acting on the particle due to the electric field E will be qE along the positive direction of the y-axis *i.e.* upward direction. The magnetic field is so adjusted that the force qv_xB in the down direction due to it is balanced by the force in the upward direction and there will be no motion of the particle along the y-axis, *i.e.* $v_y = 0$ if,

$$qv_x \, \mathrm{B} = q \, \mathrm{E}$$

or $$v_x = \frac{\mathrm{E}}{\mathrm{B}}$$

i.e. if its velocity component along the x-axis be equal to the ratio between the electric and the magnetic fields. Since this expression for v_x is independent of mass and charge, this means that it applies equally well to all charged particles irrespective of their masses and the sign and magnitude of the charges they carry.

This system of crossed fields is known as a "velocity selector" or a "velocity filter" because it is used for selecting a beam of electrons or ions having constant velocity v = E/B and filtering out other by suitably adjusting the value of either field.

13.18 BAINBRIDGE'S MASS SPECTROGRAPH

Bainbridge designed a spectrograph based on the principle of velocity selector or velocity filter.

Apparatus

Arrangement is shown in Fig. 13.26. A narrow beam of positive ions enters the slits S_1 and S_2 and is thus collimated. Finally passes through a velocity selector *i.e.* a crossed electric and magnetic field of intensity E and B respectively. E is produced by maintaining the plates P_1 and P_2 at suitable potential difference while B by an electromagnet represented by dotted circle. The only ions having their velocity v = E/B will enter the slit S_3 while the others will be prevented from reaching the slit S_3 since they will be bent away from the rectilinear path on account of the unbalanced effect of one of the two opposing fields. The ions of those emerge from S_3 at the exit selector with the same velocity are introduced into a uniform magnetic field of intensity B′ right angle to the direction of motion of ions.

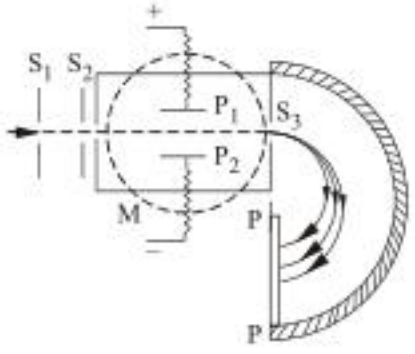


Fig. 13.26

Under the influence of B′ they travel along circular path given by the usual relation,

$$\mathrm{B}' ev = \frac{mv^2}{r}$$

where r is the radius of circular path,

$$\frac{e}{m} = \frac{v}{\mathrm{B}' r} = \frac{\mathrm{E}}{\mathrm{B}' \mathrm{B} r} \qquad \ldots(13.67)$$

The ions fall on the photographic plate and they will strike it at different points depending on different values of e/m. The value of r is obtained from the position of the line on the plate hence the value of e/m is calculated.

Advantages

It has the following advantages:

1. high resolving power than Dempster's mass spectograph
2. greater precision than Dempster's mass spectrograph
3. has symmetrical images and chiefly a linear mass scale which Aston could not get with his apparatus.

13.19 TABLE OF CHARGE AND MASS FOR COMMON PARTICLES

The charge and mass for a number of common particles are listed in Table 13.1. The mass given is the rest mass or mass at zero velocity.

Table 13.1

Particle	*Charge e Coulombs*	*Mass in kg*	*Ratio e/m Coulombs/kg*
Electron	-1.602×10^{-19}	9.107×10^{-31}	-1.76×10^{11}
Positron	$+1.602\times10^{-19}$	9.107×10^{-31}	$+1.76\times10^{11}$
Neutron	0	1.6747×10^{-27}	0
Proton (Hydrogen nucleus)	$+1.602\times10^{-19}$	1.6725×10^{-27}	$+9.6\times10^{7}$
Deuteron (heavy hydrogen nucleus)	$+1.6\times10^{-19}$	3.34×10^{-27}	$+4.8\times10^{7}$
Alpha particle (helium nucleus)	$+3.2\times10^{-19}$	6.644×10^{-27}	$+4.81\times10^{7}$

13.20 MAGNETRON

Valves in which the electrons move in a magnetic as well as in the electric field between the anode and cathode are called magnetron. Usually the electric and magnetic fields are perpendicular, such arrangement is called a plane magnetron. Cylindrical structures are generally employed. A cylindrical magnetron consists of a cylindrical anode along the axis of which is located the cathode.

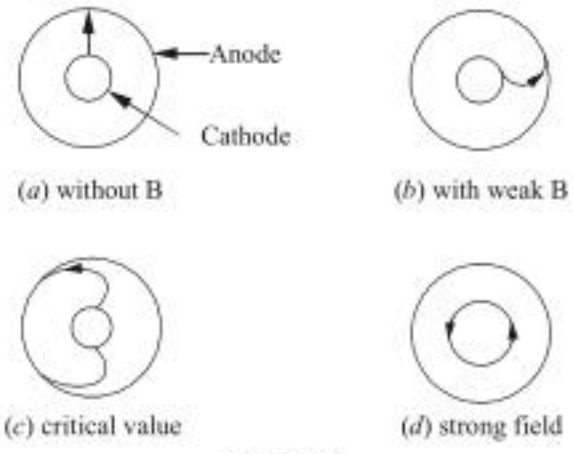


Fig. 13.27

A radial electric field is set up by applying a high d.c. potential between cathode and anode. An axial magnetic field is provided by either permanent magnet or an electromagnet.

When magnetic field is not applied, the electrons will travel only due to electric field from cathode to anode as shown in Fig. 13.27(*a*).

With weak magnetic field a deflecting force acts at right angles to the B and to the direction of motion of the electrons and produce a circular motion as shown in Fig. 13.27 (*b*). That is that electrons travel in a circular arc between anode and cathode, the radius of curvature is given by

$$\frac{mv^2}{\mathrm{R}} = e\mathrm{B}v$$

or $$\mathrm{R} = \frac{mv}{e\mathrm{B}}$$

The radius of curvature R decreases with increasing value of B. For a given value of v that corresponds to a given anode potential, there is a "*critical value of magnetic field strength*" which causes the electrons to just graze the anode as shown in Fig. 13.27 (*c*).

This value of magnetic field strength is known as the "*cut off value*". If B exceeds the cut off value, the curvature of the electron path further decreases in such a way that electron misses the anode and spiral back to the cathode as shown in Fig. 13.27 (*d*)

Anode current *vs*. magnetic field strength graph is shown in Fig. 13.28 (*a*). As the critical value of magnetic field is reached, the anode current suddenly reduces to zero.

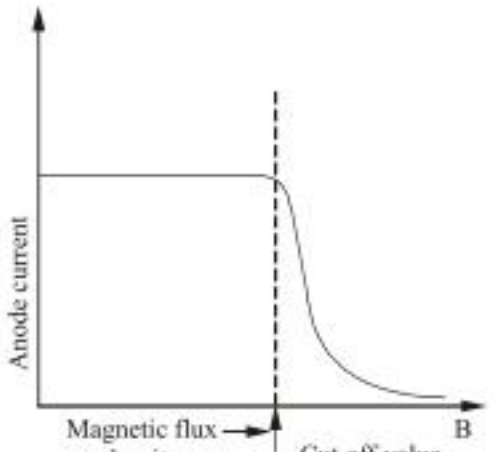


Fig. 13.28 (*a*)

Theory of cylindrical Magnetron (Calculation of Critical value of B)

Suppose the radius of anode be r_a and of cathode be r_k. The electrons are emitted from the surface of the cathode. An electron will experience an electrostatic force due to electric field intensity E as well as the magnetic force due to magnetic field B.

The force due to E is $\mathrm{F} = -e\,\mathrm{E}$ and the force due to B is $f = \mathrm{B}ev$, where v is the velocity of electron.

But, $$\mathrm{E} = -\frac{d\mathrm{V}}{dr}$$

hence $$\mathrm{F} = e\frac{d\mathrm{V}}{dr}$$

and $$f = \mathrm{B}ev$$

The velocity v can be resolved into two components v_r along the radius and v_θ perpendicular to the radius vector r. Therefore, the components of f are given by,

$$\left.\begin{aligned} f_r &= \mathrm{B}ev_\theta \\ f_\theta &= \mathrm{B}ev_r \end{aligned}\right\} \text{ as shown in Fig. 13.28 } (b) \qquad \ldots(1)$$

because magnetic force is $\perp$ to the velocity v by relation $\vec{\mathrm{F}} = e\,(\vec{v} \times \vec{\mathrm{B}})$.

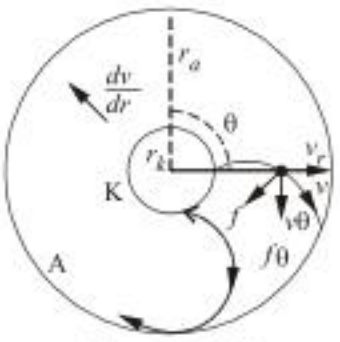


Fig. 13.28 (*b*)

Let ω be the angular velocity and radius vector r makes an angle θ with an arbitrary line in the azimuthal plane, so that,

$$\omega = \frac{d\theta}{dt} = \frac{v_\theta}{r} \text{ since } v_\theta = r\frac{d\theta}{di} \; (v = \omega r)$$

Equation of motion is obtained by Newton's second law.

$$\frac{d}{dt}\left(m\frac{dr}{dt}\right) = \mathrm{F} - f_r \qquad \ldots(2)$$

The equation of motion in the azimuthal plane is obtained by equating the moment of impressed force to the rate of change of angular momentum.

$$rf_\theta = \frac{d}{dt}(mv_\theta \cdot r) \qquad \ldots(3)$$

Substituting f_θ and v_θ from above,

$$\frac{r\,\mathrm{Be}dr}{dt} = \frac{d}{dt}(mr^2\omega)$$

Integrating with respect to t, we have,

$$\int \mathrm{Be}r\frac{dr}{dt}\,dt = \int \frac{d}{dt}(mr^2\omega)\,dt$$

$$\int \mathrm{Be}r dr = \int d\,(mr^2\omega)$$

$$\mathrm{Be}\frac{r^2}{2} = mr^2\omega + \mathrm{C} \qquad \ldots(4)$$

The value of C is determined from initial condition, we know that when electron leaves the cathode *i.e* when $r = r_k$, $\omega = 0$

$$\mathrm{C} = \frac{\mathrm{Be}r_k^{\,2}}{2} \qquad \ldots(5)$$

therefore,

$$\frac{\mathrm{Be}}{2}(r^2 - r_k^{\,2}) = mr^2\omega$$

$$\omega = \frac{\mathrm{Be}}{2m}\left(1 - \frac{r_k^{\,2}}{r^2}\right) \qquad \ldots(6)$$

Now if we ignore the small velocity with which the electron is emitted from the cathode, the velocity v of the electron corresponding to the potential V is given by,

$$\frac{1}{2}mv^2 = e\mathrm{V}$$

$$v = \sqrt{\frac{2e\mathrm{V}}{m}}$$

When the current is just cut off, the electron is grazing the anode, then,

$$\mathrm{V} = \mathrm{V}_a$$

and $v_r = 0$ since there is no radial motion.

$$\text{So,} \quad v_\theta = v = \sqrt{\frac{2eV_a}{m}}$$

$$r = r_a$$

therefore,

$$\omega = \frac{v_\theta}{r_a} = \frac{1}{r_a}\sqrt{\frac{2eV_a}{m}} \qquad \ldots(7)$$

If B_θ is the critical value of magnetic field corresponding to cut off value of anode current, then equating eqns. (6) and (7) replacing B in (6) by B_c and r by r_a,

$$B_c \frac{e}{2m}\left(1-\frac{r_k^2}{r_a^2}\right) = \frac{1}{r_a}\left(\frac{2eV_a}{m}\right)^{1/2}$$

$$B_c = \left(\frac{8V_a}{e/m}\right)^{1/2} \frac{1}{r_a\left(1-\frac{r_k^2}{r_a^2}\right)} \qquad \ldots(8)$$

B_c is not depending on potential distribution. So that the presence of any other electrode or the space charge between the cathode and anode could not affect the cut-off conditions. The magnetron are of two types:

1. Split anode magnetron
2. Cavity magnetron

13.21 β-RAY SPECTROMETER

There are two chief methods for determining the energies of β-particles.

1. The magnetic spectrograph and
2. The Wilson cloud chamber.

The Magnetic Spectrograph Method

The basic principle of this method is based on the fact that when a charge particle enters the magnetic field, then a force $\vec{F} = q \cdot (\vec{v} \times \vec{B})$ acts on it in a direction $\perp$ to $\vec{v}$ and $\vec{B}$ hence the particle describes a circular path. The experimental technique in the case of β -rays are of three types depending upon the technique adopted for collecting the β -particles after they travel through the magnetic field.

1. The photographic plate technique
2. The Geiger tube counter technique
3. The Faraday cylinder technique.

We shall discuss the 1st technique.

The Photographic Plate Technique

In this process the β-rays given off by radioactive substances under study are bent by a constant magnetic field in such a way that they traverse semicircular path and then fall on photographic plate. Their motion is governed by,

$$\frac{mv^2}{R} = Bev \qquad \ldots(1)$$

$$v = \frac{e}{m} BR. \qquad \ldots(2)$$

β-particles of different velocities are thereby focussed at different places on the plate. If the β-particles having velocities v_1, v_2,... they will move in circles of radii, R_1, R_2... governed by general relation (2). By suitably adjusting the position of the plate and the width of the slit S we can obtain 180° magnetic focussing. The β-particles of same velocity but slightly different initial directions being made to strike at the same point as shown in Fig. 13.29.

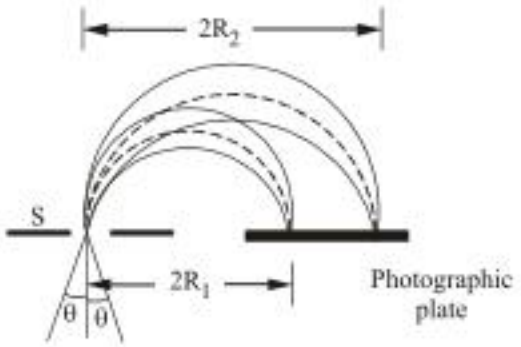


Fig. 13.29

The magnetic intensity used B can be measured. Using these values of R and B and assuming the value of *e/m*, the velocity *v* is calculated from relation (2). Hence, the kinetic energy $= \frac{1}{2}m\left(\frac{e}{3}\mathrm{BR}\right)^2$ is known.

13.22 THE BETATRON

Principle

We have discussed that a time changing magnetic field produces an electric field. When a loop of wire is placed in the changing magnetic field, an emf is induced in the loop that is equal to the line integral of the electric field around the loop (or circuit). Thus according to Faraday's law the induced emf is given by,

$$\xi_{in} = \oint \vec{\mathrm{E}} \cdot d\vec{l} = -\frac{d\phi_{\mathrm{B}}}{dt}$$

by eqn. (11.1),

$$\xi_{in} = \oint \vec{\mathrm{E}} \cdot d\vec{l} = -\int_s \frac{\delta \mathrm{B}}{\delta t} \cdot d\vec{\mathrm{S}} \qquad \ldots(1)$$

When the loop is open this emf appears at the loop terminals. *It is highly significant to note that the induced electric field* $\vec{E}$ *exists whether the wire loop is present or not*. This fact is used for accelerating an electron and the device is called betatron.*

The betatron was developed by Donald W. Kerst about 1940, however, a device of this type having been proposed earlier by Max Steenback.

In a betatron electrons are accelerated by an electric field induced by time-changing magnetic field. The magnetic field is produced by an a.c. electromagnet with iron cone. Electrons are injected between the pole pieces of the magnet and are accelerated during a fraction of one a.c. period.

Let there be a magnetic field of flux density $\vec{\mathrm{B}}$ directed out of the page from a pole piece as shown in Fig. 13.30 (*a*). The time changing $\vec{\mathrm{B}}$ induces an electric field $\vec{\mathrm{E}}$. The force $\vec{\mathrm{F}}$ on an electron due to this field $\vec{\mathrm{E}}$ is given by,

$$\vec{\mathrm{F}} = e\vec{\mathrm{E}} \qquad \ldots(2)$$

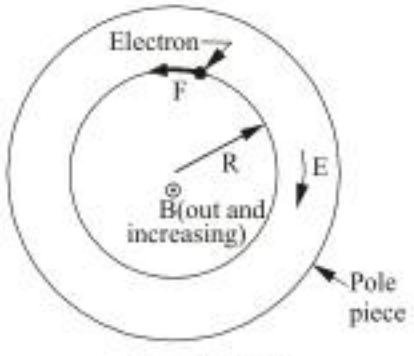


Fig. 13.30 (*a*)

Since the electron is in a magnetic field, it tends to move in a circle. Thus, if $\vec{\mathrm{B}}$ in Fig. 13.30 (*a*) is outward as shown and increasing with time, the electron will rotate counter clockwise.

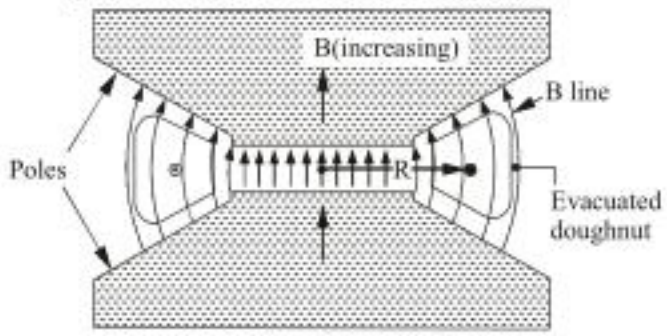
---

*The fact that an electric field is produced in empty space by a changing magnetic field is also illustrated by a travelling electromagnetic wave, which propagates through space because an electric field is produced by changing magnetic field and this electric field in turn produces another magnetic field etc.

The energy per unit charge acquired by an electron in travelling once around a circular orbit of fixed radius R is equal to the total induced emf around the path (by definition).

As $\bar{E}$ is everywhere tangential to the orbit, therefore angle between $\bar{E}$ and $d\bar{l}$ is zero. Thus,

$$\xi_{ind} = \oint \bar{E} \cdot d\bar{l} = E \oint dl$$



(*b*)

Fig. 13.30 (*b*)

$$= E\,2\pi R = -\frac{d\phi_B}{dt}$$

where ϕ_B is total magnetic flux through the electron orbit (inside radius R). Disregarding the sign, the magnitude of the electric field is,

$$E = \frac{1}{2\pi R}\frac{d\phi_B}{dt}$$

Substituting E in eq. (2) we get the magnitude of the force

$$F = eE = \frac{e}{2\pi R}\frac{d\phi_B}{dt}$$

By Newton's second law

$$F = \frac{d}{dt}(mv)$$

Equating these two expressions for force, we get,

$$\frac{d(mv)}{dt} = \frac{e}{2\pi R}\frac{d\phi_B}{dt} \qquad \ldots(13.68)$$

Suppose that the total flux ϕ_B through the electron orbit varies hormonically as given by,

$$\phi_B = \phi_{B_0} \sin \omega t$$

where ϕ_{B_0} = maximum magnetic flux

$$\omega = 2\pi f = \frac{2\pi}{T} = \text{radian frequency}$$

Electrons from a cathode may be injected into the betatron at a radius R at a time $t = 0$ when the flux ϕ_B is zero as shown in Fig. 13.31.

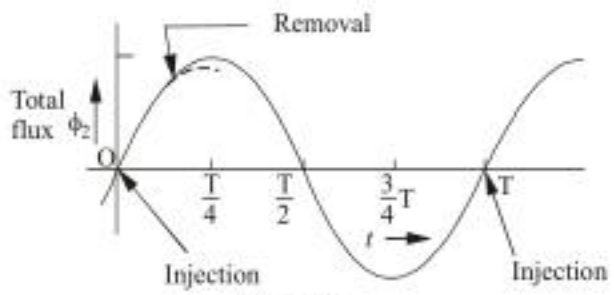


Fig. 13.31

Then, if an electron is removed at some later time t less than one quarter period (T/4), its momentum is obtained by integrating eqn. (13.68). Therefore on integrating eqn. (13.68), we get momentum.

$$mv = \frac{e\phi_B}{2\pi R} \qquad \ldots(13.69)$$

where ϕ_B = total magnetic flux at time t through the electric orbit.

If we neglect the initial momentum of the electron, eqn. (13.69) gives the momentum mv of the electron at some time t after injecting the electron at $t = 0$ (see Fig. 13.31). The velocity of the electron is

$$v = \frac{e\,\phi_B}{2\pi R m} \text{ metre/sec} \qquad \ldots(13.70)$$

However, the radius R of the curved path in which electron moves is given by

$$\frac{mv^2}{R} = evB$$

or
$$R = \frac{mv}{eB} \qquad \ldots(13.71)$$

where B = magnetic field at radius R.

Substituting mv from eqn. (13.69), in eqn. (13.71), we get

$$B = \frac{e\phi_B}{2\pi eR^2} = \frac{\phi_B}{2\pi R^2}$$

But πR^2 = area (A) of electron orbit, therefore,

$$B = \frac{1}{2}\frac{\phi_B}{A}$$

$$= \frac{1}{2} B_{av} \qquad \ldots(13.72)$$

where B_{av} = average magnetic field inside electron orbit.

Both B and B_{av} are instantaneous values. Thus according to Eqn. (13.72), the instantaneous magnetic field B at the electron orbit (Radius R) must be one-half the average magnetic field inside the orbit at the same instant when this condition is fulfilled, the radius R remains constant as the electrons are accelerated. *Therefore, the correct operation of a betatron requires that the average magnetic field inside the orbit increases at twice the rate of the magnetic field at the orbit itself.* In these circumstances, as the energy of the particle is increased by the induced electric field, the magnetic field at the orbit increases at just the rate required to keep the particle moving in a circle.

Construction

The betatron is so constructed that the condition given by eqn. (13.72) is satisfied. To obtain this the poles of the magnet can be given a proper shaping as indicated in the cross-sectional view in Fig. 13.30 (*b*). This cross-section also shows the hollow, ring-shaped glass encloser, or doughnut, inside which the electrons travel.

To utilise the accelerated electrons, they may be made to impinge on a target and produce X-rays. The target is placed near the orbit radius R, and a radial displacement of the orbit is caused by upsetting the 2 to 1 relation of B_{av} to B.

Advantage over Cyclotron

Betatron is free from the difficulty of relativistic variation of mass which prevents the use of the cyclotron for accelerating the electrons. As the betratron operates, the momentum of electron increases in proportionate to magnetic flux ϕ_B (or B_{av}) and if electron is to continue to move in its proper circle, the relation (13.71) must continue to hold and B increases in proportion to B_{av} (or momentum mv) by equation (13.72).

Energy

Energy can be estimated by knowing the average induced emf and total number of revolutions. The flux variation can be given by,

$$\phi_B = \phi_{B_0} \sin \omega t$$

The energy gained by the electrons per turn

$$e\,\xi_{ind} = e\frac{d\phi_B}{dt} = e\omega\phi_{B_0} \cos \omega t$$

The acceleration period is quarter of cycle *i.e.* T/4 where $T = \frac{2\pi}{\omega}$, hence acceleration period is

$$= \frac{\pi}{2\omega} \qquad \ldots(13.73)$$

The average value of the energy gained by electron per turn

$$= \frac{\text{Total energy during acceleration period}}{\text{Acceleration time}}$$

$$= \frac{e\omega\phi_{B_0} \int_0^{\pi/2\omega} \cos \omega t}{\frac{\pi}{2\omega}}$$

$$= \frac{e\omega\phi_{B_0} \left| \frac{1}{\omega} \sin \omega t \right|_0^{\pi/2\omega}}{\frac{\pi}{2\omega}}$$

$$= \frac{e\omega\phi_{B_0} \frac{1}{\omega} \sin \frac{\pi}{2}}{\frac{\pi}{2\omega}}$$

$$= \frac{2}{\pi} e\omega\phi_{B_0}$$

For most of the time electrons travel with a velocity close to the velocity of light. Therefore the distance travelled during the acceleration process

$$= c\frac{\pi}{2\omega}$$

The number of revolutions

$$= \frac{\text{Total distance travelled}}{\text{Distance travelled in one revolution}}$$

If R is radius of the orbit then in one revolution the electron travels a distance $2\pi R$.

Therefore the number of revolutions $= \dfrac{c\pi/2\omega}{2\pi R}$

$$= \frac{c}{4\omega R}$$

Therefore, the total energy gained by electron

= Number of revolutions × mean energy per turn

$$= e\omega\phi_{B_0} \cdot \frac{2}{\pi} \cdot \frac{c}{4\omega R} = \frac{e\,c\,\phi_{B_0}}{2\pi R} \qquad \ldots(13.74)$$

However, the energy can also be estimated by the relativistics equation for energy,

Kinetic energy = pc

$$= Be\,Rc \qquad \ldots(13.75)$$

REVIEW QUESTIONS

13.1 Show that gain in kinetic energy of a charged particle in an electric field is equal to qV where V is the potential difference between its initial and final positions. Define an electron volt.

13.2 Show that the path of charged particle moving with a uniform initial velocity in a constant transverse electric field is a parabola. Find out an expression for the direction of emergence of the particle from the field with initial direction of motion.

13.3 Discuss the motion of charged particle in a uniform magnetic field. Define and explain the terms,

(*i*) Gyro-radius or cyclotron radius

(*ii*) Gyro-frequency or cyclotron frequency.

13.4 Explain clearly the principle underlying the working of a cyclotron. What are its drawbacks and how they have been modified?

13.5 Show that the cyclotron frequency of a given kind of ions is independent of the energy. State the limiting conditions, if any.

13.6 Explain with a diagram the construction and action of a cathode ray oscillograph. Give an account of its practical applications.

13.7 Obtain expressions for deflection sensitivity of an electrostatic CRO and magnetic CRO. Compare the two results.

13.8 (a) What do you mean by time base circuit?

(b) Describe the procedure for measuring phase difference by CRO.

(c) Show how you can study the hysteresis loop of a specimen of ferro-magnetic material with CRO.

13.9 Describe J.J Thomson's method for determining e/m of an electron.

13.10 Draw detailed diagram to illustrate the principle of Thomson's mass spectrograph. What are the limitations of this method?

13.11 Show that a charged particle is moving in crossed electric and magnetic field perpendicular to both the fields can go undeflected if, V = E/B.

13.12 Describe with necessary theory Aston's mass spectrograph. Discuss how it has been used for the detection of isotopes.

13.13 Describe 180° magnetic focussing.

13.14 (a) State the limitations of the Aston's mass spectrograph.

(b) How is the instrument used for identification of isotopes?

(c) What are its advantages over the parabolic method of Thomson?

13.15 Describe Dempster's mass spectrograph and show how it is used to separate the isotopes.

13.16 Give a detailed account of Bainbridge's mass spectrograph.

13.17 What is Bainbridge's velocity selector device?

13.18 Mention different types of mass spectrographs and their merits and demerits.

13.19 Give the principle of a magnetron.

13.20 Give an account of β–ray spectrometer.

13.21 Discuss the principle and working of a Betatron.

13.22 What is necessary condition for the operation of a betatron? Obtain an expression for energy.

PROBLEMS

13.1 Calculate the intensity of electric field which would produce in an electron an acceleration equal to that due to gravity.

13.2 An electron having initial kinetic energy of 10^{-9} ergs at the surface of one of two parallel plate electrodes and moving normal to the surface is slowed down by the retarding field caused by 400 volt potential difference applied between the electrodes.

(*a*) Will the electron reach the second electrode?

(*b*) What retarding potential would be required for the electron to reach the second electrode with zero velocity?

13.3 An electron beam is accelerated through a potential difference of 1000 volts and then passed through a pair of deflecting plates 2 cm long and 0.5 cm apart. What should be the potential difference between the plates in order that the electron beam may be deflected through 5°?

13.4 An electron describes a helix of radius 6 cm and pitch 2.0 cm in a magnetic field of intensity 30 gauss. Obtain the component of its velocity along and perpendicular to the magnetic field.

13.5 An electron with a velocity 10^7 m/sec enters a region of uniform magnetic flux density of 0.10 weber/metre^2, the angle between the direction of the field and the initial path of the electron being 25°. Find the pitch of its helical path (e/m for an electron 1.8×10^{11} coul/kg).

13.6 What will be the energy of (1) a proton, (2) an electron as it emerges from a cyclotron after making 100 revolutions inside it, if the potential difference across the dees be 10^8 volts. Which of these will emerge with a higher velocity?

13.7 A CRO has deflecting plates of length 2 cm and separation 0.5 cm. Calculate the potential difference between the plates which will cause angular deviation of 0.04 radius in an electron beam of speed 8×10^8 cm/sec.

13.8 The uranium isotopes of atomic weight 235 and 238 may be separated by 180° magnetic focussing. Calculate the separation of their beams at the focus, brought in magnetic field of 10^4 gauss if the radius of curvature of the heavier isotopes be 100 cm when (a) the velocities, (b) the energies of the beams are equal.

SELF-TEST OBJECTIVE QUESTIONS

13.1 When a charged particle moves in uniform electric field, the force acting on it is

(*a*) perpendicular to the direction of field

(*b*) along the direction of field.

13.2 If a charged particle moves in uniform magnetic field, the acceleration is maximum when

(*a*) the angle between $\vec{v}$ and $\vec{B}$ is zero (*b*) the angle between $\vec{v}$ and $\vec{B}$ is 45°

(*c*) the angle between $\vec{v}$ and $\vec{B}$ is 90° (*d*) none of these

13.3 The radius of the curvature of the path of a particle moving in magnetic field can be increased by

(*a*) increasing its velocity (*b*) decreasing its velocity

(*c*) increasing the value of $\vec{B}$ (*d*) none of these

13.4 Cyclotron frequency of charged particle in non-relativistic region is independent of its

(*a*) charge (*b*) mass

(*c*) velocity (*d*) none of these

13.5 Two charged particles with same mass and charge move in circular path in a magnetic field perpendicular to their velocities. If $v_a > v_b$, then

(*a*) A takes less time to complete a revolution

(*b*) B takes less time to complete a revolution

(*c*) Both take equal time

(*d*) none of the above

13.6 The CRO uses electromagnetic focussing.

(*a*) True (*b*) False

13.7 An oscilloscope indicates

(*a*) D.C. value of a voltage (*b*) RMS value of A.C. voltage

(*c*) Peak to peak value of a voltage (*d*) none of these

13.8 A cathode ray oscilloscope (CRO)

(*a*) is another name for cathode ray tube (CRT)

(*b*) is different from CRT

(*c*) is a CRT with some additional circuitry like horizontal and vertical deflection amplifiers etc.

(*d*) none of these

13.9 In CRO by changing time base, we change

(*a*) amplitude of sawtooth voltage (*b*) frequency of saw tooth

(*c*) gain of the waveform (*d*) change the frequency of vertical amplitude

13.10 Betatron is a device for accelerating

(*a*) Proton (*b*) Neutron

(*c*) Electron (*d*) none of these

13.11 The correct operation of a betatron requires that the average magnetic field inside the orbit increases at ..... the rate of the magnetic field at the orbit itself.

(*a*) half (*b*) same

(*c*) twice (*d*) thrice

ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*c*)	**5.** (*c*)	**6.** (*b*)
7. (*c*)	**8.** (*c*)	**9.** (*b*)	**10.** (*c*)	**11.** (*c*)	

CHAPTER 14

MAGNETIC PROPERTIES OF MATTER

In Chapter 10, we have discussed techniques for finding the magnetic induction $\vec{B}$ due to distribution of currents. Thus, for example, if we are dealing with a current-carrying circuit consisting of a closed loop of wire, the magnetic field in the vacuum region surrounding the wire may be calculated with the aid of Biot-Savart law or Ampere's law. Now, let the region surrounding the wire be filled with a material medium. Will the magnetic induction be altered by the presence of the matter? The answer is yes; in this chapter we shall discuss the cause of this alteration in the value of $\vec{B}$.

To be more specific, suppose that we put a sample of material into a magnetic field. How is the field changed and what are the field equations? How do we describe what happens to the material? What new phenomena arise that can be measured and used to learn something about the material's microscopic structure? These questions are discussed in this chapter.

A study of magnetic materials involves several problems. First, the essential difference between a magnetic substance, such as iron, and a substance which shows very weak magnetic effects (called non-magnetics), such as wood, should be described. Second, the precise meaning of magnetisation must be explained. Finally, our previously stated laws of physics will need modification in order to take into account the magnetisation of materials present in the field.

14.00 MAGNETISATION (AN ELEMENTARY DISCUSSION)

We shall discuss magnetisation in the next section in detail; here is an elementary view. If a specimen is placed in a magnetic field, the field in the vicinity of the sample is changed. How much it is changed depends on the material and on the shape of the sample, but for most specimens it is not altered very much–typically only by about one part in 10^5. For certain materials, however, the field at some points near by can be increased by a factor of one hundred or so over its value in the absence of the specimen. Such materials are called ferromagnetic. Non-ferromagnetic materials, which produce only a small change in field, can be either diamagnetic or paramagnetic. If the field is decreased very slightly then the material is diamagnetic. If the field is increased slightly then the material is paramagnetic.

The most easily observed characteristics of the different materials are their behaviours in strong non-uniform fields. A ferromagnetic material is strongly pulled into the field. A diamagnetic material is repelled towards the field free regions, but so weakly that sensitive apparatus is required to measure the repulsive force A paramagnetic material is pulled into the field, although the attraction is usually weak. This attraction or repulsion can be explained in terms of the interaction of magnetic dipole moments with the field in which the materials are placed. Certain atoms or molecules may have permanent magnetic dipole moments. If they do have permanent moments, the material is paramagnetic, otherwise it is diamagnetic.

If paramagnetic molecules (or materials) are placed in a magnetic field, there is a net alignment of the dipoles in the field direction and the substance becomes magnetised. This is exactly analogous to the way in which polar molecules on average acquire an effective electric dipole moment in the direction of an applied electric field as discussed in Chapter 6. In a non-uniform magnetic field a paramagnetic material will thus experience a force in the direction of increasing field.

All atoms and molecules acquire an induced magnetic dipole moment when placed in a magnetic field. Induced magnetic dipole moments, unlike induced electric dipole moments, are in a direction opposite to the field in the specimen. Thus, if the molecules of a substance have no permanent moments *i.e.* the substance is diamagnetic, a sample inserted into a non-uniform magnetic field will experience a force in the direction of decreasing field strength.

Clearly, the process of alignment of the dipoles as well as the phenomena by which atoms or molecules acquire an induced magnetic dipole moment when a substance is placed in a magnetic field is known as magnetisation. Thus, any process by which a substance acquires (or develops) a net magnetic moment is magnetisation.

If we want to know the origin of diamagnetism and paramagnetism we must know the origin of magnetic dipole moment of an atom or molecule. There are two chief contributions to the magnetic dipole moment of an atom or molecule. The first arises from the orbital motions of the electrons, which are equivalent to current loops. The second arises from the spinning motions of the electrons. The magnetic dipole moments are closely related to the angular moments of the electrons; the orbital moment to the orbital angular momentum and the intrinsic moment (because of spin) to an intrinsic angular momentum possessed by electrons. These relations are discussed just below.

14.1 ATOMIC CURRENTS OR AMPERIAN CURRENTS

A careful study of the various magnetic phenomena associated with matter indicates that magnetism is an atomic or molecular property and that there is no intrinsic magnetic substance. In the light of this, the hypothesis is adopted that all magnetic properties of matter can be explained in terms of the motions of the electrons and positive charges associated with its constituent atoms and molecules.

All matter is made of atoms each of which consists of electrons those move in the orbits around the nucleus and rotate or spin about their own axes. These circulating electrons constitute a current without a charge transport, we call this as "Atomic currents" and is different from true-current which corresponds to charge transport. Such circulating currents were first postulated by Ampere, and are often called Amperian currents.

14.2 ABSENCE OF MONO-POLE (OR GAUSS'S LAW OF MAGNETISM)

The relation div $\vec{B} = 0$ (everywhere) is mathematically representing the fact that lines of $\vec{B}$ are closed on themselves, whenever the divergence of vector point function is zero, it means no sources or sinks exist in the region over which the divergence is zero.

In the same way Gauss's law is a formal way of stating a fundamental fact of magnetism, *i.e.* the isolated magnetic pole does not exist, the law (we have already discussed in sec 9.7) states that the magnetic flux through any closed surface is zero, *i.e.*

$$\phi_B = \oint \vec{B} \cdot d\vec{S} = 0 \qquad \ldots(A)$$

where integral is to be taken for entire surface. In contrast for electric field, we have

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{S} = q \quad \text{or} \quad \text{div}\, \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \ldots(B)$$

By the comparison of the two equations (A) and (B) it is evident that in magnetism there is no counter-part to the free charge q.

14.3 MAGNETIC MOMENT AND ANGULAR MOMENTUM (BOHRMAGNETON)

An electron is circulating in a circular orbit or radius r is equivalent to current loop with the current flowing in a direction opposite to v (the velocity of electrons). Let us examine this. An electron is moving in a orbit of radius r with velocity v as shown in Fig. 14.1.

The motion of electron in orbit is governed by Bohr's rule. The force between an electron in the orbit of radius r and the nucleus of hydrogen atom is given by Coulomb's law.

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$

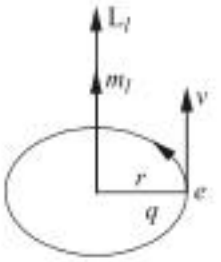


Fig. 14.1

and electron is bound by this force with nucleus, but it moves in a circular orbit of radius r. The centripetal force $F_e = \frac{m_e v^2}{r}$ holding the electron in an orbit r from the nucleus is provided by the electrostatic force $F_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$ between them, and the condition for orbit stability is $F_c = F_e$ *i.e.*,

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = \frac{m_e v^2}{r} \qquad \ldots(a)$$

or $$v = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e r}}$$

The angular velocity ω is given by

$$\omega = \frac{v}{r} = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e r^3}} \qquad \ldots(b)$$

The orbital angular momentum is

$$L_l = m_e v r$$

Substituting v from eqn. (a) we get,

$$L_l = \sqrt{\frac{e^2 m_e r}{4\pi\varepsilon_0}} \qquad \ldots(14.1)$$

Its direction is determined by cross-product rule, as $\vec{L}_l = \vec{r} \times \vec{p}$, where $\vec{p} = m\vec{v}$. Thus L_l is perpendicular to $\vec{r}$ and $\vec{v}$, if $\vec{r}$ is along x-axis, $\vec{v}$ is along y-axis then L_l is along z-axis. L_l is directed perpendicular to the plane of the orbit as shown upward.

Now the rotating electrons constitute the current, and the current is determined by the rate at which the charge passes the given point. If v is the velocity of the electron, it will make $\frac{v}{2\pi r}$ revolution per second and is equivalent to a current given by

$$i = \frac{ev}{2\pi r}$$

$$= e\frac{\omega}{2\pi}, \text{ substituting } \omega \text{ from above Eqn. } (b)$$

$$= \frac{e}{2\pi}\sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e r^3}}$$

$$= \sqrt{\frac{e^4}{16\pi^3\varepsilon_0 m_e r^3}}$$

The close orbit will behave as a magnetic dipole and its magnetic moment is given by

$$m_l = i\,\mathrm{A}$$

$$= \sqrt{\frac{e^4}{16\pi^3\,\varepsilon_0 m_e r^3}}\cdot \pi r^2$$

$$= \frac{e^2}{4}\sqrt{\frac{r}{\pi\varepsilon_0 m_e}} \qquad ...(14.2)$$

Its direction is given by right hand curl rule. Let the fingers of the RH curl around the loop in the direction of the current, the extended right thumb will then point in the direction of $\vec{m}_l$. However, by relation $j = nev$ and $i = j/A$ the direction of current is in the direction of v.

Thus, m_l is also directed perpendicular to the plane of the orbit, so L_l and m_l are in the same direction. The ratio

$$\frac{m_l}{\mathrm{L}_l} = \frac{e^2}{4}\frac{\sqrt{r}}{\sqrt{\frac{e^2 m_e r}{4\pi\varepsilon_0}}} \qquad \text{(substituting from eqns. 14.1 and 14.2)}$$

$$= \frac{e}{2}m_e \qquad ...(14.3)$$

which shows that the orbital magnetic moment of an electron is proportional to its orbital angular momentum.

14.4 ORBITAL GYROMAGNETIC RATIO

The ratio of the magnetic dipole moment of the electron due to its orbital motion and the angular momentum of the orbital motion is called orbital gyromagnetic ratio, or sometimes orbital magneto mechanical ratio of an electron. Thus,

$$\frac{\text{Magnetic moment}}{\text{Angular momentum}} = \frac{e}{2m_e}$$

From Bohr's postulates, the electron chooses such orbits, for which the angular momentum L_l is an integral multiple of $h/2\pi$ where h is Planck's constant *i.e.*,

$$m_l = \frac{nh}{2\pi}\frac{e}{2m_e}$$

$$= n\left(\frac{eh}{4\pi m_e}\right) \text{ where } n = 1, 2, 3, ... \text{ etc.} \qquad ...(14.4)$$

All the quantities in $\frac{eh}{4\pi m_e}$ are constants and together form a unit of magnetic moment called the Bohr magneton, *i.e.*,

$$\text{Bohr magneton} = \frac{eh}{4\pi m_e} \qquad ...(14.5)$$

Substituting e, h and m, we get,

$$\text{Bohr magneton} = 9.1\times10^{-24} \text{ amp.m}^2$$

And for first orbit $n = 1$, from equation (14.4), the magnetic moment is one Bohr magneton.

14.5 ELECTRON SPIN

S.A. Goudsmith and G.E. Uhlenbeck proposed in 1925 that an electron while moving in circular orbit round the nucleus also rotates about its own axis just as the earth rotates about its axis. In other words, the electron has a spin motion over and above the orbital motion. The rotation involves angular momentum, and because the electron is negatively charged it has a magnetic moment m_s opposite in direction to its angular momentum vector L_s as shown in Fig. 14.2.

With the idea of spin the electron possesses an intrinsic angular momentum independent of any orbital angular mometum it might have and associated with this angular momentum is a certain magnetic moment m_s. The detailed theory which is purely quantum mechanical is beyond the scope here, however, the ratio of magnetic moment m_s to the angular momentum L_s for the electron spin is twice as that for its orbital motion, *i.e.*,

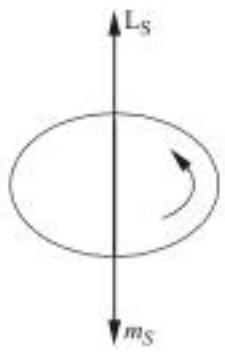


Fig. 14.2

$$\frac{m_s}{L_s} = \frac{e}{m_e}, \qquad \text{[from equation (14.3)]} \qquad ...(14.6)$$

The magnitude of L_s always comes out to be $\frac{1}{2}\left(\frac{h}{2\pi}\right)$ where h is Planck's constant.

In an atom there are several electrons. Each one has angular momentum as well as magnetic moment for both orbital and spin motion. Thus, an atom has a total angular momentum and a total magnetic moment. For an atom as a whole the ratio of magnetic moment to angular momentum on the basis of quantum physics is e/m, or $e/2m$, or in between these two because there is a combination of the contributions from the orbits and spins,

$$m_A = g\left(\frac{e}{2m_e}\right)L \qquad ...(14.7)$$

where m_A is total magnetic moment and L is total angular momentum and g is called Lande g-factor which is the characteristic of the atom. The value of g is one for a pure orbital moment, two for a pure spin moment and for an atom g lies between one and two because electrons in atoms have both orbital and spin motion.

In nuclei there are protons and neutrons and they also move around in orbits and spin like electrons. The magnetic moment is parallel to the angular momentum. Therefore, for nuclei, we have,

$$m_{\text{nuclei}} = g\left(\frac{q_e}{2m_p}\right)L \qquad ...(14.8)$$

where q_e is charge, m_p is mass of proton and g is called the nuclear-g factor, g is a number near one to be determined for each nucleus. However, the spin magnetic moment of the proton does not have a g factor of two, *i.e.*, the value for the electron. For proton $g = 2.79$. The neutron although neutral also has a spin magnetic moment. The neutron is not exactly neutron in the magnetic sense.

14.6 THE ATOMIC MODEL AND MAGNETISM

An electron revolving in its orbit around the nucleus of an atom forms a tiny electric current loop. Since a current loop has a magnetic field and *all atoms have revolving electrons, we might suppose that all substances would exhibit magnetic effects*. It is reasonable to imagine that magnetic field associated with a piece of iron is due to tiny electric currents within the iron itself. Such electric currents are known as Amperian currents. It can be shown that "magnetic moment" of a current loop is $m = Ia$ where a is the area of the orbit. This is often referred to as the "orbital magnetic moment".

Another feature of the atomic model, which is based on experimental and theoretical results, is that the electron in the atom also rotates about an axis through the electron (for detail see art. 14.5--electron spin). The electron spin is considered as the equivalent of a tiny current (amperian current), consequently there is an associated magnetic moment with the electron spin. This is referred to as the "spin magnetic moment".

The resultant magnetic moment of the atom is obtained by the proper vector summation of the orbital and spin magnetic moments. These are combined according to quantum theory rules, which will not be discussed here. It is important to note, however, that the *magnetic properties of bulk matter can be described in terms of the orbital and spin motions of the atoms which make up the matter.*

Classification of Magnetic Material

In many atoms the electrons are completely paired, that is, for each electron spinning in one direction, there is an electron spinning in the opposite direction. And the same situation exists in regard to the orbital motion of the electrons. Thus, the net current circulating about any given axis is zero and there are no Amperian current in such substances, consequently the resultant magnetic moment of the atom is also zero. *Such substances show very "weak magnetic effects" that they are called "non-magnetic". However, vacuum is the only truly non-magnetic medium.*

All materials show some magnetic effects with the exception of the ferromagnetic group these effects are weak. It has been established both theoretically and experimentally that all matter may be classified into three groups, according to its fundamentally different behaviour under the action of a magnetic field. These are classified as (1) diamagnetic, (2) paramagnetic and (3) ferromagnetic.

1. Diamagnetic Substances

Diamagnetic substances are those which, when placed in a magnetic field, *become weakly magnetised. The resultant magnetisation is opposed to the applied field*. The magnetism which is thus established, in general, varies directly with the impressed field and is independent of the absolute temperature. A diamagnetic substance contains no dipoles, but only dipoles that are induced by an external field. We shall discuss in detail this effect in article 14.22.

2. Paramagnetic Substances

Paramagnetic substances are those which, when placed in a magnetic field, *become weakly magnetised in the direction of the applied field*. The magnetism thus established, in general, is proportional to the impressed field and varies inversely with the absolue temperature. A paramagnetic substance contains permanent dipoles. For further detailed discussion see Art. 14.22. When paramagnetism exists, it is usually great enough to mask the diamagnetism which is always present, and which has the opposite sign.

3. Ferromagnetic Substances

Ferromagnetic substances are those which, when placed in a magnetic field, *become strongly magnetised in the direction of the applied field,* and which also exhibit the phenomena of retentivity and hysteresis. The magnetism which is thus established, in general, is proportional to the applied field and varies with the quantity $T-\theta$, where T is the absolute temperature and θ is the critical temperature and is characteristic of the material. The variation is known as Curie Weiss law. *Ferromagnetic materials are sometimes classed as strongly paramagnetic.* All ferromagnetic substances are paramagnetic substances but all paramagnetic substances are not ferromagnetic substances. The ferromagnetism disappears if the temperature is increased above a critical value called Curie temperature and substance becomes paramagnetic. For further details see Art. 14.14.

14.7 MAGNETISATION

When a magnetic material is placed in a magnetic field, the atomic currents due to uncompensated spinning electrons are aligned and the process is called magnetisation. To be more specific, since circulating electrons is equivalent to a circular coil which in turn is equivalent to a small magnet, the alignment of atomic currents is same as alignment of small magnets because it is sufficient to describe them by their magnetic moment which can be expressed either as $Q_m l$ or IA, (For detail see equivalence of circular coil to a magnet). *For simplicity we define the magnetisation as the alignment of elementary (or atomic) magnetic dipoles.*

Uniform Magnetisation

Magnetisation of any substance is uniform if all the atomic magnetic dipoles are uniformly distributed throughout the volume of the substance and are oriented in the same direction as the magetising field. Thus, magnetisation is constant throughout the material. Magnetisation is uniform if the substance is *homogeneous and isotropic.*

Magnetisation Vector $\vec{M}$

If a specimen is placed in a magnetic field then by the process of magnetisation the atoms or molecules are magnetised on the atomic scale. Precisely, the magnetisation is the interaction of magnetic dipole moments with the field in which the materials are placed. This interaction (*i.e.* the effect of magnetising field on the magnetic dipoles) can conveniently be described by a quantity called magnetisation vector $\vec{M}$. The microscopic magnetisation can vary from atom to atom but can be averaged over volume which contain very many atoms to give a function which is smoothly varying with position on a macroscopic scale. This function is called the magnetisation of the medium and is represented by symbol $\vec{M}$. It is defined as the magnetic dipole moment per unit volume, and is given by

$$\vec{M} = \frac{\text{Magnetic dipole moment}}{\text{volume}} \qquad \ldots(14.9)$$

Magnetisation at a point

The value of $\vec{M}$ in above equation (14.9) is an average for the volume. If magnetisation is uniform, the substance has a continuous distribution of infinitesimal atomic magnetic dipoles and the dipoles are of discrete, finite size. Then, for continuous magnetisation, the value of $\vec{M}$ at a point can be defined as the net dipole moment $\Sigma\vec{m}$ of a small volume Δv divided by the volume with the limit taken as Δv shrinks to zero around the point, thus,

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\Sigma \vec{m}}{\Delta v} \qquad \ldots(14.10)$$

If the substance is not magnetised, the individual moment $\vec{m}$ will be oriented at random and the vector sum $\Sigma\vec{m}$ will be practically zero. In this case $\vec{M}$ is zero.

Units of $\vec{M}$. Dimensionally eqn. (14.10) has the form,

$$\text{Magnetisation} = \frac{\text{Magnetic dipole moment}}{\text{volume}}$$

$$= \frac{\text{Current} \times \text{Area}}{\text{volume}}$$

$$= \frac{\text{Amp. metre}^2}{\text{metre}^2} = \frac{\text{Amp.}}{\text{metre}}$$

Thus, the RMKS units of magnetisation is ampere per metre.

The currents equivalent to a magnetised body

In this section we will discuss how a magnetised body can be replaced by currents in order to determine the magnetic field arising from the magnetisation. Let us first estimate the equivalent current in a simple example *i.e* a bar magnet.

Consider a bar magnet in which the magnetisation is everywhere in the direction of the positive z-axis. Let us consider the volume Δv to be a small rectangular box as shown in Fig. 14.3. Since all the atomic currents within the box may be imagined to flow around the area $\Delta x\, \Delta y$, a resultant atomic current (or fictitious current also amperian current) I_m may be considered to flow in one direction over a face. The magnetisation $\vec{M}$ is given at a point by,

$$M = \frac{I_m}{\Delta x} \frac{\Delta x}{\Delta y} \frac{\Delta y}{\Delta z} = \frac{I_m}{\Delta z} = I' \qquad \ldots(14.11)$$

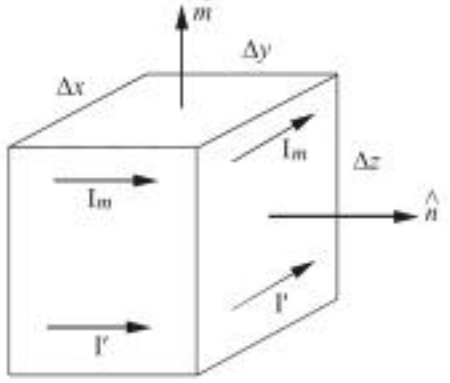


Fig. 14.3

Since magnetic dipole moment = current × area

where I' is the current flowing per unit length over the sides of the volume. Thus we must assume a surface current (I') at the surface of the magnetised volume numerically equal to the magnetisation. We can indicate the direction as well as the magnitude of this surface current by the vector equation,

$$\text{Surface current } I' = \frac{I_m}{\Delta z} = \vec{M} \times \hat{n} \qquad \ldots(14.12)$$

where $\hat{n}$ is the out-drawn normal to the surface. We call I' also as the *linear current density*.

Non-uniform Magnetisation

In this case all the atomic magnetic dipoles are distributed in a non-uniform manner. *The magnetisation varies from point to point in the material.* $\vec{M}$ is a function of position in a non-uniformly magnetised material. Then the total magnetic moment,

$$\Sigma \vec{m} = \int_v \vec{m}\, dv \text{ amp. metre}^2 \qquad \ldots(14.13)$$

where integration is carried out over the volume of the substance. Non-uniform magnetisation occurs when substances are *inhomogeneous and non-isotropic*. If at two points magnetisation is $\overline{M}$ and $\overline{M} + \Delta\overline{M}$, then

$$\text{Surface current} = \Delta\overline{M} \times \hat{n} \qquad \ldots(14.14)$$

where $\hat{n}$ is normal to the surface.

If the magnetisation is inhomogenous *i.e.* magnetisation varies from point to point in the material, there will not be the complete cancellation of the circulating currents between the boundaries of the magnetised material and there will be a net current flowing throughout the volume of the material.

Let us find the resultant current density. Consider two adjacent rectangular current loops 1 and 2 in a magnetised material. Since magnetisation is not uniform, it is different for the two loops. If $\overline{M}$ is magnetisation vector, *i.e.* magnetic moment per unit volume. The moment $(\overline{m})$ of a single loop in Fig. 14.4 is given by,

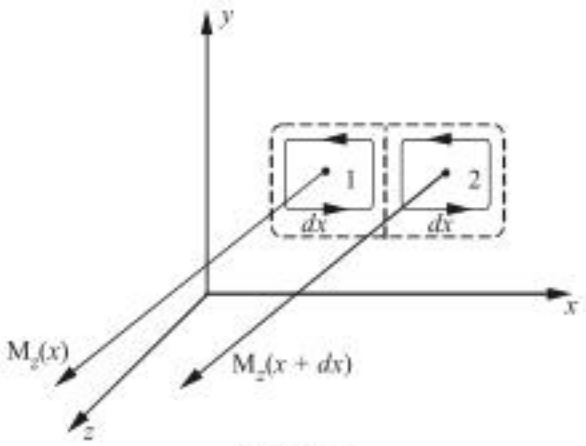


Fig. 14.4

$$\overline{m} = \overline{M}\, dx\, dy\, dz$$

where $dx\, dy\, dz$ is the volume of the loop.

Suppose rectangle has a magnetisation Mz in the z-direction. But we know that

Magnetic moment = current × area (IA)

Therefore, the current (amperian or magnetising) in the rectangle 1 for the considered case if denoted by I_{m1} is given by,

$$I_{m1} = \frac{M_z\, dx\, dy\, dz}{dx\, dy}$$

where $dx\, dy$ is the area of the coil.

If magnetisation is changing at the rate of $\frac{\delta M_z}{\delta x}$ along x-axis, then magnetisation at a point where second rectangle is situated is

$$= M_z + \left(\frac{\delta M_z}{\delta x}\right) dx$$

Therefore the current in the rectangle two is

$$I_{m2} = \left[M_z + \left(\frac{\delta M_z}{\delta x}\right) dx \right] \frac{dx\, dy\, dz}{dx\, dy}$$

The difference $I_{m1} - I_{m2}$ is the net current at the mutual boundary of rectangle 1 and 2. If it is denoted by I_m, then,

$$I_m = I_{m1} - I_{m2} = -\frac{\delta M_z}{\delta x}\, dx\, dz$$

the direction is determined by equation (14.12) by vector product, rule I_m is normal to the plane of $\overline{M}$ and $\hat{n}$ *i.e.* in the y-direction. However, if $\overline{M}_x$ *i.e.* x component of $\overline{M}$ varied with z, then we should

also have a component of I_m in y-direction, *i.e.*

$$= \frac{\delta M_z}{\delta z} dx\, dz$$

Thus / combining these two we have the resultant current in y-direction,

$$J_{my} = \left[\frac{\delta M_x}{\delta z} - \frac{\delta M_z}{\delta x} \right] dx\, dz$$

$dx\, dz$ is the area of coil in x-z plane. Therefore the current density in y-direction,

$$J_{my} = \frac{I_{my}}{dx\, dz} = \left[\frac{\delta M_x}{\delta z} - \frac{\delta M_z}{\delta x} \right]$$

$$= (\text{curl } M)\, y$$

We have similar expressions for x and z components *i.e.*,

$$J_{mx} = (\text{curl } M)_x \text{ and } J_{mz} = (\text{curl } M)_z$$

Hence, we can generalise,

$$\vec{J}_m = \text{curl } \vec{M} \qquad \ldots(14.15)$$

Important Points

1. In fact, the amperian currents (or atomic currents) are fictitious and are replaced by the effective magnetising currents that give the same magnetic effect of the material. We shall discuss this in the next article.
2. The magnetisation $\vec{M}$ is analogous to the electric polarisation $\vec{P}$. *The amperian current in magnetic materials corresponds to the induced charges in the dielectrics.*

14.8 THREE MAGNETIC VECTORS

If we place a magnetic material in a magnetic field, the elementary magnetic dipoles will be aligned and set up their own field, which will modify the original field.

Let us consider a toroidal coil (Rowland ring) carrying a current I_0 in it. The ring is so constructed that its core can be easily replaced. If the current in the winding is constant, the value of B will be greater when the core is in place than without it. The large value of B is due to magnetisation of the material of the core. If I_0 is the current through toroid of radius r_0, N_0 are the number of turns, then, according to Ampere's law, the value of $\vec{B}$ without iron core:

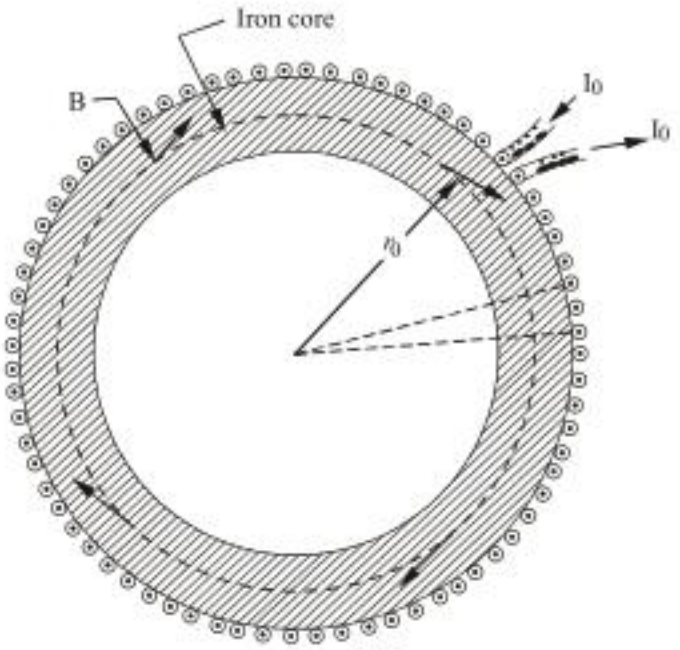


Fig. 14.5

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \qquad \ldots(14.16)$$

$$(B)\, 2\pi r_0 = \mu_0 N_0 I_0 \qquad \ldots(14.17)$$

Let us now put the core. The value of $\vec{B}$ determined experimentally is different when an iron core is placed than without it. As I_0 is the same then according to Ampere's law B must be same, which is not in agreement with experiment. *Thus Ampere's law fails.*

However, the value of $\vec{B}$ can be increased and equal to its value with core, even in the absence of the core by increasing the current in the toroid say by an amount I_{mo}. *The magnetisation of the iron core is thus equivalent in its effect on B to such a hypothetical current increase.* We modify the Ampere's law by adding magnetic current term I_m for fictitious amperian (or atomic) currents. To make it clear we can say as the Gauss's law was modified in the presence of dielectric, similarly the Ampere's law is modified in the presence of magnetic material, *i.e.*,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (I + I_m) \qquad \ldots(14.18)$$

or $$(B)\, 2\pi r_0 = \mu_0 N_0 I_0 + \mu_0 (N_0 I_{m0}) \qquad \ldots(14.19)$$

In fact I_{mo} is related to the magnetisation $\vec{M}$ [see eq. (14.12)]. *It is the value of the current which has the magnetic dipole moment equal to the magnetic moment produced due to the alignment of atomic dipoles.* If A is the area of the toroid, then magnetic moment for the element *dl* of the toroid,

$$d\vec{m} = \vec{M}\,(A\,dl)$$

But magnetic moment is defined by,

$$m = NIA$$

where N are the number of turns, I is the current through the loop. For element *dl*, the number of turns

= Number of turns per unit length X*dl*

$$= \frac{N_0}{2\pi r_0}\, dl$$

therefore the magnitude of magnetic moment for element *dl*

$$M\,(A dl) = \frac{N_0}{2\pi r_0}\, dl \cdot I_{m0}\, A$$

or $$N_0 I_{m0} = M\, 2\pi r_0 \qquad \ldots(14.19\ a)$$

Substituting this value of $N_0 I_{m0}$ in equation (14.19), we get,

$$(B)\, 2\pi r_0 = \mu_0 (N_0 I_0) + \mu_0 (M\, 2\pi r_0)$$

or in general $$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \oint \vec{M} \cdot d\vec{l}$$

or, $$\oint \left(\frac{\vec{B} - \mu_0 \vec{M}}{\mu_0} \right) \cdot d\vec{l} = I \qquad \ldots(14.20)$$

The quantity $\frac{\vec{B} - \mu_0 \vec{M}}{\mu_0}$ is given a special name, *magnetic field strength* $\vec{H}$, hence,

$$\vec{H} = \frac{\vec{B} - \mu_0 \vec{M}}{\mu_0} \qquad \ldots(14.21)$$

or, $$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} \qquad \ldots(14.22)$$

or $$\vec{B} = \mu_0 \left(1 + \frac{\vec{M}}{\vec{H}} \right) \vec{H} \qquad \ldots(14.23)$$

Restatement of the Ampere's law

Ampere's law, in the presence of magnetic material, is given by eqns. (14.20) and (14.21), *i.e.*,

$$\oint \vec{H} \cdot d\vec{l} = I \qquad \ldots(14.24)$$

where I is true current and does not include the magnetising current. The equation states the most important property of field intensity $\vec{H}$ — *the line integral of* $\vec{H}$ *around any closed path equals the total conduction current encircled regardless of the presence of magnetised material.*

The find $\vec{H}$ in our toroid case, let us apply Ampere's law as given in equation (14.24).

$$H(2\pi r_0) = N_0 I_0$$

or, $$H = \left(\frac{N_0}{2\pi r_0}\right) I_0 = n \cdot I_0$$

where n are the number of turns per unit length.

Magnetising force or intensity $\vec{H}$

The relation $\oint \vec{B} \cdot d\vec{l} = \mu I$ shows that the value of $\vec{B}$ depends on the permeability of the medium.

From relation $\oint \vec{H} \cdot d\vec{l} = I$, it can be seen that $\vec{H}$ does not depend on permeability μ but only the currents. $\vec{H}$ is independent, therefore, of the medium in which the conductors are situated. It is this reason that $\vec{H}$ is regarded as being due to the currents only. $\vec{H}$ is then a "cause" which gives rise to a flux density $\vec{B}$ given by $\mu\vec{H}$, and so $\vec{B}$ is dependent on the medium used. Because of this interpretation, $\vec{H}$ is often called the "magnetising force" or magnetising intensity. $\vec{H}$ is a vector in the direction of $\vec{B}$. The term magnetic intensity is not particularly appropriate since it implies that $\vec{H}$ is analogous to the electric field intensity $\vec{E}$, which is not precisely correct since in electric field $\vec{E}$ is defined in terms of force, whereas in magnetic field it is $\vec{B}$ that is defined in terms of force. Therefore, the name magnetising force is sometimes used for $\vec{H}$ which is more appropriate.

Unit of $\vec{H}$

The unit of $\vec{H}$ is obtained by equation (14.24) and is Ampere-turn per metre.

14.9 IMPORTANT CONCLUSIONS ABOUT $\vec{B} \cdot \vec{H}$ and $\vec{M}$

Here, we are summarising the previous results.

(1) First property of $\vec{B}$ is derived from Gauss's law, *i.e.* div $\vec{B} = 0$, which means $\vec{B}$ lines are closed circles and do not have sources or sinks. In other words absence of monopole.

(2) Second property of $\vec{B}$ is expressed by the Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$, which means $\vec{B}$ is *non-conservative* field *i.e.* do not have sources or sinks in contrast to the electric field $\vec{E}$ which originates from charges. For details see article 3.11.

(3) We have restated Ampere's law as,

$$\oint \vec{H} \cdot d\vec{l} = I \text{ (see eqn. 14.24)} \qquad \ldots(A)$$

where I is the true conduction current. According to this relation $\vec{H}$ *is also a non-conservative field* (for details of conservative nature see Chapter 3).

By Stoke's theorem,

$$\oint \vec{H} \cdot d\vec{l} = \int_S \text{Curl}\, \vec{H} \cdot d\vec{S} \qquad \ldots(B)$$

And current,

$$I = \int_S \vec{J} \cdot d\vec{S} \quad \text{(see eqn. 8.17)} \qquad \ldots(C)$$

By comparison of equations A, B and C, we get,

$$\text{Curl } \vec{H} = \vec{J} \qquad \ldots(14.25)$$

This is the basic equation valid everywhere.

(4) We have eatablished the relation

$$\oint \vec{M} \cdot d\vec{l} = I_m$$

where I_m is amperian current, equation states that the line integral of the magnetisation around a closed path is equal to the total amperian currents through the surface encircled by the path.

Now using Stoke's theorem,

$$\oint \vec{M} \cdot d\vec{l} = \int_S \nabla \times \vec{M} \cdot d\vec{S}$$

and $$I_m = \int_S \vec{J}_m \cdot d\vec{S}$$

therefore, $\nabla \times \vec{M} = \vec{J}_m$ $\qquad \ldots(14.26)$

which is the same as proved previously in equation (14.15).

(5) **Important point** to be noted is that sometimes the relation, $\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right)$ is taken as,

$$\vec{B} = \vec{H} + \mu_0 \vec{M}$$

Then is RMKS, $\vec{H}$ will have the same unit as $\vec{B}$ *i.e.* weber/metre^2. However, in our relations $\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right)$, $\vec{H}$ is expressed in the same unit as $\vec{M}$ *i.e.* ampere per metre. *This unit is more convenient in engineering for designing transformers, magnets etc.*

Further, in our relation $\vec{B}$ and $\vec{H}$ are expressed in different units. There is no difficulty with this definition of $\vec{H}$ in RMKS units. Hence we prefer relation $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ than to $\vec{B} = \vec{H} + \mu_0 \vec{M}$.

14.10 PERMEABILITY AND SUSCEPTIBILITY

Relative permeability

We can measure $\vec{B}$, $\vec{H}$ and $\vec{M}$ for different magnetic materials. For non-ferromagnetic that is, for paramagnetic and diamagnetic materials it is found experimentally that $\vec{B}$ is directly proportional to $\vec{H}$

or $\qquad \vec{B} \propto \vec{H}$

or $\qquad \vec{B} = \mu_r \, \mu_0 \, \vec{H}$ $\qquad \ldots(14.27)$

where μ_r (sometimes written as K_m) is the relative permeability of the magnetic medium. $\vec{B}$ is vector in the direction of $\vec{H}$. Putting the value of $\vec{B}$ from eqn. (14.27) in eqn. (14.22), we get

$$\mu_r \mu_0 \vec{H} = \mu_0 \vec{H} + \mu_0 \vec{M}$$

or $\qquad \vec{M} = (\mu_r - 1) \vec{H}$ $\qquad \ldots(14.28)$

Now in vacuum, the magnetisation $\overrightarrow{M}$ is zero, because there are no magnetic dipoles present, hence putting $\overrightarrow{M} = 0$, equation (14.22) reduces,

$$\overrightarrow{B} = \mu_0 \overrightarrow{H} \qquad \ldots(14.29)$$

Since in vacuum $\overrightarrow{M} = 0$, hence from equation (14.28)

$$(\mu_r - 1) = 0$$

or $\mu_r = 1$ for vacuum

Magnetic susceptibility

For para and diamagnetic substances the magnetisation is proportional to the magnetising field H,

$$M \propto H$$

or $$\overrightarrow{M} = \chi_m \overrightarrow{H} \qquad \ldots(14.30)$$

where χ_m is known as the magnetic susceptibility.

Since, $\chi_m = \dfrac{M}{H}$. It may be defined as ratio of magnetisation to the magnetising field. Comparing equations (14.28) and (14.30), we get,

$$\chi_m = \mu_r - 1$$

or $$\mu_r = 1 + \chi_m \qquad \ldots(14.31)$$

The magnetic susceptibility of the medium is the measure of the capability of the medium to take up magnetisation. It is analogous to dielectric susceptibility.

Permeability

Substituting equation (14.31) in eqn. (14.27), we get,

$$\overrightarrow{B} = (1 + \chi_m)\, \mu_0 \overrightarrow{H}$$

or $$\overrightarrow{B} = \mu \overrightarrow{H} \qquad \ldots(14.32)$$

where $\mu = \mu_0 (1 + \chi_m)$, is the permeability of the medium, the dimensionless quantity,

$$\frac{\mu}{\mu_0} = 1 + \chi_m = \mu_r \quad \text{(by eqn. 14.31)} \qquad \ldots(14.33)$$

or $$\mu = \mu_0 \left(1 + \frac{M}{H}\right), \sin \chi_m = \frac{M}{H} \qquad \ldots(14.34)$$

where μ_r is relative permeability of the medium and is defined as the ratio of the number of field lines per square metre taken at right angles in the medium to that in air or free space.

The permeability μ *measures the degree of penetration of magnetic field through the substance or, in other words, it measures the capacity of the substance to take magnetisation.* In isotropic medium $\overrightarrow{M}$ and $\overrightarrow{H}$ are in the same direction, hence μ is scalar. However, in non-isotropic media, $\overrightarrow{M}$ and $\overrightarrow{H}$ are, in general, not in the same direction and μ is not a scalar but becomes a nine components quantity of tensor. Therefore relation (14.22) is general relation and (14.32) is concise expression only for isotropic media.

Units

From equation (14.32), the units of μ

$$= \frac{\text{Units of B}}{\text{Units of H}}$$

$$= \frac{\text{Webers/metre}^2}{\text{Ampere/metre}}$$

$$= \frac{\text{Webers}}{\text{Ampere.metre}}, \text{ since henry} = \frac{\text{Webers}}{\text{Ampere}}$$

$$= \frac{\text{Henrys}}{\text{Metre}}$$

or from equation (14.33), the unit of μ is same as μ_0

$$= \frac{\text{Weber}}{\text{Ampere.metre}} = \frac{\text{Henrys}}{\text{metre}}$$

Since μ_r is ratio it is dimensionless quantity. From eqn. (14.30) since $\overline{M}$ and $\overline{H}$ are expressed in the same unit, *i.e.* ampere/metre, the ratio,

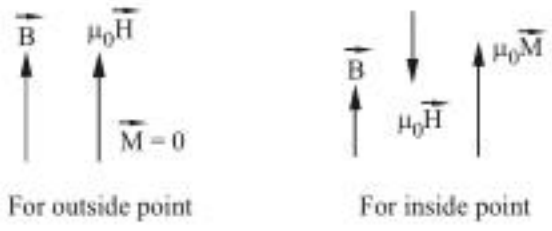
χ_m is dimensionless quantity.

14.11 REPRESENTATION OF $\overline{B}$, $\overline{H}$ and $\overline{M}$

Since magnetisation occurs inside the materials. According to equation (14.22),

$$\overline{B} = \mu_0 \overline{H} + \mu_0 \overline{M}$$

At the boundary $\overline{H}$ is reversed. Thus within the magnets $\overline{M}$ and $\overline{H}$ point in opposite direction, as shown in Fig. 14.6. How $\overline{H}$ is reversed? Let us examine it as follows.



For outside point For inside point

Fig. 14.6

Magnetic Scalar Potential

Taking the divergence of equation (14.22), *i.e.*

$$\overline{B} = \mu_0 \overline{H} + \mu_0 \overline{M}$$

$$\nabla \cdot \overline{B} = \nabla \cdot \mu_0 \overline{H} + \nabla \cdot \mu_0 \overline{M}$$

Since $\nabla \cdot \overline{B} = 0$, we have,

$$\nabla \cdot \overline{H} = -\nabla \cdot \overline{M} \qquad \ldots(14.35)$$

Since $\vec{M}$ and $\vec{H}$ having the dimensions of current/length, (ampere/metre). Then $\nabla \cdot \vec{H}$ or $\nabla \cdot \vec{M}$ have the dimensions as current per area (ampere/metre^2) which can be conveniently expressed as,

$$\frac{\text{Ampere.metres}}{\text{Metres}^3} = \frac{\text{Pole strength}}{\text{Volume}}$$

As we have proved that a current carrying coil is equivalent to a magnet (*i.e.* $IA = Q_m l$ where Q_m is the pole strength), hence dimensions of pole strength Q_m are (current $\times$ length) *i.e.* (Amp. metre). Therefore $\nabla \cdot \vec{H}$ or $\nabla \cdot \vec{M}$ can also be expressed as pole strength per volume.

We can write,

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M} = \rho_m \qquad \ldots(14.36)$$

where ρ_m = pole volume density $\left(\frac{\text{Ampere.metres}}{\text{metres}^3}\right)$ the quantity ρ_m is like the polarisation volume density ρ_p.

If the divergence of a vector field is not zero, means the field has a source, or place of origin. Let us recall the polarised dielectric case, *i.e.* $\nabla \cdot \vec{P} = \rho_p$, which indicates that the polarization field originates on the polarisation charge at the dielectric surface. In an analogous manner equation (14.35) *indicates that the $\vec{H}$ field originates where the magnetisation field $\vec{M}$ ends and the $\vec{H}$ field ends where the $\vec{M}$ field originates.* This occurs at the surface of the substance. Thus, if we consider points entirely outside of the current regions, and choose a path which does not enclose any current, thus when no current is enclosed,

$$\oint \vec{H} \cdot d\vec{l} = 0 \qquad \text{(by Ampere's law)}$$

Under this condition $\vec{H}$ is conservative field and can then be derived from a scalar magnetic potential function (V_m).*

Thus when path does not enclose current, the above relation by Stoke's theorem can be written as

$$\nabla \times \vec{H} = 0$$

Then $\vec{H}$ can be expressed as negative gradient of a scalar in the same manner as in electrostatics $\nabla \times \vec{E} = 0$, then $\vec{E} = -\nabla V$.

$$\vec{H} = -\nabla V_m \qquad \ldots(14.37)$$

Taking div. of (14.37), we get,

$$\nabla \cdot \vec{H} = -\nabla^2 V_m$$

Therefore, from equation (14.35),

$$\nabla^2 V_m = \nabla \cdot \vec{M} \qquad \ldots(14.38)$$

This indicates that magnetic scalar potential V_m is related to source of magnetisation. *This is the "Poisson's equation" solution of which gives the magnetic scalar potential in current free regions.*

Vector Potential

Taking the curl of the relation,

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$$

*See comments in article 14.12.

$$\nabla \times \vec{B} = \mu_0 \nabla \times \vec{H} + \mu_0 \nabla \times \vec{M} \qquad \ldots(14.39)$$

when there is no magnetisation, equation (14.39) reduces to,

$$\nabla \times \vec{B} = \mu_0 \nabla \times \vec{H}, \quad \left[\text{since } \nabla \times \vec{H} = \vec{J} \text{ by eqn. (14.25)}\right]$$

$$= \mu_0 \vec{J} \qquad \ldots(14.40)$$

From relation (14.39), the *curl of* $\vec{M}$ *has the dimensions of current density (amp.metre*^{2}*) and represents the equivalent current of density* $\vec{J}_m$ *(amp.metre*^{2}*) flowing, in a very thin layer around the surface of a uniformly magnetised substance (as rod).*

Hence $\nabla \times \vec{M} = \vec{J}_m$ again the same conclusion as given in equations (14.15) and (14.26).

Thus equation (14.39) reduces,

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \vec{J}_m\right) \qquad \ldots(14.41)$$

where $\vec{J}$ = actual current density, as in a current carrying wire (amp./metre^2).

$\vec{J}_m$ = equivalent current density, as at the surface of a magnetised substance (bar) amp/metre^2

Potential due to a Magnetisation

We have defined the vector potential as,

$$\vec{B} = \nabla \times \vec{A}$$

and $\vec{A} = \dfrac{\mu_0}{4\pi} \int_0 \dfrac{\vec{J}}{r} dv'$ see eqn. (10.40)

If both conduction and magnetisation currents are present, we have

$$\vec{A} = \frac{\mu_0}{4\pi} \int_0 \frac{\vec{J} + \vec{J}_m}{r} dv' \qquad \ldots(14.42)$$

where, $\vec{J} = \nabla \times \vec{H}$ amp./metre^2

$\vec{J}_m = \nabla \times \vec{M}$ amp./metre^2

14.12 COMMENT ON THE FIELD $\vec{H}$

We have restated Ampere's law as,

$$\oint \vec{H} \cdot d\vec{l} = 1$$

From the discussion of the preceding section it is clear that $\vec{H}$ can be regarded as sum of two components.

(1) H_1 due to true currents and it is non-conservative and

$$\oint \vec{H}_1 \cdot d\vec{l} = 1$$

By non-conservative means it does not have sources or sinks.

(2) However, if we consider the magnetisation and if H_c is the field due to magnetisation, then it is a conservative field that satisfies the relation,

$$\oint \vec{H}_c \cdot d\vec{l} = 0$$

In fact H_c represents the conservative field due to fictitious point charge those are assumed when substance is magnetised and then we have the magnetic scalar potential only in current free regions (As when $\oint \vec{E} \cdot d\vec{l} = 0$, we express $\vec{E} = -\nabla V$).

14.13 BOUNDARY CONDITIONS AT THE SURFACE OF A MAGNETIC MATERIAL

Let us determine the relation between the values of $\vec{B}$ and $\vec{H}$ inside a magnetic material to the value of $\vec{B}$ and $\vec{H}$ just outside the material. The relation is obtained by two basic laws, the first is that the field $\vec{B}$ consists of closed lines, *i.e.* the flux of $\vec{B}$ through a closed surface is zero which is precisely Gauss's law,

$$\nabla \cdot \vec{B} = 0$$

or $$\oint \vec{B} \cdot d\vec{S} = 0 \qquad \ldots(1)$$

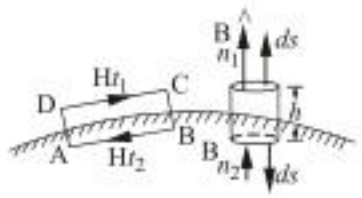


Fig. 14.7

The second basic law is Ampere's law, *i.e.*,

$$\oint \vec{H} \cdot d\vec{l} = 1 \qquad \ldots(2)$$

Applying equation (1) to the cylindrical surface of height h shown in Fig. 14.7 we obtain,

$$B_{n1}\,\Delta S - B_{n2}\,\Delta S = 0 \qquad \ldots(14.43)$$

where B_{n1} is the average component of $\vec{B}$ normal to top face of the cylinder and B_{n2} is the average component of $\vec{B}$ normal to the lower face in inward direction, that is B_{n2} and ΔS are oppositely directed. The integral of $\vec{B}$ over the curved surface of the cylinder is neglected because by making height h approaching zero, integral zero even though there may be finite of $\vec{B}$ normal to the surface, by eq. (14.43)

$$B_{n1} = B_{n2} \qquad \ldots(14.44)$$

Therefore, the normal component of $\vec{B}$ inside the substance is the same as the normal component of $\vec{B}$ just outside the substance. If the boundary separates two media of different permeability, even then equation (14.44) is valid. *Thus, the normal component of the $\vec{B}$ is continuous across the boundary between the two media.*

Let us now apply Ampere's law to the path ABCD, $AB = \Delta x$.

Let the average value of $\vec{H}$ tangent to the boundary for upper half is H_{t1} and for lower half path H_{t2}, then according to Ampere's law,

$$H_{t1}\,\Delta x - H_{t2}\,\Delta x = I$$

where I is current enclosed by ABCD. The integral of $\vec{H}$ over the segments BC and DA becomes zero even though a finite field may exist normal to the boundary by choosing BC and DA as small as possible and approaching zero. The second term is negative because H_{t2} and Δx are oppositely directed.

$$H_{t1} - H_{t2} = \frac{I}{\Delta x} = I' \text{ amp/metre} \qquad \ldots(14.45)$$

where I' is the linear density of current *i.e.* current per unit length flowing in an infinitesimally thin sheet at the surface. According to equation (14.45), the change in tangential component of $\vec{H}$ across a boundary is equal in magnitude to the linear current density. If $I' = 0$,

$$H_{t1} = H_{t2} \qquad \ldots(14.46)$$

According to equation (14.46) the tangential component of $\vec{H}$ inside the magnetic material is the same as the tengential component of $\vec{H}$ just outside the material. If the boundary is separating two media of different permeability, even then equation (14.46) is valid.

Thus, the tangential components of $\vec{H}$ are continuous across the boundary between two media provided the boundary has no current sheet of infinitesimal thickness.

Example 14.1 *Consider a plane boundary between two media of permeability* μ_1 *and* μ_2 *as shown in Fig. 14.8. Find the relation between the angles* α_1 *and* α_2 *. Assume that the media are isotropic with B and H in the same direction.*

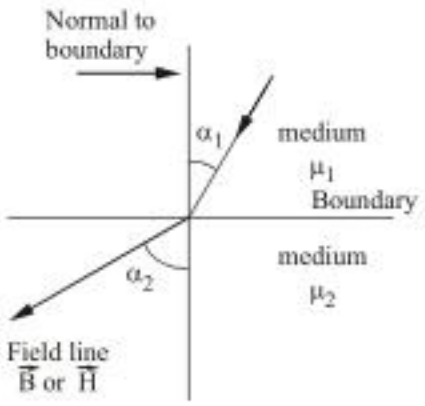


Fig. 14.8

Solution. From the boundary relations,

$$B_{n1} = B_{n2} \text{ and } H_{t1} = H_{t2} \quad \ldots(1)$$

From Fig. 14.8,

$$B_{n1} = B_1 \cos \alpha_1 \text{ and } B_{n2} = B_2 \cos \alpha_2 \quad \ldots(2)$$

$$H_{t1} = H_1 \sin \alpha_1 \text{ and } H_{t2} = H_2 \sin \alpha_2 \quad \ldots(3)$$

Substituting (2) and (3) into (1) and dividing yields

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}} \quad \ldots(4)$$

where μ_{r1} = relative permeability of medium 1 (dimensionless)

μ_{r2} = relative permeability of medium 2 (dimensionless)

14.14 CLASSIFICATION OF MAGNETIC MATERIALS

Faraday divided the magnetic materials into three groups in terms of susceptibility χ_m and relative permeability μ_r.

Paramagnetic Substances

For paramagnetic substances $\vec{M} = \chi_m \vec{H}$ and susceptibility χ_m is positive and less than unity and the relation $\mu_r = 1 + \chi_m$ indicates that relative permeability is slightly greater than unity. Susceptibility χ_m is independent of the field, when $\vec{H} = 0$, $\vec{M} = 0$ in this case. Paramagnetic substances when placed *in magnetic field acquire a feeble induced magnetism in the same direction as the external field as* $\mu_r > 1$. In a non-uniform field it will experience an attractive force toward the strongest point. The susceptibility χ_m varies inversely with the absolute temperature. Mathematically,

$$\chi_m \propto \frac{1}{T}$$

or $$\chi_m T = \text{Const.} \qquad \ldots(14.47)$$

This is called Curie law.

Examples. Some examples of paramagnetic substances are oxygen, platinum, aluminium, chromium, manganese and solutions of iron and nickel.

Diamagnetic Substances

Such substances acquire feeble induced magnetism in a direction opposite to the external field. In non-uniform field, they have tendency to move from stronger to weaker parts of the field. The magnetisation M is proportional to the magnetising field,

$$\vec{M} = \chi_m \vec{H}$$

Here, the susceptibility χ_m is independent of the temperature. χ_m is negative and relative permeability μ_r is slightly less than unity *i.e.* $\mu_r < 1$. *Examples* of diamagnetic substances are bismuth, antimony, copper, mercury, gold, water, air, hydrogen and alcohol.

Ferromagnetic Substances

For ferromagnetic substances, the susceptibility χ_m is positive and is larger than paramagnetic substances and the relative permeability is also much greater than unity. However, the relation $\vec{M} = \chi_m \cdot \vec{H}$ (*i.e.* the magnetisation proportional to the field) does not hold for ferromagnetic materials. The ferromagnetic substances having spontaneous magnetisation even without the field, when $\vec{H} = 0, M \neq 0$ in this case. Due to greater value of μ_r, they are able to acquire high degree of magnetisation even they can develop north and south pole.

Distinction between Para and Ferro

Susceptibility decreases with increase in temperature. If the temperature is increased above a certain critical value, called Curie temperature, the substance loses its ferromagnetism, that is, the spontaneous magnetisation disappears and substance becomes paramagnetic. Above the Curie temperature the susceptibility follows the Curie Weiss law, *i.e.*,

$$m = \frac{C}{T - \theta} \qquad \ldots(14.48)$$

where C is Curie-constant and temperature θ is Curie-temperature. *Hence we conclude that all ferromagnetics are basically paramagnetic but all para are not ferromagnetics. Above the Curie temperature substance is paramagnetic and below it, it is ferromagnetic.*

Examples. The five elements-iron, cobalt, nickel, Gadolinium and dysprosium- and variety of alloys of these and other elements.

14.15 MAGNETISATION CURVES (B-H CURVES)

The permeability μ of a substance is given by equations (14.32) and (14.33) *i.e.*,

$$\mu = \frac{B}{H} = \mu_r \mu_0$$

But the permeability μ of the ratio B/H is not a constant for ferromagnetic materials. In other words B is not proportional to $\vec{H}$. Therefore, in order to know the relation of $\vec{B}$ to $\vec{H}$ a graph showing the variation of B with H is plotted by taking B as ordinate and H as abscissa. The line or curve showing B as a function of H as such B – H curve is called magnetisation curve.

Ballistic Method for Plotting B-H Curve

To plot a magnetisation curve for an iron sample, take the sample in the form of a ring. A primary coil P is wound uniformly and closely spaced over the ring, forming an iron-cored toroid as in Fig. 14.9.

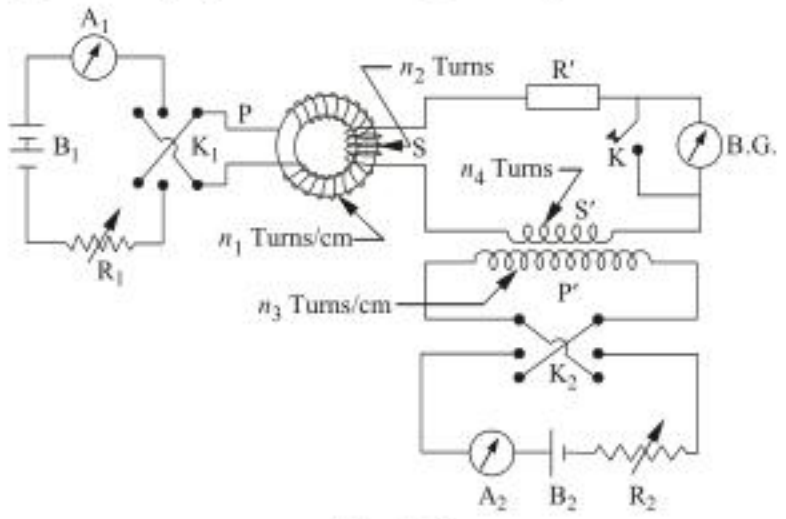


Fig. 14.9

The ends of the primary coil are connected to a battery B_1, through a resistance box R_1 and a reversing key K_1.

The small coil S, called the secondary coil, is wound over the primary coil P and is connected to ballistic galvanometer through rheostat R' and secondary coil S'. The secondary coil S' itself wound over a primary P', which is further connected to battery B_2 through a reversing key K_2 (a commutator) for calibrating the B.G as shown in Fig. 14.9.

First the reversing key K_1 is switched on. A steady current I passes through the primary coil P. If the number of turns per unit length are n_1, then, the value of H applied to the ring is,

$$H = n_1 I \text{ amp/metre or amp. turns/metre} \qquad \ldots(14.49)$$

(Since, for toroid $B = \mu_0 n_1 I$ and there is no magnetisation so far, hence $\overline{M} = 0$, then relation $\overline{B} = \mu_0 \overline{H} + \mu_0 \overline{M}$ is $B = \mu_0 H$, hence $\overline{H} = \dfrac{B}{\mu_0} = n_1 I$).

This value of H applied to the ring may be called the "*magnetising force*". The magnetic induction B in the ring may be regarded as the result of applied field H and is measured by placing the secondary coil S over the ring. The magnetisation of the specimen develops a magnetic flux density (or magnetic induction) B inside the ring which is transferred to the secondary S. This transfer of flux to the secondary coil causes an induced emf.

Suppose the flux density developed in the ring in the time $dt = dB$.

Therefore, the flux developed per turn = AdB where A is the area of cross-section of the ring.

The flux linked with secondary = $dBAn_2$ where n_2 are the number of turns of the secondary.

Since initially flux in the secondary is zero.

The change in magnetic flux in the secondary $= d\phi = d\,B \cdot An_2$

Rate of change of flux $\dfrac{d\phi}{dt} = An_2 \dfrac{dB}{dt}$

According to Faraday's law $\frac{d\phi}{dt}$ is equal to the induced emf in the secondary.

$$\xi = An_2 \frac{dB}{dt} \text{ volts}$$

And corresponding current in the secondary,

$$I_2 = \frac{\xi}{R} = \frac{An_2}{R} \frac{dB}{dt} \text{ amps}$$

where R is the total resistance of the galvanometer circuit.

Since current is rate of flow of change, hence,

$$\frac{dq}{dt} = \frac{An_2}{R} \frac{dB}{dt}$$

$$dq = \frac{An_2}{R} dB \text{ Couls}$$

Since initially magnetic induction B was zero it rises as current increases from zero to a saturation value B. Thus total charge developed in the secondary,

$$q = \int_0^B dq = \int_0^B \frac{An_2}{r} dB = \frac{An_2 B}{R} \text{ Coul}$$

Now if θ is the first throw in B.G, then,

$$q = K\theta\left(1 + \frac{\lambda}{2}\right), \qquad \text{K is B.G. constant.}$$

From these two expressions, we get

$$\frac{An_2 B}{R} = K\theta\left(1 + \frac{\lambda}{2}\right) \qquad \ldots(1)$$

Determination of Galvanometer Constant

The value of K can be determined by the use of a standard solenoid. A known current I' is passed through the primary of the standard solenoid and suddenly this will cause change in magnetic flux and is calculated as follows. The magnetic field in the primary $P' = n_3 I'$ where n_3 are the number of turns per unit length of the primary P'.

This creates a magnetic flux through the secondary,

$$\phi' = \mu_0 n_3 I' \alpha n_4$$

where α is the area of cross-section of the secondary S' and n_4 are the number of turns in the secondary S'. This change in flux will give rise to an induced emf in the secondary hence proceeding in the manner as above, we get,

$$\frac{\mu_0 n_3 I' \alpha n_4}{R} = K\theta'\left(1 + \frac{\lambda}{2}\right) \qquad \ldots(2)$$

where θ' is the deflection produced. The constant K can be eliminated from eqns. (1) and (2) *i.e.* dividing eqn. (1) and (2),

$$\frac{B n_2 A}{\mu_0 n_3 I' \alpha n_4} = \frac{\theta}{\theta'}$$

or $$B = \frac{\mu_0 n_3 I' \alpha n_4}{A n_2} = \frac{\theta}{\theta'} \text{ weber/metre}^2 \qquad \ldots(14.50)$$

and $$H = n_1 I \text{ amp/metre}$$

These eqns. give the value of B and H in terms of known quantities. *Experimentally the B-H curve may be plotted either (1) by steps or (2) by reversal.*

(1) *Step method*: In this method the current is increased in steps and ballistic throw is observed. The total current and magnetic field are obtained by adding together the individual values in steps.

(2) *Reversal method:* This method is better because when the current is reversed from + I to – I, the value of B is double as the flux change will be double, hence the ballistic throw θ is also doubled. Further, when calibrating B-G, the reversal of current I′ doubles the flux density B, consequently the ballistic throw θ' is also doubled. Therefore, the value of B does not change as the values of equations (1) and (2) are doubled, hence the ratio of equations (1) and (2) given in equation (14.50) for B remains unchanged.

14.16 DETERMINATION OF SUSCEPTIBILITY χ_m AND PERMEABILITY μ

From equation (14.32),

$$\vec{B} = \mu \vec{H} \qquad \ldots(14.51)$$

where $$\mu = \mu_0 (1 + \chi_m)$$

$$\therefore \quad \vec{B} = \mu_0 (1 + \chi_m) \vec{H} \qquad \ldots(14.52)$$

Thus, by knowing $\vec{B}$ and $\vec{H}$, we can calculate the susceptibility χ_m by using equation (14.52) and B from equation (14.51). The $\vec{B} - \vec{H}$ curve is shown in Fig. 14.10.

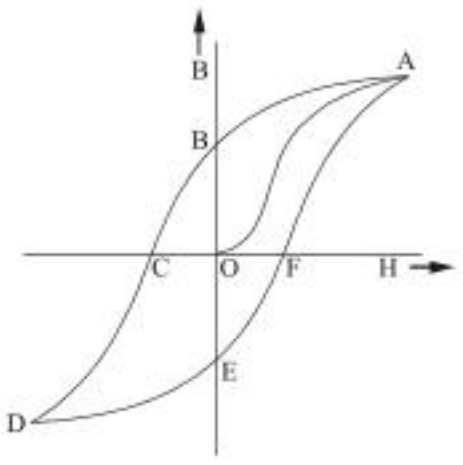


Fig. 14.10

$\vec{M}$ - $\vec{H}$ Curve

Since $\vec{B}$ and $\vec{H}$ are known, we can also find the value of magnetisation M by the relation,

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

Then a curve between $\vec{M}$ and $\vec{H}$ can be drawn. It is called $\vec{M} \cdot \vec{H}$ curve and is identical to $\vec{B}$ - $\vec{H}$ curve.

Variation of Permeability

Although $\mu = \frac{B}{H}$, for ferromagnetic materials, μ depends on the magnetising field H. *Various kinds of permeability can be defined from a hysteresis loop.*

1. **Ordinary or normal permeability.** It is the ratio B/H for a point on the normal magnetisation curve, *i.e.* ratio B/H for the tip of the loop.
2. **Differential permeability.** It is the slope $\frac{dB}{dH}$ of the magnetisation curve.
3. **Initial permeability.** The value of $\frac{dB}{dH}$ for $H = B = 0$ is called the initial permeability.
4. **Incremental permeability.** If a steady value of H is applied to a specimen and then small alternating H is superimposed, a minor hysteresis loop is obtained that has its centre near the original point on B-H graph. The slope of the line connecting tips of this minor loop is the incremental permeability. All these are shown in Fig. 14.11.

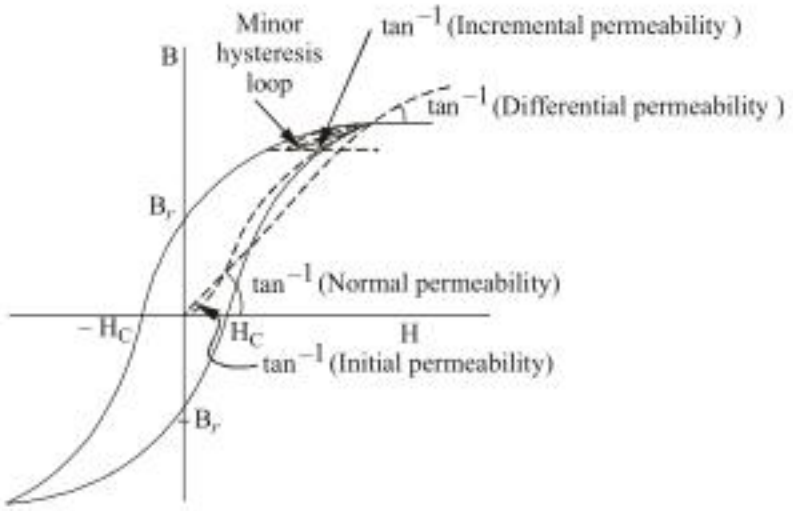


Fig. 14.11

14.17 CYCLE OF MAGNETISATION HYSTERESIS

Let us consider an unmagnetised ferromagnetic substance an iron bar in a magnetising field. The magnetising field is slowly increased as in the case of B-H curve. The magnetisation $\overline{M}$ increases as represented by the part OA, as shown in Fig. 14.12. After A any further increase in H would not increase $\overline{M}$. That is, the substance reaches the "*magnetic saturation*" $\overline{M}$. In other words, all elementary dipoles have been aligned along the field direction. Now the magnetic field H is gradually decreased to zero. The magnetisation decreases but does not become zero when H = 0. The magnetisation $\overline{M}$ is represented by OB. This value of $\overline{M}$ corresponding to H = 0 is called "*Residual Magnetism*" M_r or "*Retentivity*" or "*Remanence*". The remanence gives the state of permanent magnetisation of the sample. If we consider the B – H curve, the value of B corresponding to H = 0 *i.e.* after reducing the H from saturation to zero is known as "*remanence induction*" B_r.

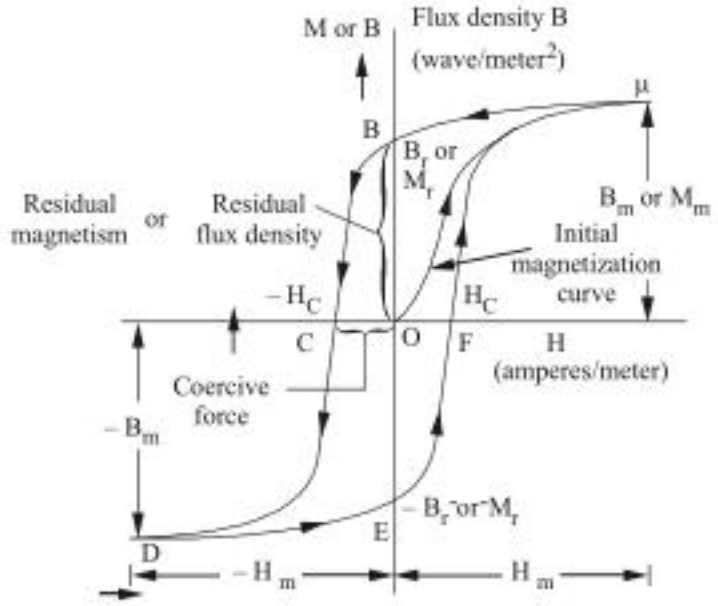


Fig. 14.12

If the direction of the field H is now reversed and the magnitude gradually increased, the curve BCD is obtained. The residual magnetism vanishes at C for the value OC of magnetic field in the negative direction. So *OC gives the magnetic field H needed to reduce the magnetisation to zero or in other words this is the magnetising force required to completely demagnetise the sample.*

This is called the "*coercive force*" or "*coercivity*" H_c. The three quantities referred above, saturation value, the residual magnetism together with the area of the hysteresis loop, are able to explain up to some extent the nature of magnetic material.

Hysteresis

If we further increase H in the negative direction up to the saturation point D symmetrical to A. Now again, all the dipoles are oriented along the field direction but opposite to their direction of orientation at the saturation point A. If we now gradually reduce H to zero and reverse its direction at E, *i.e.* restore its original direction, then increasing the field again in steps, we reach back to the point A along the path DEFA. We trace out a path ABCDEFA is called the "hysteresis loop" and is characteristic of the magnetic properties of the material. It is clear from M-H curve shown in Fig. 14.12 that the descending part of the curve always lies above the ascending part. *Hence the magnetisation* $\overline{M}$ *or* $\overline{B}$ *(flux density) will become zero at a 'later point"* of the curve than the field H. It means the *magnetisation lags behind the magnetising field. This lag of magnetisation behind the magnetising field is called hysteresis.*

14.18 ENERGY LOSS PER CYCLE OF MAGNETISATION

The process of magnetisation of a ferromagnetic substance through a cycle involves the expenditure of the energy. This energy is spent in aligning the elementary dipoles in the direction of magnetising field. If we gradually reduce the magnetising field to zero, the magnetisation does not fall to zero and magnetic field has to be applied in the opposite direction to completely demagnetise the sample. Thus energy spent in magnetising a specimen is not completely recoverable and there is a loss of energy in taking a sample through a cycle of magnetisation and it appears in the form of heat. This loss of energy is called hysteresis loss.

Hysteresis Loss from B-H Curve

Let us consider the sample of ferromagnetic material in the shape of toroidal solenoid. The magnetising field H is obtained by modified form of Ampere's law and is given in equation (14.24) *i.e.* $\oint \vec{H} \cdot d\vec{l} = I$, $Hl = NI$ or $H = \frac{NI}{l}$ where N are total number of turns over toroidal winding and l is length of toroid

thus $H = n_1 I$

where n_1 are the number of turns per unit length, then we get,

$$\vec{H} = \frac{NI}{l} \qquad \ldots(14.53)$$

Let us calculate the amount of work that has to be done in establishing a current I in the toroidal coil by which magnetising field H is obtained and the substance is magnetised giving rise to the magnetic flux density (or induction) $\vec{B}$. If A is the area of toroidal ring, the flux inside the ring is,

$$\phi = NAB$$

Now as current I increases, the magnetic flux also increases and according to Faraday's law an induced emf develops in the coil, the magnitude is given by,

$$\xi = \frac{d\phi}{dt} = NA\frac{dB}{dt}$$

The rate of doing work,

$$P = \xi I = NAI \cdot \frac{dB}{dt}$$

Substituting I from equation (14.53),

$$P = H\,Al\frac{dB}{dt}$$

The product Al is the volume of the sample *i.e.* ring. Hence the rate of doing work per unit volume,

$$P = H\frac{dB}{dt}$$

This gives the rate at which energy is transferred per unit volume to the sample in magnetising it is,

$$\frac{dw}{dt} = P = H\frac{dB}{dt}$$

Therefore, the energy transferred in time dt per unit volume,

$$dw = H\,dB$$

The energy transferred to the sample per unit volume per cycle

$$= \oint H\,dB \qquad \ldots(14.54)$$

Now be careful to the fact that though it has been deduced for an iron ring, this expression is true in general.

Let us evaluate the integral for ferromagnetic materials. Ferromagnetic materials are "inherently irreversible" as they retain magnetisation even if the magnetising field is reduced to zero. Fig. 14.13 represents a part of B-H loop. For ferromagnetic materials the small change in H produces a large change in $\vec{B}$ (or $\vec{M}$). But for the change dH in H, the corresponding change dB in B will be very

small. Therefore, the value of H may be taken to be same for the points *a* and *b*. The product H*d*B is the energy transferred per unit volume to the sample for a change *d*B in B is equal to the area $ab \times cb$ which is area *abcd* shown shaded. Therefore, the energy transferred for the path OA per unit volume

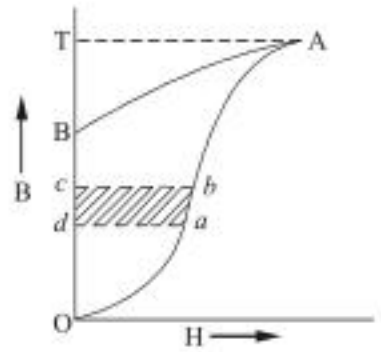


Fig. 14.13

$$= \int_{OA} H dB$$

= area OATO

Now H is gradually decreased reduced to zero as the process is irreversible. The trace does not follows AO but AB. Hence, the energy recoverable during demagnetisation

= area BATB

Therefore the energy	Energy transferred	–Energy recovered
not recoverable	= during magnetisation	during demagnetisation
i.e. energy loss	= area OATO - area BATB	
	= area OABO	

If we consider the hysteresis loop, then the energy not recoverable, *i.e.* energy loss,

$$\oint H\, dB = \text{area ABCDEFA}$$

$$= \text{area B} - \text{H loop} \qquad \ldots(14.55)$$

The energy loss is converted into heat.

Energy density

The energy required to magnetise the sample per unit volume per cycle is given by eqn. (15.54), *i.e.*

$$= \oint H\, dB$$

It is assumed that the relation $B = \mu H$ is valid over the entire range of B with $\mu = \mu_0 \mu_r$ then energy stored in the sample if B changes from 0 to B.

$$\int_0^B H dB \int_0^B \frac{B}{\mu} dB = \frac{B^2}{2\mu} = \frac{\mu H^2}{2} = \frac{BH}{2} \text{ Joules/metre}^2$$

Magnetic energy per unit volume is called,

$$\text{Magnetic energy density} = \frac{1}{2}\frac{B^2}{\mu} \text{ Joules/metre}^3$$

$$= \frac{1}{2}\mu H^2 \text{ Joules/metre}^2 \qquad \ldots(14.56)$$

$$= \frac{1}{2} BH \text{ Joules/metre}^3$$

where the permeability of the medium is μ. In case the medium is vacuum (or air), $\mu_r = 1, \mu = \mu_0$.

$$\text{Energy density} = \frac{1}{2}\frac{B^2}{\mu_0}$$

which is same as derived earlier in equation (11.27). This expression for energy is analogous to the electrostatic energy expression *i.e.*

$$\frac{1}{2}\varepsilon_0 E^2 \text{ or } \frac{1}{2}\varepsilon E^2$$

Hysteresis Loss from M-H Cycle

Suppose the behaviour of the ferromagnetic substance is represented by M-H curve. Using the relation,

$$\overline{B} = \mu_0 (\overline{H} + \overline{M})$$

or $$dB = \mu_0 (dH + dM)$$

Integrating for whole hysteresis loop, we have after multiplying by H,

$$\oint H dB = \mu_0 \oint H dH + \mu_0 \oint H dM$$

If H is plotted against dH we will get a straight line, the integral $\oint H dH$ *represents the area enclosed by a straight line which is zero.* So, we have,

$$\oint H dB = \mu_0 \oint H dM$$

$\oint H dB$ represents the energy loss per unit volume per cycle and equals to area of $\overline{B} - \overline{H}$ loop. Similarly $\oint H dM$ equals to area of $\overline{M} - \overline{H}$ loop or curve.

Therefore the energy loss $= \mu_0$ area of M - H loop ...(14.56 a)

This is called Warburg's law.

Thus, the *energy loss per unit volume of the material per cycle is equal* to μ_0 times the area of M - H loop and is dissipated in the form of heat.

14.19 DEMAGNETISATION

Ferromagnetic materials are not fully demagnetised by removing the magnetising field. They retain magnetisation corresponding to H = 0. The capacity to retain magnetisation depends on the degree up to which the substance is magnetised. It is maximum corresponding to saturation value of magnetisation. The portion of a saturation hysteresis loop for which M is positive and H is negative is called "demagnetisation curve". The retentivity is different for different values of H below saturation value. In fact retentivity decreases with decreasing H below saturation value. Hence it is possible to reduce retentivity to zero value by gradually decreasing H, through repeated cycles as shown in Fig. 14.14.

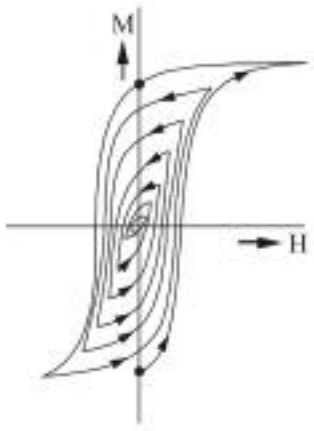


Fig. 14.14

However, the demagnetisation is also possible by heating the substance above the Curie temperature, but this method causes the permanent change in the magnetic behaviour of the substance.

Soft and Hard Mgnetic Materials

The shape of hysteresis loop is a characteristic of the magnetic materials under test. Fig. 14.15 shows M–H curves for iron and steel. We can draw the following conclusions from the curves.

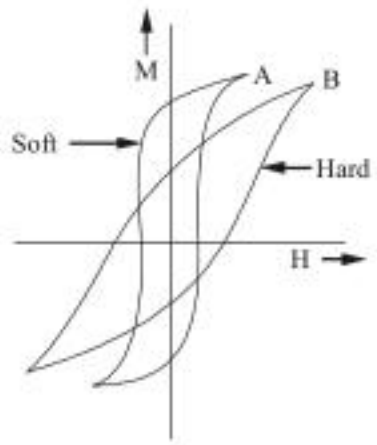


Fig. 14.15

Coercivity

Substances with large value of coercive force are said to be magnetically hard, whereas those with small coercive force are called soft.

Retentivity

The retentivity is greater for soft materials than for hard materials.

Susceptibility

It is greater for soft materials than for hard materials.

Permeability

It is greater for soft materials than for hard materials.

Hysteresis Loss

The area of the hysteresis loop which measures the energy loss per unit volume per cycle is less for soft materials than for hard materials.

14.20 USES OF HYSTERESIS CURVE

Choice of Materials

The hysteresis curve and its properties, saturation, retentivity, coercivity and hysteresis loss provide very useful information for choosing the material for practical uses in industry.

Permanent Magnets

For permanent magnets the materials should have

(1) Large coercive force because we want that its magnetism should be permanent and not easily destroyed by magnetic fields, mechanical ill-treatment or by temperature change, and
(2) High residual magnetism so that the magnet may be strong.

These requirements are satisfied by steel. Though the retentivity of iron is greater than steel but iron has very small value of coercivity, *hence steel is preferred for permanent magnets.* The fact that hysteresis loss is also less for soft iron is immaterial because permanent magnets are never put to cyclic change of magnetisation. The mateials used are cobalt-steel, tungsten steel and other number of alloys. Very recently an alloy of iron, cobalt and vanadium called vicalloy having highest coercivity has been developed.

Electromagnets

The materials for the construction of core of electromagnets should have

1. maximum induction B with comparatively small magnetising field H.
2. high initial permeability.
3. low hysteresis loss.

Considering these facts, the *soft iron is an ideal material for constructing an electromagnet.*

Transformer Cores, telephone diaphragms, Armature of dynamos and motors and cores of chokes

When these are used practically the material is subjected to cyclic changes. The chief requirements are that the material must have

1. high initial permeability in order to obtain large B for low value of H since $(B = \mu H)$,
2. low hysteresis loss which will produce a small heating effect hence winding is secure of any breakdown of insulation, and
3. high specific resistance to reduce eddy current losses.

Soft iron is better than steel for these purposes. It has initial permeability about 250, initial permeability can be increased by adding 4 per cent silicon and alloy is called transformer steel which is a very good material for cores of transformers. There are some other alloys containing nickel iron called Perm alloys, Mumetal (Ni 76%, Cu 5%, Fe 17.5%; Cr 1.5%) having initial permeability between 10,000 and 100,000. An alloy called radio-metal (N 48%, Fe 48.5%, Cu 3%, Mn 0.5%) has an initial permeability between 2,000 and 15,000.

14.21 MAGNETOMOTIVE FORCE AND MAGNETIC CIRCUITS

We are familiar with an electrical circuit and Ohm's law which connects the emf with the current in the circuit. There is flow of electrons which constitutes the electric current. The line integral of electric field $\vec{E}$ *i.e.* $\oint \vec{E} \cdot d\vec{l}$ over a circuital path is the electromotive force and is equal to the work done in carrying a unit positive charge once round the circuit. But in magnetism, since there is nothing like a separate north pole and a thing as magnetic current is meaningless. However, the magnetic lines of flux form closed loops. Such closed paths are known as magnetic circuit and we have Ampere circuital law that gives the line integral of magnetic field H, *i.e.*,

$$\oint \vec{H} \cdot d\vec{l} = \text{enclosed current}$$

Hence by analogy we can say that the line integral $\oint \vec{H} \cdot d\vec{l}$ *is equal to the work done in carrying unit north pole once round the close path if north pole is free to do so and termed as mmf (magnetomotive force)*

Hence, from Ampere's law,

$$\oint \vec{H} \cdot d\vec{l} = H \oint dl = I$$

Here $\vec{H} \cdot d\vec{l} = Hdl$ as H is everywhere parallel to the path $d\vec{l}$ and $\oint dl$ = length of the path.

Let us consider the case of toroidal solenoid of N turns. The total current enclosed in NI, the magnetic field H inside, is given by,

$$H = \frac{NI}{l} \text{ amp/metre}$$

The flux density inside the ring is,

$$B = \mu H$$

$$= \frac{\mu NI}{l}$$

Total magnetic flux $\phi_B = \oint_s \vec{B} \cdot d\vec{S}$

$$\phi_B = BA = \frac{\mu NIA}{l}$$

where A is area of cross-section of ring. Therefore,

$$NI = \frac{l}{\mu A}\phi_B \qquad \ldots(14.57)$$

Returniing now to the consideration of electric field, we know that emf,

$$\xi = \oint \vec{E} \cdot d\vec{l}$$

In a magnetic field we may write an "*analogous relation*", based on Ampere's law, that when current is enclosed by the path of integration,

$$\oint \vec{H} \cdot d\vec{l} = NI = \xi_m \qquad \ldots(14.58)$$

where the quantity ξ_m, called "magneto motance", or "magneto-motive force" *i.e.* mmf is equal to the current enclosed. The product NI is expressed in ampere turns.

Unit of mmf

From above equation (14.58), the mmf has the same unit as NI *i.e.* ampere turns. If N = 1, then simply ampere. Comparing equations (14.57) and (14.58),

$$\text{mmf} = NI = \phi_B \frac{l}{\mu A} \qquad \ldots(14.59)$$

In an electrical circuit we have similar relation, namely,

$$\text{emf} = \text{current} \times \text{resistance}$$

(by Kirchhoff's law the total emf in the circuit is equal to the total IR drop). We can write,

$$\text{Current} = \frac{\text{emf}}{\text{Resistance}}, \; i.e. \; I = \frac{\xi}{R} \qquad \ldots(14.60)$$

and from equation (14.59),

$$\text{Flux} = \frac{\text{mmf}}{l/\mu A}, \; i.e. \; \phi_B = \frac{\xi m}{R_e} \qquad \ldots(14.61)$$

By analogy we define a quantity called the "reluctance" R_e which is considered to oppose the setting up of magnetic flux in magnetic circuit as resistance opposes the setting up electric current. R_e is expressed in amp. turns/weber which is reciprocal of inductance. Therefore, the units for reluctance are:

$$\frac{\text{amp.turns}}{\text{Weber}} = \frac{1}{\text{Henries}}$$

Let us consider the reluctance of portion of a magnetic circuit, since in an electric circuit we have Ohm's law,

$$R = \frac{V}{I}$$

where V is potential difference between the points.

In the analogous magnetic case reluctance R_e between two points,

$$R_e = \frac{V_m}{\phi_B} \qquad \ldots(14.62)$$

where V_m = magnetic potential difference between the points, and

$$V_m = \int_1^2 \vec{H} \cdot d\vec{l}, \text{ just as } V = \int_1^2 \vec{E} \cdot d\vec{l}$$

Thus, we call equation (14.62) AS "*Ohm's law*" in magnetism.

Permeance

The reciprocal of reluctance is known as permeance

$$\frac{1}{R_e} = \text{Permeance, its unit is } \frac{\text{Weber}}{\text{amp.}} = \text{henrys}$$

and $\phi_B = \text{mmf} \times \text{permeance}$

$$\text{Reluctance} = \frac{l}{\mu A}$$

and $$\text{Resistance} = \rho \frac{l}{A} = \frac{1}{\sigma} \frac{l}{A}$$

where σ is electrical conductivity, μ plays the same part in magnetic circuit as σ does in electrical circuit.

Combination of reluctances

If a magnetic circuit consists of different materials, with different length and cross-section. The combined reluctance is obtained by adding individual reluctances, as we do for resistances in series. Thus in a magnetic circuit made up of different materials of permeabilities $\mu_1, \mu_2 \ldots$ of lengths $l_1, l_2 \ldots$ and of cross-section $A_1, A_2, \ldots$, the total reluctance is given by,

$$R_e = \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2}$$

or $$R_e = \Sigma \frac{l_1}{\mu A} \qquad \ldots(14.63)$$

Points of difference in two circuits

In spite of the above analogy in two circuits with regard to various terms, they differ on the following points:

1. In an electric circuit, the current represents the actual flow of electrons or ions whereas in magnetic circuit there is no such flow because free poles do not exist.
2. In an electric circuit, energy is required to create as well as maintain the electric current. But in magnetic circuit the energy is required only for creation of the flux.
3. Though apparently μ corresponds to σ, the specific conductivity but actually μ in magnetic process corresponds to ε which is the absolute permittivity of the medium in electrical process.
4. An important point of difference is, that the resistance is independent of the current in the circuit. As μ is not constant, the reluctance of magnetic circuit depends upon the flux.

Magnetic circuit with Air Gap

Consider an iron ring of permeability μ, having mean circumference $l\ (= 2\pi r)$ and area of cross-section A and a narrow air gap of thickness d as shown in Fig. 14.16. The field H_g in the gap, $H_g = \frac{B}{\mu_0}$ while the field H_i in the iron, $H_i = \frac{B}{\mu}$

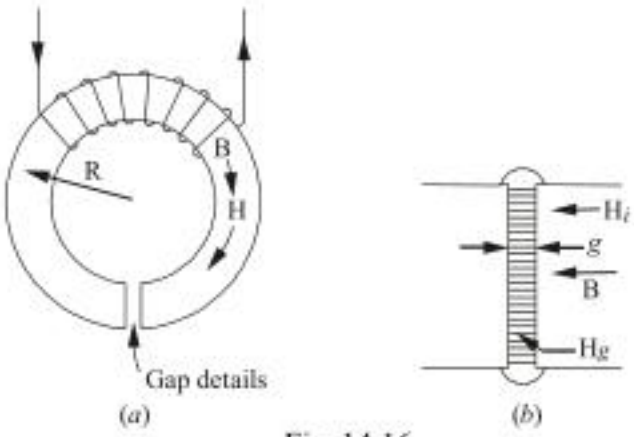


Fig. 14.16

$$H_i = \frac{B}{\mu} = \frac{B}{\mu_r \mu_0} = \frac{H_g}{\mu_r}$$

or $$\frac{H_g}{H_i} = \mu_r \qquad \ldots(14.64)$$

The reluctance of air gap $= \frac{d}{\mu_0 A}$

The reluctance of iron $= \frac{l-d}{\mu A}$

Total reluctance $= \frac{d}{\mu_0 A} + \frac{l-d}{\mu A}$

The magnetic flux, $\phi_B = \frac{NI}{\frac{d}{\mu_0 A} + \frac{l-d}{\mu A}}$ $\qquad \ldots(14.65)$

In the absence of air gap,

$$\phi_B = \frac{NI}{l/\mu A} \qquad \ldots(14.66)$$

The comparison of equations (14.65) and (14.66) shows that the magnetic flux in the circuit with air gap is less than that in the circuit without any air gap. *Thus with air gap flux is considerably reduced.*

Magnetic Circuit of an Electromagnet

Fig. 14.17 shows an electromagnet with an air gap and with different ferromagnetic materials of various shapes and lengths. With the idea of magnetic circuit it is easy to calculate the flux by knowing the total reluctance of the circuit. The total reluctance is given by,

$$\frac{l_1}{\mu_1 A_1} + \frac{2l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A_3}$$

where A_1, A_2 and A_3 are areas of cross-section for three parts of the electromagnet. Hence by equation (14.61),

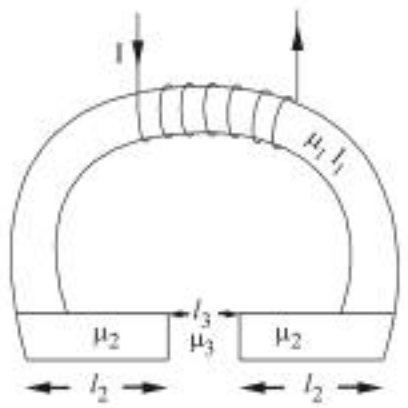


Fig. 14.17

$$\text{Flux } \phi_B = \frac{\text{mmf}}{\frac{l_1}{\mu_1 A_1} + \frac{2l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A}} = \frac{\text{mmf}}{\Sigma \frac{l}{\mu A}} \qquad \ldots(14.67)$$

As μ_3 corresponds to air gap $\mu_3 = \mu_0$, because μ_r the relative permeability for air is one. However, the calculation is only an approximation because leakage will take place in air gap. If air gap is very

thin leakage is negligible and calculations are more reliable. *The small air gap l_3 increases the reluctance of the circuit by a large amount.*

Calculation of Flux due to a Close Ring

Such a ring is shown in Fig. 14.18. With different areas of cross-section, A_1, A_2, A_3, ... at different points the corresponding flux densities are B_1, B_2, B_3. Assuming that there is no flux leakage and using Gauss's law for magnetism, *i.e.*,

$$\oint_s \vec{B} \cdot d\vec{S} = 0$$

i.e., the number of lines of $\vec{B}$ leaving any volume enclosed by a surface will be the same as that entering it. Hence we get,

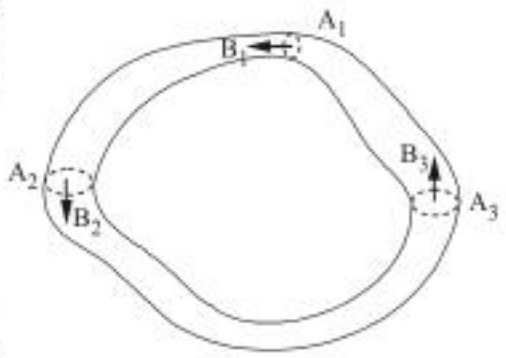


Fig. 14.18

$$\phi_B = B_1 A_1 = B_2 A_2 = B_3 A_3 \ldots = B_n A_n$$

Example 14.2 *Find the reluctance and permeance between the ends of the rectangular block of iron shown in Fig. 14.19. Assuming that B is uniform throughout the block and normal to the ends, permeability of the block is also uniform and has a value = 500 μ_0.*

Solution. $l = 0.1$ metre, $A = 3\times5\times10^{-4}$ metre^2

Reluctance,

$$R_e = \frac{l}{\mu A} = \frac{0.1}{500\times4\pi\times10^{-7}\times15\times10^{-4}}$$

$$= 1.06\times10^5 \frac{1}{\text{Henries}}$$

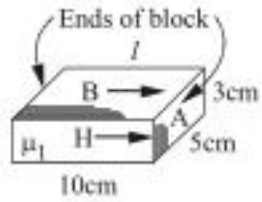


Fig. 14.19

$$\text{The permeance} = \frac{1}{R_e}$$

$$= \frac{1}{1.06\times10^5} = 9.4\times10^{-6} \text{ henry}$$

Example 14.3 *Find the total reluctance and permeance between the ends of the series connected rectangular iron blocks shown in Fig. 14.20. Assuming that B is uniform throughout the block and normal to the ends, given $u_1 = 500\mu_0$, $\mu_2 = 2000\mu_0$.*

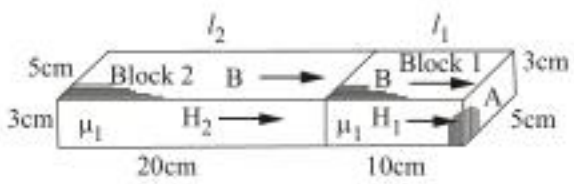


Fig. 14.20

Solution. The reluctance R_{e1} of block 1 is given in previous example. The reluctance of block 2,

$$R_{e2} = \frac{l_2}{\mu_2 A} = \frac{0.2}{2000\times4\pi\times10^{-7}\times15\times10^{-4}}$$

$$= 0.53\times10^5 \frac{1}{\text{Henries}}$$

The total reluctance $= R_{e1} + R_{e2}$

$$R_{eT} = (1.06 + 0.53)\,10^5 \frac{1}{\text{Henries}}$$

$$= 1.59 \times 10^5 \frac{1}{\text{Henries}}$$

The total permeance $= \frac{1}{R_{eT}} = \frac{1}{1.59 \times 10^5}$ Henry

$$= 6.3 \times 10^{-6} \text{ henry}$$

Example 14.4 *If in example 14.2, H = 1000 amp./metre then calculate the magnetic potential difference between the ends of the bar and total flux.*

Solution. $V_m = \int \vec{H} \cdot d\vec{l}$, as H is along the length of the bar, then $\vec{H} \cdot d\vec{l} = H dl$. Magnetic potential between the ends,

$$V_m = \int_0^{0.1} H dl = 1000 \times 0.1 = 100 \text{ amp.}$$

The flux density $B = \mu_1 H = 500\,\mu_0 \times 1000$ weber/metre^2

$$= 6.28 \times 10^{-1} \text{ weber/metre}^2$$

The total flux $= BA$

$$= 6.28 \times 10^{-1} \times 15 \times 10^{-4}$$

$$= 9.4 \times 10^{-4} \text{ weber}$$

Example 14.5 *In example 14.3 if B = 1 weber/metre^2, then calculate the potential difference between the ends of blocks.*

Solution. From boundary condition the normal component of B is continuous in block 1

$$H_1 = \frac{B}{\mu_1} = \frac{1}{500\mu_0} = 1.59 \times 10^3 \text{ amp./metre}$$

and potential difference

$$V_{m1} = H_1 l_1 = 1.59 \times 10^3 \times 0.1 = 159 \text{ amp.}$$

In block 2,

$$H_2 = \frac{B}{\mu_2} = \frac{1}{20000\mu_0} = 3.97 \times 10^2 \text{ amp/metre}$$

and potential difference V_m between the end faces of block 2,

$$V_{m2} = H_2 l_2 = 3.97 \times 10^2 \times 0.2 = 79.4 \text{ amp.}$$

The total potential difference across both blocks is then given by,

$$V_m = V_{m1} + V_{m2}$$

$$= 159 + 79.4 = 238.4 \text{ amp.}$$

Example 14.6 *Find the total reluctance and permeance between the ends of parallel-connected rectangular iron blocks shown in Fig. 14.21* $\mu_1 = 500\,\mu_0$ *and* $\mu_2 = 2000\,\mu_0$.

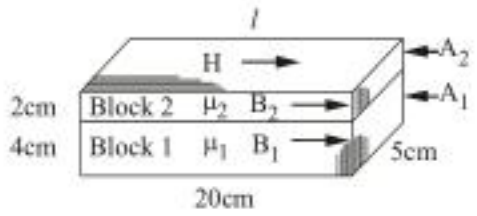


Fig. 14.21

Solution. Since the blocks are connected in parallel, it is convenient to calculate the total permeance first.

$$\text{Permeance of block 1} = \frac{\mu_1 A_1}{l}$$

$$= \frac{500 \times 4\pi \times 10^{-7} \times 20 \times 10^{-4}}{0.2}$$

$$= 6.28 \times 10^{-6} \text{ henry}$$

$$\text{Permeance of block 2} = \frac{\mu_2 A_2}{l}$$

$$= \frac{2000 \times 4\pi \times 10^{-7} \times 10 \times 10^{-4}}{0.2}$$

$$= 12.6 \times 10^{-6} \text{ henry}$$

$$\text{Total permeance} = (6.28 + 12.6)\,10^{-6} \text{ henry}$$

$$= 1.89 \times 10^{-5} \text{ henry}$$

The total reluctance is given by

$$= \frac{1}{1.89 \times 10^{-5}} = 5.3 \times 10^{4} \frac{1}{\text{henry}}$$

Example 14.7 *In the above example 14.6 if H = 1000 amp/metre, calculate the total flux.*
Solution. Since from boundary condition the tangential component of H is continuous that of H is same in both blocks, then,

$$B_1 = \mu_1 H = 500\mu_0 \times 1000 = 0.628 \text{ weber/metre}^2$$

$$\phi_{B1} = B_1 A_1 = 0.628 \times 20 \times 10^{-4} = 1.26 \times 10^{-3} \text{ weber}$$

and $$B_2 = \mu_2 H = 2000\mu_0 \times 1000 = 2.52 \text{ weber/metre}^2$$

$$\phi_{B_2} = B_2 A_2 = 2.52 \times 10^{-3} \text{ weber}$$

$$\text{Total flux} = \phi_{B1} + \phi_{B2} = 3.78 \times 10^{-3} \text{ weber.}$$

Example 14.8 *An iron ring has a cross-sectional area 10 cm² and air gap of width d = 2 mm and mean length* $l = 2\pi R = 60$ *cm including the air gap. Find the number of turns required to produce flux density. B = 1 weber/metre², ,* $\mu_{iron} = 10^{-3}$ *henry/metre.*

Solution. From equation (14.65),

$$\text{Total flux} = \frac{\text{mmf}}{\text{Reluctance}}$$

$$\phi_B = \frac{NI}{\frac{d}{\mu_0 A} + \frac{l-d}{\mu A}}$$

$$\text{Flux density, } \phi_B = BA = \frac{NI}{\frac{d}{\mu_0 A} + \frac{l-d}{\mu A}}$$

or
$$NI = BA\left(\frac{d}{\mu_0 A} + \frac{l-d}{\mu A}\right)$$

$$= B\left(\frac{d}{\mu_0} + \frac{l-d}{\mu}\right)$$

$$= 1 \times \left(\frac{2\times10^{-6}}{4\pi\times10^{-7}} + \frac{598\times10^{-6}}{10^{-3}}\right)$$

$$= 2188 \text{ amp.turns.}$$

Example 14.9 *An iron rod of length 50 cm and cross-sectional area 4 sq. cm is in the form of a closed circular ring. If the permeability of iron is* 65×10^{-4} *h/m, calculate the number of ampere turns required to produce a flux of* 4×10^{-4} *weber.*

Solution. Reluctance $= \frac{l}{\mu A} = \frac{0.5}{65\times10^{-4}\times4\times10^{-4}}$

$$= 1.92\times10^{5} \text{ amp. turns/weber}$$

$$\text{mmf} = NI$$

$$\text{Flux } \phi_B = \frac{\text{mmf}}{\text{Reluctance}}$$

$$4\times10^{-4} = \frac{NI}{1.92\times10^{5}}$$

$$NI = 77 \text{ amp turns}$$

Example 14.10 *The magnetic susceptibility of medium is 948 × 10*$^{-11}$*. Calculate the permeability (or absolute permeability) and relative permeability.*

Solution. Hint: $\chi_m - 948\times10^{-11}$

Absolute permeability,

$$\mu = \mu_0 (1+\chi_m)$$

$$= 4\pi\times10^{-7} (1+948\times10^{-11}) \text{ henry/m}$$

$$= 4\pi\times10^{-7}\times949\times10^{-11} \text{ henry/m}$$

$$= 1.186\times10^{-14} \text{ henry/metre}$$

and relative permeability,

$$\mu_r = \frac{\mu}{\mu_0}$$

$$= \frac{4\pi \times 10^{-7} \times 949 \times 10^{-11}}{4\pi \times 10^{-7}}$$

$$= 949 \times 10^{-11}$$

14.22 THEORIES OF MAGNETISM

The fields of permanent magnets (outside the magnets) may be described either in terms of equivalent currents or equivalent poles*. Since the entire description of magnetism has been based on the assumption that they are produced by moving charges, we are led to believe that the interpretation of the field of a permanent magnet in terms of electric charges giving rise to atomic currents is a more fundamental one than the concept of magnetic charges. The electrons have at least two motions in an atom giving rise to two kinds of currents listed below:

1. The currents due to orbital motion of the electrons in atoms.
2. The currents due to spinning of electrons about their own axes.

Such circulating currents were first postulated by Ampere, and are often called amperian currents.

The question now arises as to which is more important, the orbital motion of the electrons or the electronic spin. The electronic spin takes precedence in an explanation of the behaviour of magnetic materials.

In many atoms the electrons are completely paired, *i.e.* for each electron spinning in one direction there is an electron spinning in the opposite direction. And the same situation exists in regard to the orbital motion of the electrons. Thus, the net current circulating about any given axis is zero and there are no amperian currents in such substances. Such materials exhibit very weak magnetic effect, they are called "non-magnetic". In other words, the electron spins and orbital motions are exactly balanced out, so that any particular atom has no average amperian current. Such substances when placed in magnetic field little extra amperian currents are generated inside the atom by induction, due to the changes in the angular velocity of the electrons in their orbits. According to Lenz's law, these currents are in such a direction as to oppose the applied field. This effect is known as "diamagnetism".

In a magnetic substance, however, the cancellation is incomplete and the amperian currents are not zero. Obviously those substances whose atoms have an odd number of electrons must have at least one uncompensated spinning electron. These electrons have the same effect as tiny electric currents known as amperian currents.

In other words, a magnetic material contains tiny amperian currents distributed throughout its entire volume. *Magnetisation consists in the alignment of these amperian currents*. If the magnetic field due to this amperian current has the same direction as the applied magnetic field, obviously the field is increased by the presence of the amperian current. This magnetic effect is known as "paramagnetism".

However, the paramagnetism arising from the spins of the electrons responsible for conduction in a metal constitutes an exception, we will not be discussing the phenomena here. Paramagnetism is generally fairly weak because lining up forces are relatively small comparaed with the forces from the thermal motions which try to change the order.

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*There is no basic objection to the existence of magnetic poles. Their fields are simply not considered here since there is no experimental evidence as to their existence.

In the case of iron there are as many as four uncompensated spinning electrons within each atom of molecule and consequently, iron tends to be "strongly magnetic". Substances that behave like iron are called "ferromagnetic".

Langevin Theory of Diamagnetism

Let us now examine how the motion of the electron in its obrit is affected by an applied field $\vec{B}$ perpendicular to the plane of the orbit. If A is the area of the orbit, the magnetic flux is BA. According to Faraday's law of electromagnetic induction induced emf ξ is given by,

$$\xi = \frac{d\phi}{dt} = A\frac{dB}{dt} \qquad \ldots(1)$$

This emf changes the motion of the electrons in their orbits. Correspondnig to this induced emf, there is an electric field which acts tangentially to the direction of the motion of the electrons. The line integral to this field E around the orbit is equal to the induced emf.

$$\xi = \oint \vec{E} \cdot d\vec{l} = E \oint dl = 2\pi r E$$

or $$E = \frac{\xi}{2\pi r}$$

where r is the radius of the orbit. The electrons are accelerated by this field and experience a force eE. Hence by Newton's law,

$$m_e \frac{dv}{dt} = eE = \frac{e\xi}{2\pi r}$$

and ξ is given by equation (1), hence

$$m_e \frac{dv}{dt} = \frac{e}{2\pi r}\frac{d\phi}{dt} \qquad \ldots(2)$$

where m_e is mass of the electron and v is velocity. Consider, during the time flux changes from 0 to ϕ, the velocity of electron changes from v_0 to $v_0 + \Delta v$. Integrating the equation (2), we get,

$$\int_{v=v_0}^{v=v_0+\Delta v} \frac{dv}{dt}\,dt = \frac{e}{2\pi r}\int_0^{\phi} \frac{d\phi}{dt}\,dt$$

or $$m_e \Big| \; v \; \Big|_{v_0}^{v_0+\Delta v} = \frac{e}{2\pi r}\Big| \; \phi \; \Big|_0^{\phi}$$

or $$\Delta v = \frac{e}{2\pi m_e r}\phi \qquad \ldots(3)$$

This change Δv gives rise to change in angular velocity of the electron. Thus, the change in angular velocity of the electron.

$$\Delta\omega_L = \frac{\Delta v}{r} = \frac{e}{2\pi m_e r^2}\phi$$

But $\phi = BA$ and $\pi r^2 = A$, where A is the area of the orbit, we get,

$$\Delta\omega_L = \frac{eBA}{2m_e A} = \frac{e}{2m_e}B \qquad \ldots(14.68)$$

This change in angular velocity is called "**Larmar angular velocity**" or in terms of frequency "**Larmar frequency**". However, the change in angular velocity acts in sense opposite to that of electron.

Larmar has shown that the effect of the applied field is a "precessional motion" of the orbit without change in its form. The centripetal force necessary to hold the electron in its orbit is given by,

$$F = \frac{m_e v_0^{\,2}}{r} = m_e r\,\omega_0^{\,2}$$

where ω_0 is the angular frequency of the electron in the orbit without magnetic field. When a magnetic field is applied the angular frequency changes by $\Delta\omega_L$. *The condition of stable motion in the same orbit i.e. if electron is still in the orbit of the same radius, its centripetal force must increase correspondingly i.e. the centripetal force,*

$$F' = m_e r\,(\omega_0 + \Delta\omega_L)^2$$

$$= m_e r\omega_0^{\,2} + 2m_e r\omega_0\Delta\omega_L + m_e r\,\Delta\omega_L^{\,2}$$

Assuming that $\Delta\omega_L \ll \omega_0$ and this corresponds to small value of B thus, term with $\Delta\omega_L^{\,2}$ can be neglected. Hence,

$$F' = m_e r\omega_0^{\,2} + 2m_e r\omega_0\,\Delta\omega_L \qquad \ldots(14.69)$$

when a magnetic field is applied an additional force acts on the electron.
The magnitude of this force is given by,

$$F_B = evB$$

or $$F_B = er\,(\omega_0 + \Delta\omega_L)\,B \qquad \ldots(14.70)$$

Since $v = \omega r$ and $\omega = \omega_0 + \Delta\omega_L$, F_B is perpendicular to both $\vec{v}$ and $\vec{B}$. If electron is moving in anti-clockwise direction and $\vec{B}$ directed out from the orbit, the magnetic force F_B will be directed towards the centre of the orbit (*i.e.* $\vec{B}$ along *z*-axis, $\vec{v}$ along negative *y*-axis hence $F_B = e(\vec{v}\times\vec{B})$ is along + *x*-axis). If electron is moving in clockwise direction then this force is outward as shown in Fig. 14.22.

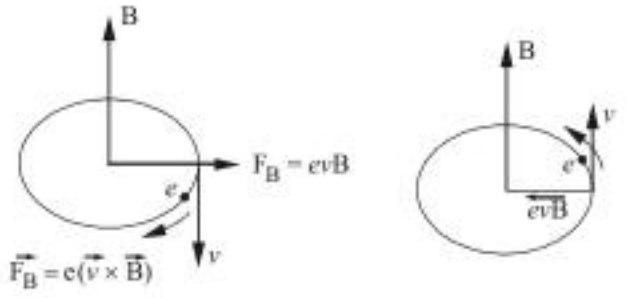


Fig. 14.22

Putting the value of B from equation (14.68) in eqn. (14.70),

$$F_B = er\,(\omega_0 + \Delta\omega_L)\,\frac{2m_e}{e}\,\Delta\omega_L$$

$$= 2m_e r\omega_0\,\Delta\omega_L + 2m_e r\,\Delta\omega_L^{\,2}$$

The term $\Delta\omega_L^{\,2}$ is neglected because of the same reasons. Thus,

$$F_B = 2\, m_e\, r\omega_0\, \Delta\omega_L \qquad \ldots(14.71)$$

By equation (14.69), the increase in centripetal force,

$$F' - m_e r\omega_0^{\,2} = 2m_e r\omega_0 \Delta\omega_L \qquad \ldots(14.72)$$

The comparison of equations (14.72) and (14.71) shows that the increase in centripetal force is completely balanced by the additional inward magnetic force. Hence the centripetal *force does not change by the application of B. Therefore the radius of the orbit will remain the same.*

A charge e revolving in a circle at the rate of n revolutions per second is "*equivalent*" to a current ne. If A is the area of the circle, the "*magnetic moment*" of the electron orbit is given by,

$$m = ne\,A, \quad (\text{since } m = IA)$$

The change in the magnetic moment dm due to change $\Delta\omega_L$ in angular velocity is given by,

$$dm = e\,A\,\Delta n. \text{ Since } n = \frac{\omega}{2\pi},\ \Delta n = \frac{\Delta\omega_L}{2\pi}$$

$$= eA\frac{\Delta\omega_L}{2\pi}$$

Substituting $\Delta\omega L$ from eqn. (14.68), we get,

$$dm = eA\frac{eB}{4\pi m_e}$$

$$= \frac{e\pi r^2 eB}{4\pi m_e} \quad (\text{since } A = \pi r^2)$$

$$= \frac{e^2 r^2}{4m_e} B \qquad \ldots(14.73)$$

But diamagnetism is an induced effect. This magnetic moment is induced and is always opposite to $\vec{B}$, hence in vector form we can write,

$$\vec{dm} = -\frac{e^2 r^2}{4m_e}\vec{B} \qquad \ldots(14.74)$$

We have derived this equation (14.74) for one electron. It can be extended to an atom having more than one electron. However, in deriving this equation, we have assumed that the orbit is perpendicular to the field. Since the orbit can have any orientation with respect to the field, the radius of the orbit in this equation should be replaced by its projection r_1 on the plane perpendicular to the field, *i.e.*

$$\vec{dm} = -\frac{e^2 r_1^2}{4m_e}\vec{B} \qquad \ldots(14.75)$$

As there are number of randomly oriented orbits within the atom, thus the total magnetic moment induced,

$$\vec{M}_A = \frac{e^2\vec{B}}{4m_e}\Sigma r_1^2 \qquad \ldots(14.76)$$

where summation is taken for all orbits. Suppose (x, y, z) are the coordinates of any point on an orbit of radius r, then,

$$r^2 = x^2 + y^2 + z^2$$

If we choose z-axis in the plane perpendicular to the applied magnetic field, then

$$r_1^2 = x^2 + y^2$$

Let the mean value of the radii of all orbits be $\bar{r}$,

$$\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$$

For spherically symmetrical atom,

$$\bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \frac{1}{3}\bar{r}^2$$

hence, $r_1^2 = \bar{x}^2 + \bar{y}^2 = \frac{2}{3}\bar{r}^2$

Substituting the value of r_1^2 in eqn. (14.76), we get,

$$\vec{M}_A = -\frac{e^2 \vec{B}}{6m_e} \Sigma \bar{r}^2 \qquad \ldots(14.77)$$

For a substance containing n molecules per unit volume, the total magnetic moment induced per unit volume of the substance,

$$\vec{M}_v = -\frac{e_n^2 \vec{B}}{6m_e} \Sigma \bar{r}^2 \qquad \ldots(14.78)$$

By equation (14.30), the susceptibility,

$$\chi_m = \frac{\text{Magnetic moment}}{H}$$

The diamagnetic susceptibility per atom is called Atomic susceptibility denoted by χ_{mA}. Then,

$$\chi_{mA} = \frac{\text{Induced magnetic moment per atom } M_A}{H}$$

Substituting M_A from equation (14.77), we get,

$$\chi_{mA} = -\frac{e^2 B}{6m_e H} \Sigma \bar{r}^2$$

For diamagnetic substance, we have linear relation,

$$B = \mu H$$

Hence, $\chi_{mA} = -\frac{\mu e^2}{6m_e} \Sigma \bar{r}^2 \qquad \ldots(14.79)$

Similarly, the volume susceptibility,

$$\chi_{mv} = -\frac{n\mu e^2}{6m_e} \Sigma \bar{r}^2 \qquad \ldots(14.80)$$

The scrutiny of the above results leads to the following conclusions:

1. The expression for susceptibility is independent of temperature.
2. The diamagnetism arises from the effect of magnetic field on the motion of electrons in the orbit and every substance having moving electrons in the orbit. Hence, every substance possesses diamagnetism, but diamagnetism is a feeble effect and overridden by the paramagnetism and ferromagnetism producing a flux in the same direction as that externally applied field.
3. The negative sign indicates that diamagnetism is an induced opposing effect, that disappears as soon as the field is removed.

14.23 LANGEVIN'S THEORY OF PARAMAGNETISM

As we have pointed out earlier that the molecules or atoms with permanent magnetic dipole moment are termed paramagnetic. For such molecules there is net atomic (or amperian) current which gives rise to the magnetic dipoles having magnetic dipole moment. They have amperian current, because the orbit of the electrons is so oriented that the amperian currents due to their orbital motion as well as spin do not cancel out completely. This happens with all the atoms and molecules having odd number of electrons for them the total spin is not zero. In most bulk materials, there is a net magnetic moment only if there are atoms present whose inner electron shell is not filled.

Langevin theory of paramagnetism is just like the Langevin theory of dielectrics (polar-molecules). Suppose each atom or molecule possesses a permanent magnetic moment m. When an external magnetic field is applied, the *dipoles align along the direction of applied field*. Let the axis of dipole be making an angle θ with the direction of applied field B. The magnetic potential energy is given by,

$$U = -\vec{m} \cdot \vec{B} \quad \text{(see eq. 9.14)}$$

$$= -m\mathrm{B}\cos\theta$$

However, the alignment of dipoles will be counteracted by the collisions of thermal motion between the molecules which knock the axes out of line. As a result, there will be an equilibrium distribution of the axes with reference to the direction of the field, which can be determined with the help of the kinetic theory of gases as follows.

According to the kinetic theory principle, the number of molecules distributed with the axis of the dipole pointing in all directions within a solid angle Ω at an angle θ with a given line of reference is proportional to $d\Omega$. And on the basis of Boltzmann theorem, the number of molecules whose potential energy is v is proportional to $e^{-U/KT}$.

Considering unit volume of the gas, if dN out of the total number N of the dipoles in it have their magnetic axes inclined at the angle θ with the field direction, then, applying the principles of kinetic theory stated above, we get,

$$d\mathrm{N} \propto e^{-U/KT} d\Omega$$

$$= \mathrm{C} e^{-U/KT} d\Omega$$

where C is constant depending on the number of molecules considered, U is potential energy, K is Boltzmann constant and T is absolute temperature. We have calculated in equation (6.86 a) that

$$d\Omega = 2\pi \sin\theta \, d\theta \text{ and } U = -m\,\mathrm{B}\cos\theta$$

After substitution, we get

$$d\mathrm{N} = \mathrm{C} \exp\left(\frac{m\mathrm{B}\cos\theta}{\mathrm{KT}}\right) 2\pi \sin\theta \, d\theta$$

putting,

$$\frac{m\mathrm{B}}{\mathrm{KT}} = x \qquad \ldots(14.81)$$

The component of the magnetic moment of one molecule in the direction of field is $m\cos\theta$. Therefore, the total magnetic moment of dN molecule, along the field direction, is,

$$d\mathrm{M} = d\mathrm{N}\, m\, \cos\theta$$

$$= \mathrm{C} e^{x\cos\theta} 2\pi \sin\theta \, m \cos\theta \, d\theta$$

$$= 2\pi\, \mathrm{C}\, m \sin\theta \cos\theta \, e^{x\cos\theta} d\theta \qquad \ldots(14.82)$$

Since the possible orientations are contained between 0 and π, hence the total magnetic moment M of all molecules in unit volume in the direction of field is obtained by integrating the above expression,

$$M = \int_0^{\pi} 2\pi C m \sin\theta \cos\theta \, e^{x\cos\theta} d\theta$$

$$= 2\pi C m \int_0^{\pi} e^{x\cos\theta} \sin\theta \cos\theta \, d\theta \qquad \ldots(14.83)$$

The total number of molecules N in unit volume can be found by integrating dN,

$$N = \int_0^{\pi} dN = \int_0^{\pi} 2\pi C e^{x\cos\theta} \sin\theta \, d\theta$$

The mean effective magnetic moment,

$$\overline{M} = \frac{M}{N} = \frac{2\pi C m \int_0^{\pi} e^{x\cos\theta} \sin\theta \cos\theta \, d\theta}{\pi}$$

$$= 2\pi C \int_0^{\pi} e^{x\cos\theta} \sin\theta \, d\theta$$

$$= m \frac{\int_0^{\pi} e^{x\cos\theta} \sin\theta \cos\theta \, d\theta}{\int_0^{\pi} e^{x\cos\theta} \sin\theta \, d\theta}$$

To evaluate the integral, put $\cos\theta = a$ and proceeding exactly as in the case of Langevin theory of dielectrics, discussed in Article 6.25, we get

$$\frac{\overline{M}}{m} = \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x}$$

$$= \text{Coth}\, x - \frac{1}{x} = L(x) \qquad \ldots(14.84)$$

where $L(x)$ is called "Langevin function". When x is small,

$$L(x) = \frac{x}{3} - \frac{x^3}{4\times 5} + \cdots$$

Under normal experimental conditions x is very small,

$\therefore$ $$L(x) = \frac{x}{3}$$

$$\frac{\overline{M}}{m} = L(x) = \frac{mB}{3KT}, \text{ since } x = \frac{mB}{KT}$$

or $$\overline{M} = \frac{m^2 B}{3KT} \qquad \ldots(14.85)$$

For paramagnetic substances,

$$B = \mu H$$

Hence,

$$\overline{M} = \frac{m^2 \mu H}{3KT} \qquad \ldots(14.86)$$

The volume susceptibility of the paramagnetic substance,

$$\chi_{mv} = \frac{m^2 \mu n}{3KT}, \left(\text{since } \frac{M}{H} = \chi_m \right) \qquad \ldots(14.87)$$

where n are the number of molecules per unit volume.

Mass susceptibility is the susceptibility per unit mass of the substance,

$$\chi_m = \frac{n_m \mu m^2}{3KT} \qquad \ldots(14.88)$$

where n_w are the number of molecules per unit mass of the gas.

The molecular susceptibility,

$$\chi_m = \frac{N_w \mu m^2}{3KT} \qquad \ldots(14.89)$$

where N_w are the number of molecules in a gram molecule (Avogadro's number),

or $$\chi_m = \frac{C_w}{T} \qquad \ldots(14.90)$$

where C_w is called the "Curie constant" given by,

$$C_w = \frac{N_m^{\ 2} \mu m^2}{3R}, \text{ since } K = \frac{R}{N_w} \qquad \ldots(14.91)$$

χ_m is referred as "molar susceptibility" *i.e.* susceptibility for one gram molecules of the substance.

Thus, we see that susceptibility depends on the temperature. However, the total magnetisation of a paramagnetic material must include the induced moments, and adding their contribution given in Eq. (14.77) to Eq. (14.85), we get,

$$\vec{M} = \left(\frac{m^2}{3KT} - \frac{e^2}{6m_e} \Sigma \bar{r}^2 \right) \vec{B} \qquad \ldots(14.91\ a)$$

Similarly, the total susceptibility of a paramagnetic material must include the induced susceptibility.

Failure of Langevin Theory

Langevin theory could not explain a more complicated dependence of susceptibility upon temperature exhibited by several paramagnetics such as highly compressed and cooled gases, very concentrated solutions of salts, solid salts and crystals. Langevin theory also did not explain the intimate relation observed between para and ferromagnetism.

14.24 WEISS MOLECULAR FIELD THEORY OF PARAMAGNETISM

As stated above, the Langevin theory did not explain the intimate relation observed between para and ferromagnetism. In an attempt to explain these points, Weiss in 1907, improved upon Langevin's treatment by the introduction of a new concept of internal molecular field. Weiss assumed that, since in a real gas the molecules are mutually influenced by their magnetic moments, there should exist

within the gas a molecular field, produced at any point by all the molecules in the neighbourhood, proportional to and acting in the same sense as the magnetisation vector $\overline{M}$. Therefore, we have,

$$\text{Molecular field} \propto \text{Magnetisation } \overline{M}$$

$$= \gamma \overline{M}$$

where γ is a constant (molecular field coefficient). Then the effective field may be regarded as the vector sum of the external applied field $\overline{B}$ and internal molecular field *i.e.*

$$B_{eff} = \overline{B} + \gamma \overline{M} \qquad \ldots(14.92)$$

Replacing B in equation (14.85) by B_{eff}, we get,

$$M = \frac{m^2 (B + \gamma M)}{3KT}$$

The magnetic moment in a gram molecule,

$$M = \frac{N_m m^2 (B + \gamma M)}{3KT}$$

where N_m are the number of molecules in a gram molecule. Rearranging the terms

$$M\left(1 - \frac{N_m m^2 \gamma}{3KT}\right) = \frac{N_m m^2 B}{3KT}$$

$$M = \frac{N_m m^2 B}{3KT\left(1 - \frac{N_m m^2 \gamma}{3KT}\right)}$$

$$= \frac{N_m m^2 B}{(3KT - N_m m^2 \gamma)}$$

$$= \frac{N_m m^2 B}{3K\left(T - \frac{N_m m^2 \gamma}{3K}\right)}$$

But $B = \mu H$

Therefore, $M = \dfrac{N_m m^2 \mu H}{3K\left(T - \dfrac{N_m m^2 \gamma}{3K}\right)}$

Molecular susceptibility,

$$\chi_m = \frac{M}{H} = \frac{N_m m^2 \mu}{3K\left(T - \frac{N_m m^2 \gamma}{3K}\right)}$$

$$= \frac{N_m^{\,2} m^2 \mu}{3R\left(T - \frac{N_m^{\,2} m^2 \gamma}{3R}\right)}, \text{ since } K = \frac{R}{N_m}$$

$$C_w = \frac{N_w^{\,2} m^2 \mu}{3R}$$ is Curie constant (see eqn. 14.91)

and $$\theta = \frac{N_w^{\,2} m^2 \gamma}{3R}$$

then, $$\chi_m = \frac{C_m}{T - \theta} \qquad \ldots(14.93)$$

This is the Curie-Weiss law. According to this law, the susceptibility of a paramagnetic substance having molecular field varies inversely as the excess of temperature not from absolute zero, as in the case of Curie's law, but from a certain critical temperature, the Curie point. The Curie point is able to explain the intimate relation observed between paramagnetic and ferromagnetic substances. Below the Curie point θ, the ferromagnetism exists, but above the Curie point, the ferromagnetic properties disappear and the substance becomes paramagnetic.

14.25 DOMAIN THEORY OF FERROMAGNETISM

Magnetic effects in most substances are weak. However, a group of substances known as ferromagnetic materials exhibit strong magnetic effects. The ferromagnetism is really an extreme case of paramagnetism. If the permanent dipoles, generally those resulting from electron spin, are very close together in the substance, there proves to be an effect (explained only by Quantum Theory), called "exchange", which results in a strong tendency for the spins of adjacent atoms or molecules to line up parallel to each other, even in the absence of an external field. Such a parallel orientation can extend, in an unmagnetised substance, over volumes of a considerable scale on an atomic order of magnitude (with physical dimensions of the order of 10^{-3} to 10^{-4} cm), though a small volume by ordinary standards. Such a volume is called a "domain", and an ordinary unmagnetised ferromagnetic substance consists of many domains. A strong permanent moment is associated with each domain. The domains are oriented in different directions.

In the presence of an external magnetic field, the domains change the orientation of their permanent moments, lining them up with the external fields. As the applied field becomes stronger, the moment increases, until finally with a very large external field the moment reaches a limit when all moments are parallel. This limit is called the "saturation moment". Reversing the field reverses the moments but there is an effect similar to friction, hindering this reorientation, so that by the time the external field is reduced to zero, the substance still retains a considerable moment. This is the origin of "permanent magnetism". If the external field is reversed alternately between one direction and the other, the moment lags behind the field, resulting in the phenomenon of the "hysteresis".

These properties of ferromagnetic substances are very complicated in contrast to the diamagnetic and paramagnetic substances, in which the moment is proportional to the field. The ferromagnetism decreases as the temperature increases because the individual domains lose their moments at a critical temperature called the Curie temperature (about 770° C for iron). Above the Curie temperature the substance becomes paramagnetic, but the moment instead of being proportional to 1/T, where T is the absolute temperature, is proportional to $1/T - \theta$, where θ is the Curie temperature. The physical explanation of this decrease of ferromagnetism as the temperature increases is that the tendency towards orientation of the magnetic moments which lines then up at low temperature is opposed by thermal agitation at high temperature.

Thus, the domain theory of ferromagnetism furnishes a clear picture of the phenomena that occur in a ferromagnetic material.

Table 14.1: Three Magnetic Vectors

Name	*Symbol*	*Associated with*	*Boundary condition*
Magnetic induction of flux density	$\vec{B}$	All currents	Normal component continuous (Eq. 14.44)
Magnetic field strength	$\vec{H}$	True currents only	Tangential component continuous (Eq. 14.46)
Magnetisation (Magnetic dipole moment per unit volume)	$\vec{M}$	Magnetisation currents only	Vanishes in a vacuum (Eq. 14.28)
Defining equations for $\vec{B}$		$\vec{F} = q\vec{v} \times \vec{B}$	Eq. (9.2)
		$= I\vec{l} \times \vec{B}$	Eq. (9.6)
General relation among the three vectors		$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$	Eq. (14.22)
Ampere's law when magnetic materials are present		$\oint \vec{H} \cdot d\vec{l} = I$ (I = true current only)	Eq. (14.24)
Empirical relations for certain magnetic materials		$\vec{B} = \mu_r \mu_0 \vec{H}$	Eq. (14.27)
		$\vec{M} = (\mu_r - 1)\vec{H}$	Eq. (14.28)

Example 14.11 *In the row land ring 2.0 amp current (i_0) is passing in the windings and the number of turns per unit length (n) in the toroid are 10 turns/cm. B measured is 1.00 weber/metre^2. Calculate (a) $\vec{H}$, (b) $\vec{M}$ and (c) the magnetising current, both when the core is present and when it is removed. (d) What is μ_r?*

Solution. (*a*) As $\vec{H}$ is independent of the core material and is given by Ampere's law

$$\oint \vec{H} \cdot d\vec{l} = I$$

$\vec{H}$ is tangential to l

$$H \oint dl = NI$$

where N are the total number of turns on the toroidal windings and NI is enclosed current.

$$Hl = NI$$

$$H = \frac{N}{l} I = nI$$

$$= (10 \text{ turns/cm})\,(2.0 \text{ amp})$$

$$= 2.00 \text{ amp} \times 10 \times 100 \text{ turns/metre}$$

$$= 2000 \text{ amp/metre}$$

(*b*) The magnitude of $\vec{M}$ is obtained by relation

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$$

$$M = \frac{B - \mu_0 H}{\mu_0}$$

$$= \frac{1 - 4\pi \times 10^{-7} \times 2000}{4\pi \times 10^{-7}}$$

$$= 7.94 \times 10^5 \text{ amp/metre}$$

(*c*) The effective magnetising current from Eq. (14.19*a*),

$$i_m = M \cdot \left(\frac{2\pi r}{N}\right)$$

$$= \frac{M}{n}$$

$$= \frac{7.9 \times 10^5 \text{ amp/metre}}{2.0 \times 10^3 \text{ turns/metre}}$$

$$= 390 \text{ amp.}$$

Thus, an additional current of this amount in the windings would produce the same value of B in the absence of a core as that obtained with the core in place.

(*d*) The permeability

$$\mu_r = \frac{B}{\mu_0 H}$$

$$= \frac{1 \text{ weber/metre}^2}{(4\pi \times 10^{-7} \text{ web/amp.metre})\,(2 \times 10^3 \text{ amp/metre})}$$

$$= 397$$

14.26 COMPARISON OF STATIC ELECTRIC AND MAGNETIC FIELDS

In order to understand clearly, it is necessary to compare electric and magnetic fields and to note both their differences and their similarities.

1. *Analogy of $\vec{B}$ to $\vec{E}$*

By comparison, we see certain analogies. For example we have noted in electric fields that $\vec{E}$ is defined in terms of force, while in magnetic fields $\vec{B}$ is also defined in terms of force. Hence, $\vec{B}$ may be considered as the magnetic quantity that is analogous to the electric field intensity $\vec{E}$.

2. *Analogy of $\vec{H}$ to $\vec{D}$*

For example, in a capacitor $\vec{D}$ is directly related to the electric charge on the plates and is independent of the medium, while near a long current-carrying wire $\vec{H}$ (By $\vec{B} = \mu\vec{H}$) is directly related to the current ($B = \frac{\mu_0 I}{2\mu R}$ in vacuum and in any medium of permeability μ, $B = \frac{\mu I}{2\pi R}$, hence $H = \frac{I}{2\pi R}$) and is independent of the medium.

Thus, $\vec{D}$ and $\vec{H}$ may be regarded as analogous quantities.

3. *Analogy of $\vec{H}$ to $\vec{E}$*

However, in some other instances we may note an analogy of $\vec{H}$ to $\vec{E}$. For example, the line integral around a closed path of the total electric field intensity $\vec{E}$ is emf $\left(\oint \vec{E} \cdot d\vec{l} = \xi\right)$, while the line integral

of $\vec{H}$ around a closed path is magnetomotive force $\left(\text{mmf} = \oint \vec{H} \cdot d\vec{l}\right)$. Thus $\vec{E}$ and $\vec{H}$ may be regarded as analogous quantities.

4. *Analogy of $\vec{B}$ to $\vec{D}$*

Furthermore, in some instances we may note an analogy of $\vec{B}$ to $\vec{D}$. For example, the divergence relations,

$$\nabla \cdot \vec{D} = \rho$$

and $\nabla \cdot \vec{B} = 0$, everywhere.

For charge free space $(\rho = 0)$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \cdot \vec{B} = 0$$

We note a mathematical similarity of $\vec{B}$ to $\vec{D}$.

Important Comment

If the analogy of $\vec{H}$ to $\vec{E}$ and $\vec{B}$ to $\vec{D}$ is pursued, it is possible to achieve a formal or mathematical symmetry between many of the electric and magnetic field equations. *However, electric and magnetic fields are fundamentally different, and the analogy of $\vec{B}$ to $\vec{E}$ and $\vec{H}$ to $\vec{D}$ has more meaningful physical significance. Static electric fields are due to electric charge, a scalar quantity while static magnetic fields are due to electric current, a vector quantity.*

In Table 14.2 many of electric and magnetic field relations developed are summarised. The analogy of $\vec{B}$ to $\vec{E}$ and $\vec{H}$ to $\vec{D}$ will be noted in many of the equations while the other analogy may be observed in a few of the relations. The relations apply to static and slowly time varying situations. They also apply in more rapidly time varying situations with the exception of those relations involving the curl or the line integral ($\vec{H} = -\nabla V_m$ also does not apply in rapidly time varying case).

Table 14.2: A Comparison of Static Electrical and Magnetic Field Equations

Description of Equation	*Electric Fields*		*Magnetic Fields*	
Force	$\vec{F} = q\vec{E}$	Eq. (4.10)	$d\vec{F} = I d\vec{l} \times \vec{B}$	Eq. (9.5)
			$\vec{F} = Q_m \vec{B}$	Eq. (9.9)
Basic relations for lamellar or solenoid field	$\nabla \times \vec{E}_c = 0$*	Eq. (5.19)	$\nabla \times \vec{B} = 0$	Eq. (10.29)
Derivation from scalar or vector potential	$E_c = -\nabla V$	Eq. (5.18)	$\vec{B} = \nabla \times \vec{A}$	Eq.(10.35)
	$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r} dv'$	Eq. (5.14)	$\vec{A} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{j}}{r} dv'$	Eq. (10.40)

Relations for $\vec{D}$ and $\vec{H}$	$\vec{D} = \varepsilon \vec{E}$	Eq. (6.47)	$\vec{H} = \dfrac{\vec{B}}{\mu}$	Eq. (14.32)
	$\nabla \cdot \vec{D} = \rho f$	Eq. (6.42)	$\nabla \times \vec{H} = \vec{j}$	Eq. (14.25)
Energy density	$W_e = \dfrac{1}{2} \varepsilon E^2$	Eq. (5.48)	$W_m = \dfrac{1}{2} \dfrac{B^2}{\mu}$	Eq. (11.27)
Capacitance and inductance	$C = \dfrac{\text{Charge}}{\text{Potential diff.}}$	Eq. (6.2)	$L = \dfrac{\text{Flux linkage}}{\text{Current}}$	Eq. (11.14)
Closed path of integration	$\oint \vec{E} \cdot d\vec{l} = \text{emf}$	Eq. (8.5)	$\oint \vec{H} \cdot d\vec{l} = \text{NI} = \text{mmf}$	Eq. (14.58)
	$\oint \vec{E} \cdot d\vec{l} = 0$	Eqns (5.8) and (5.9)	$\oint \vec{H} \cdot d\vec{l} = 0$ (no current enclosed)	
Derivation from scalar potentials	$\vec{E}_c = -\nabla V$	Eq. (5.18)	$H = -\nabla V_m$ (In current free region)	Eq. (14.37)

* $\vec{E}_c$ is the static electric field intensity (due to charges)

$\vec{E}$ (without subscript) implies that emf producing fields (not due to charges) may also be present.

14.27 COMPARISON OF ELECTRIC AND MAGNETIC RELATIONS INVOLVING POLARISATION AND MAGNETISATION

It is interesting to compare the magnetic relations where magnetisation $\vec{M}$ is present with the corresponding electric relations where polarisation $\vec{P}$ is present (see Chapter 6). This is done in Table 14.3.

Table 14.3: Comparison of Equations involving Polarisation $\vec{P}$ and Magnetisation $\vec{M}$

Description of Equation	*Electric Fields*		*Magnetic Fields*	
Dipole moment relation	$\vec{P} = \dfrac{\vec{p}}{v} = \dfrac{Q\vec{d}}{v}$	Eq. (6.29)	$\vec{M} = \dfrac{\vec{m}}{v} = \dfrac{Q_m \vec{l}}{v}$	Eq. (14.9)
Flux density	$\vec{D} = \left(\varepsilon_0 + \dfrac{\vec{P}}{\vec{E}} \right) \vec{E}$	Eq. (6.37)	$\vec{B} = \mu_0 \left(1 + \dfrac{\vec{M}}{\vec{H}} \right) \vec{H}$	Eq. (14.23)
Permittivity and permeability	$\varepsilon = \varepsilon_0 + \dfrac{\vec{P}}{\vec{E}}$	Eq. (6.39)	$\mu = \mu_0 \left(1 + \dfrac{\vec{M}}{\vec{H}} \right)$	Eq. (14.34)
Relation to polarisation charge density and to equivalent current density	$\nabla \cdot \vec{P} = -\rho_p$	Eq. (6.32a)	$\nabla \times \vec{M} = \vec{J}_m$	Eq. (14.26)

Poisson's eqn.	$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ in vacuum Eq. (5.20) $\nabla^2 V = -\frac{\rho}{\varepsilon}$ in medium	$\nabla^2 V_m = \nabla \cdot \vec{M} = -\nabla \cdot \vec{H}$ Eqs. (14.37 & 14.38)
Scalar and vector potentials	$V = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho - \nabla' \cdot P}{r} dv'$ Eq. (6.102)	$\vec{A} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J} + \nabla \times \vec{M}}{r} dv'$ Eq. (14.42)
Magnetisation curve	$\vec{P} - \vec{E}$ curve (see Fig. 6.25)	$\vec{M} - \vec{H}$ curve identical to $\vec{P} - \vec{E}$ curve (see Fig. 14.12)
Susceptibility	$\varepsilon = \varepsilon_0 (1 + \chi_{er})$ (χ_{er} is dimensionless) Eq. (6.46)	$\mu = \mu_0 (1 + \chi_m)$ (χ_m is dimensionless) Eq. (14.33)

14.28 DETERMINATION OF MAGNETIC SUSCEPTIBILITY OF DIA AND PARAMAGNETIC SUBSTANCES

Let a small piece of a given material be suspended in the magnetic field of an electromagnet as shown in Fig. 14.23. Further, let the magnetic field between pole pieces be non-uniform, *i.e.*, it varies from place to place. The suspended substances will experience a force. As the diamagnetism is an induced effect, *i.e.*, magnetisation is in the opposite direction to the magnetising field, hence if the sample under consideration is of *diamagnetic material it will tend to move to the region of smaller magnetic field*. On the other hand, since for paramagnetic substances the magnetisation is in the same direction as the magnetising field hence if the sample is of *paramagnetic material it will move towards the region of higher magnetic field*. Suppose a force F acts on the sample placed in the non-uniform magnetic field and causes the sample to move through a distance *dx*. Then the work done by this force will be F.*dx*. *This work done will be equal to the change in potential energy caused by the displacement of the sample*. Let a small volume *v* of the given material be situated at the place where B is magnetic field and the permeability of the medium (air) occupying the field when the sample is outside the field is μ_a. The energy density of the magnetic field on the medium (air) is given by eq. (14.56), *i.e.*

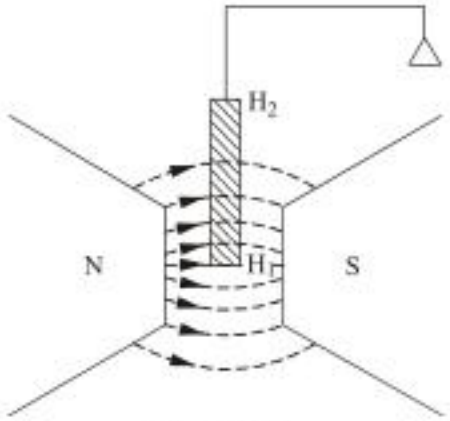


Fig. 14.23

$$\frac{B^2}{2\mu_a} = \frac{\mu_a H^2}{2}$$

Thus the volume *v* which is to be occupied by the sample contains magnetic energy of amount,

$$U_1 = \frac{\mu_a H^2}{2} v$$

Now, the sample of permeability μ is placed in the magnetic field, then the magnetic energy contained by the volume *v* is

$$U_2 = \frac{\mu H^2}{2} v$$

Therefore, when a volume v of the given material replaces the same volume of air, it contains an *excess amount of energy given by*,

$$U_2 - U_1 = \frac{(\mu - \mu_a)\, v H^2}{2}$$

As the magnetising field H is not uniform, then the sample experiences a force and consequently moves through a distance dx. The change in the magnetic energy during this motion is,

$$\frac{d}{dx}\left[\frac{(\mu-\mu_a) v H^2}{2}\right] dx = \frac{(\mu-\mu_a)}{2} v 2 H \frac{dH}{dx} dx$$

$$= (\mu - \mu_a)\, vH \frac{dH}{dx} dx$$

But by Eq. (14.33),

$$\frac{\mu}{\mu_0} = \chi_m + 1 \quad \text{or} \quad \mu = \mu_0(\chi_m + 1)$$

where χ_m is magentic susceptibility of material, for air we have

$$\mu_a = \mu_0\, (\chi_{ma} + 1)$$

where χ_{ma} is the magnetic susceptibility of air. Therefore,

$$(\mu - \mu_a) = \mu_0 \left[\chi_m - \chi_{ma}\right].$$

Hence $(\mu - \mu_a)\, vH \frac{dH}{dx} dx = \mu_0\, (\chi_m - \chi_{ma})\, vH \frac{dH}{dx} dx$

This change in magnetic energy is equal to the work done on the sample. Thus,

$$F dx = \mu_0 \left[\chi_m - \chi_{ma}\right] vH \frac{dH}{dx} dx$$

or $$F = \mu_0 \left[\chi_m - \chi_{ma}\right] vH \frac{dH}{dx} \qquad \ldots(1)$$

By using this equation we can determine the susceptibility χ_m of a dia or paramagnetic material, provided magnetic susceptibility of air is known. We shall discuss two methods.

1. Gouy Method when the substance is available in the form of solids.
2. Quincke's method when the substance is available in the form of liquids.

1. Gouy method for measuring susceptibility of solids

The sample under test is taken in the form of thin rod and is suspended from one arm of a sensitive micro balance in the magnetic field of electromagnet as shown in Fig. 14.23. The field will vary rapidly along the vertical direction, hence the pole pieces are wedge-shaped. If the pole pieces are sufficiently wide apart, the variation in the field in the horizontal direction will be small. Hence the field is uniform along horizontal direction.

As the vertical dimension of the pole pieces is small, the magnetic field falls rapidly in the vertical direction. Procedure is as follows.

Put the specimen at the central part of the field where the field is uniform horizontally. First the specimen is balanced by putting weights on the other arm of the balance. Now the magnetic field is swtiched on and the force (upward or downward, depending on whether the specimen is dia or

paramagnetic) acts on the specimen due to the non-uniform field in vertical direction. This force is counter balanced by adjusting the weights to balance the specimen. If the weights added or removed in the second case be m, then the force experienced by specimen due to magnetic field is mg.

If A is the area of the cross-section of rod, then volume of dx length of the rod is Adx where x is measured vertically from upper end. Then the force on dx length of the rod is by eq. (1),

$$dF = \mu_0 \left[\chi_m - \chi_{ma}\right] A dx H \frac{dH}{dx}, \quad [v = A dx]$$

Therefore, the force experienced by the whole rod is

$$F = \mu_0 \left[\chi_m - \chi_{ma}\right] A \int_{H_2}^{H_1} H dH$$

$$F = \frac{1}{2} \mu_0 (\chi_m - \chi_{ma}) A \left(H_1^2 - H_2^2\right) \quad \ldots(2)$$

where H_1 and H_2 are the values of magnetic field at the lower and upper ends of the rod.

But, $F = mg$, hence

$$mg = \frac{1}{2} \mu_0 A (\chi_m - \chi_{ma}) (H_1^2 - H_2^2)$$

As magnetic field falls very rapidly upward from the centre, hence H_2 is very very small and it can be neglected. Therefore,

$$mg = \frac{1}{2} \mu_0 A (\chi_m - \chi_{ma}) H_1^2 \quad \ldots(3)$$

If χ_{ma} is neglected as it is very small in comparison to χ_m, then,

$$mg = \frac{1}{2} \mu_0 A \chi_m H_1^2 \quad \ldots(4)$$

Hence by using eq. (3) or (4) χ_m can be calculated. Magnetic field H_1 is measured by fluxmeter.

2. Quincke's method for measuring susceptibility of liquids

Quincke directly used Gouy method for liquids. The liquid under test is taken in a U tube, one end of which is capillary section and the other end as wide tube.

As shown in Fig. 14.24, the capillary section is placed between the pole pieces of electromagnet. The liquid meniscus is placed in the centre of the pole pieces where the field is uniform. When the current energising the electromagnet is switched on under the action of magnetic field, the level of meniscus falls or rises depending on whether liquid is diamagnetic or paramagnetic. This change in the height of the meniscus is measured by a travelling microscope. If it is h, then h is the height of the liquid column supported by the force due to non-uniform magnetic field.

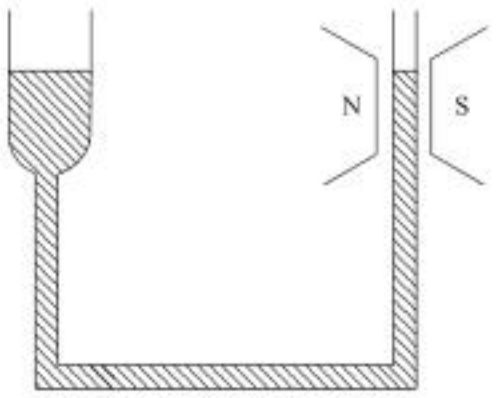


Fig. 14.24

If ρ be the density of the liquid and σ be that of air, then change in pressure of the liquid corresponding to the change h is $(\rho - \sigma)\, gh$. Therefore, force acting downwards on the liquid

$$= (\rho - \sigma)\, gh A \quad \ldots(5)$$

where A is area of cross-section of the capillary tube.

The vertical force exerted by magnetic field is given by eqn. (1) in,

$$dF = \mu_0 (\chi_m - \chi_{ma}) A dx H \frac{dH}{dx}$$

where v has been replaced by Adx. The total force can be obtained by integrating the above equation from $H = 0$ *i.e.* the initial zero magnetic field to $H = H_m$, the final value of magnetic field,

$$F = \mu_0 [\chi_m - \chi_{ma}] \int_{H=0}^{H=H_m} A dx \frac{H dH}{dx}$$

or $$F = \frac{1}{2} \mu_0 (\chi_m - \chi_{ma}) A H_m^2 \quad \ldots(6)$$

Equating (5) and (6), we get,

$$(\rho - \sigma) gh = \frac{1}{2} \mu_0 (\chi_m - \chi_{ma}) H_m^2 \quad \ldots(7)$$

If the susceptibility of air χ_{ma} is known, we can calculate χ_m from equation (7). The H_m can be measured by fluxmeter.

REVIEW QUESTIONS

14.1 What are atomic currents? How will you distinguish between magnetic and non-magnetic materials?

14.2 Prove that the magnetic moment due to orbital motion of an electron must be an integral multiple of $eh / 4\pi m$.

14.3 What do you mean by magnetisation? What is difference between uniform and non-uniform magnetisation? Show that for non-uniform magnetisation $\nabla \times \vec{M} = \vec{j}$.

14.4 Derive the relation between the resultant angular momentum and the resultant magnetic moment of atom. What is gyromagnetic ratio?

14.5 Give a short account of our present idea about dia. para. and ferromagnetism.

14.6 How the Ampere's law is modified in presence of magnetic materials? Establish the relation $\oint \vec{H} \cdot d\vec{l} = 1$.

14.7 Define $\vec{B} \cdot \vec{M}$ and $\vec{H}$. Establish the relation $\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$.

14.8 Define permeability and magnetic susceptibility. Show that $\mu = \mu_0 (1 + \chi_m)$.

14.9 Define magnetic scalar and vector potentials. Show that Poisson's equation is

$$\nabla^2 V_m = \nabla \cdot \vec{M}$$

and vector potential due to a magnetised material is

$$\vec{A} = \frac{\mu_0}{4\pi} \int_v \frac{\vec{j} + \vec{j}_m}{r} dv'$$

14.10 Distinguish dia-para- and ferro-magnetic materials. Give the Langevin theory of diamagnetism.

14.11 What are magnetisation curves? Explain residual magnetism, coercive force and hysteresis.

14.12 What is meant by hysteresis and cycle of magnetisation? Prove that

(1) the area of $\vec{B} - \vec{H}$ cycle denotes the energy dissipated per cc of the magnetisation.

(2) the energy dissipated is μ_0 times the area of $\vec{M} - \vec{H}$ curve.

14.13 Describe the method (B.G) of tracing of the hysteresis curve for a sample of iron in the form of an anchor ring.

14.14 Discuss the importance of the hysteresis curve. How would you use the hysteresis curve to select the material for the construction of

(a) Permanent magnet

(b) An electromagnet
(c) A transformer
(d) A dynamo.

14.15 Explain the magnetic circuit, magnetomotive force and reluctance. How a magnetic circuit differs from an electric circuit?

14.16 Distinguish between para. and ferromagnetic materials. Discuss the Langevin theory of paramagnetism.

14.17 What is Curie-Weiss law.

14.18 Give domain theory of ferromagnetism.

PROBLEMS

14.1 An iron ring of 10 cm radius and 5 sq. cm. cross-section has an air gap of 2 mm width. A coil is wound uniformly over the ring. A flux of 10^{-5} weber is produced in the gap. If a current of 1.82 amp. passes through the coil, find the total number of turns in the coil if the permeability of the core is 65×10^{-4} henry/metre.

14.2 An iron ring of mean radius 10 cm and area of cross-section 5 sq. cm. has an air gap 1 cm wide. The ring is wound uniformly with coil of 900 turns. A flux of 10^{-4} weber is required in the air gap. Find the strength of current required to produce. Given $\mu_{iron} = 65 \times 10^{-4}$ henry/metre.

14.3 An iron ring of mean length 30 cm area of cross-section 1 cm^2 is wound with 300 closely space turns of wire. It is found that a current equal to 0.032 amp. flowing in the winding produces in the ring a flux equal to 2×10^{-6} weber. Calculate:
(1) Flux density in the ring,
(2) The magnetic intensity *i.e.* $\overline{M}$,
(3) The permeability,
(4) The relative permeability, and
(5) The magnetic susceptibility of the material of the ring.

14.4 Find the number of ampere turns needed to produce a flux of 40,000 Maxwells through a closed soft-iron ring formed of a rod of square cross-section of side 2 cm, bent into a circle of outside diameter of 22 cm. Find also the number of amp. turns required for the same flux if the ring be cut into two halves each being separated by an air gap of 1 mm. Given that the permeability of iron is 1900. Assume that there is no leakage of flux (10^8 Maxwell = 1 weber).

SELF-TEST OBJECTIVE QUESTIONS

14.1 When a substance is placed in a magnetic field, its ability to get magnetised depends upon its
(*a*) permeability (*b*) susceptibility
(*c*) magnetic viscosity (*d*) none of the above.

14.2 Platinum has permeability greater than unit and a small positive value for susceptibility. It must be
(*a*) paramagnetic (*b*) diamagnetic
(*c*) ferromagnetic (*d*) none of the above.

14.3 On investigation the susceptibility of a substance was found to vary in inverse proportionality to temperature. The substance must have been
(*a*) ferromagnetic (*b*) diamagnetic
(*c*) ferromagnetic above Curie temp (*d*) none of the above

14.4 Diamagnetic substances are attracted by magnetic field. The attraction is
(*a*) very strong (*b*) weak
(*c*) zero (*d*) negative

14.5 For a magnetic substance the hysteresis loss per unit volume is equal to

(*a*) area enclosed by $\vec{B} - \vec{H}$ curve

(*b*) 4π times area enclosed by $\vec{B} - \vec{H}$ curve

(*c*) $1/4\pi$ times area enclosed by $\vec{B} - \vec{H}$ curve

(*d*) none of the above

14.6 Reluctance is measured in

(*a*) Ampere. Weber (*b*) Ampere-turn / Weber

(*c*) Ampere. Weber (*d*) None of the above

14.7 The ratio of magnetomotive force round the circuit to the magnetic flux is known as

(*a*) reactance (*b*) conductance

(*c*) reluctance (*d*) resistivity

14.8 Permeability μ in magnetic phenomenon is analogous to one of the following in electric phenomena:

(*a*) inductance (*b*) current density

(*c*) conductivity (*d*) resistance

14.9 Which one is correct?

(*a*) $\vec{B} = \vec{H} + \vec{M}$ (*b*) $\vec{B} = \mu_0(\vec{H} + \vec{M})$

(*c*) $\vec{B} = \vec{H} + \mu_0\vec{M}$ (*d*) None of the above

14.10 The energy density in static magnetic field is

(*a*) $W_m = \frac{1}{2}\frac{B}{\mu}$ (*b*) $W_m = \frac{1}{2}\frac{B^2}{\mu}$

(*c*) $W_m = \frac{1}{2}\frac{BH}{\mu}$ (*d*) $W_m = \frac{1}{2}\frac{H^2}{\mu}$

14.11 According to Curie-Weiss law the susceptibility of a substance varies as

(*a*) T (*b*) $1/T-\theta$

(*c*) T^2 (*d*) T^{-2}

14.12 The MMF along a given line is given by the line integral of the vector H *i.e.* $\xi = \int \vec{H} \cdot d\vec{l}$. This is known as

(*a*) Faraday's law (*b*) Maxwell's law

(*c*) Ampere's law (*d*) Gauss' law

14.13 The source of H is

(*a*) Q (*b*) M (*c*) I (*d*) B

14.14 The effect of inserting iron core within a current carrying coil is to

(*a*) weaken the field (*b*) change the direction of the field

(*c*) strengthen the field (*d*) concentrate the magnetic lines in the centre.

ANSWERS

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*d*)	**5.** (*a*)	**6.** (*b*)
7. (*c*)	**8.** (*c*)	**9.** (*b*)	**10.** (*b*)	**11.** (*b*)	**12.** (*c*)
13. (*c*)	**14.** (*c*)				

CHAPTER 15

MAXWELL'S EQUATIONS AND ELECTROMAGNETIC THEORY

At this point in out study of electricity and magnetism, a survey of the preceding chapters will help and indicate the direction for our future discussion of electromagnetic theory. We have discussed so for the basic laws of electrostatics, magnetostatics and the effect of changing magnetic field.

However, we still, do not know the basic laws of electricity and magntism that apply to the general case—namely, the situation in which the variations in the electric and magnetic fields are so rapid, that we are not justified in assuming that they constitute small departure from the electrostatics and magnetostatics cases. This general case occurs in many practical situations. Such as all ratio and television broadcasts are propagated through space by means of electromagnetic waves that consist of rapidly varying electric and magnetic fields.

The objective of this chatper are:

(1) To reformulate or extend the laws of electricity and magnetism to include rapidly varying as well as slowly varying fields and

(2) To use these extended laws to study some of the properties of electromagnetic waves.

15.1 BASIC EQUATIONS

The basic equations of electricity and magnetism which we have studied are summarised in differential form by the following four equations.

(1) Gauss's law for electric field of change, *i.e.*

$$\text{div } \vec{D} = \nabla \cdot \vec{D} = \rho \quad \text{(see eq. 6.42)} \qquad \ldots(15.1)$$

(2) Gauss's law for magnetic field *i.e.*,

$$\text{div } \vec{B} = \nabla \cdot \vec{B} = 0 \quad \text{(see eq. 10.29 and art. 14.2)} \qquad \ldots(15.2)$$

(3) Faraday's law for induced emf produced by the rate of change of magnetic flux, *i.e.*

$$\text{Curl } \vec{E} = \nabla \times \vec{E} = \frac{\delta \vec{B}}{\delta t}, \quad \text{(see eq. 11.6)} \qquad \ldots(15.3)$$

(4) Ampere's law for magnetic field due to steady current *i.e.*,

$$\text{Curl } \vec{B} = \nabla \times \vec{B} = \mu_0 \vec{j} \quad \text{(see eq. 10.32)} \qquad \ldots(15.4)$$

The first three equations are valid for static as well as dynamic field and they apply in general. However, the fourth law *i.e. Ampere's law was derived for steady current and does not hold for time varying field.*

15.2 TYPES OF CURRENTS

We shall here discuss currents in material in a manner similar to that used to treat charges in material medium. In general, currents may be classified in two categories:

(1) *True currents* due to the motion of true charges.

(2) *Other currents* which are due to medium itself.

This separationis analogous to the separation that was discussed in the electrostatic theeory between the potentials (or fields) or ture changes and the potentials of polarisation charges. Similarly here, there are two types of magntic fields, one derived from the true currents and the other derived from the combined effect of all the current whatever may be their origin. It is this latter field, namely, the field of magnetic induction $\vec{B}$, that can be considered to be space-time average of the interatomic fields.

All currents are classified as follows:

(1) *True currents* indentical with the physical transport of true charges.

(2) *Polarisation currents.* Currents those arise from the change of polarisation with time.

(3) *Magnetisation currents.* These are stationary currents that flow within regions that are inaccessible to observation but might give rise to net boundary or volume currents, due to imperfect orbit cancellation on an atomic scale.

(4) *Conective currents.* If a material medium is in motion and contains charge of various types. Additional currents and obtained those arise from the motion of both true and polarisation charges contained in the medium. Such currents are called convective currents.

Therefore in a stationary medium the total current is given by the sum of the first three types of currents given above. Thus,

$$\vec{J}_{total} = \vec{j}_{true} + \frac{\delta \vec{P}}{\delta t} + \nabla \times \vec{M} \qquad \ldots(15.5)$$

15.3 VACUUM DISPLACEMENT CURRENT

Taking the divergence of equation (15.4), we get,

$$\text{div curl } \vec{B} = \mu_0 \text{ div } \vec{J} \qquad \ldots(15.6)$$

Here $\vec{J}$ it total current as given in equation (15.5).

This means that the total flux of current out of any closed surface is zero. Phuysically, this means that current is always closed and there are no sources or sinks. Thus we arrive at a contradiction. Since for time vary case *i.e.* for changing electric fields such as the discharge of a charged condenser through a conducting wire joining its plates. The current starts at positively charged plate, whose charge gradully decreases as the current flows through the conducting wire to the negatively charged plate and neutralises the negative charge on it. Thus the condenser plates are regarded as sources or sinks. *Hence the Ampere's law is valid only for steady state phenomena and not for chaning fields.*

Equation (15.6) is also in conflict with the equation of continuity which gives the complete relation

$$div\ \vec{J} = \frac{\delta \rho}{\delta t} \qquad \ldots(15.7)$$

Maxwell concluded that equation (15.6) is incomplete. The contradication of eq. (15.6) with the equation of continuity *i.e.* (15.7) could be removed by converting the equation of continuity into a form whose divergence is zero by adding something to $\vec{j}$ in equation (15.7). Hence to assume the conservation of charge, it is necessary that total current should obey the equation of continuity.

Gauss's law is

$$\nabla \cdot \vec{D} = \rho$$

Differentiating with respect to time

$$\frac{\delta}{\delta t}\nabla\cdot\vec{\vec{D}}=\frac{\delta\rho}{\delta t}$$

$$\nabla\cdot\frac{\delta\vec{\vec{D}}}{\delta t}=\frac{\delta\rho}{\delta t}$$

Adding $\nabla\cdot\vec{J}$ on both sides,

$$\nabla\cdot\vec{J}+\frac{\delta\rho}{\delta t}=\nabla\cdot\vec{J}+\nabla\cdot\frac{\delta\vec{\vec{D}}}{\delta t}$$

$$=\nabla\cdot\left(\vec{J}+\frac{\delta\vec{\vec{D}}}{\delta t}\right)$$

where $\vec{\vec{D}}=\varepsilon_0\vec{\vec{E}}$ is displacement vector in free space. According to equation of continuity,

$$\nabla\cdot\vec{J}+\frac{\delta\rho}{\delta t}=\nabla\cdot\left(\vec{J}+\frac{\delta\vec{\vec{D}}}{\delta t}\right)=0$$

$$\left.\begin{array}{l}\text{Thus } \nabla\cdot\vec{J}=0, \text{ eq. (15.6) for steady current}\\ \nabla\cdot\left(\vec{J}+\dfrac{\delta D}{\delta t}\right)=0, \text{ everywhere}\end{array}\right\} \quad \ldots(15.8)$$

i.e. for steady currents as well as currents correspoding to time varying fields.

So that Maxwell replaced $\vec{J}$ is Ampere's law by,

$\vec{J}+\dfrac{\delta D}{\delta t}$ and Ampere's law becomes

$$\text{curl }\vec{\vec{B}}=\mu_0\left(\vec{J}+\frac{\delta\vec{\vec{D}}}{\delta t}\right) \quad \ldots(15.9)$$

The term $\dfrac{\delta\vec{\vec{D}}}{\delta t}$ is called the *displacement current density* since it arises when the electric displacement $\vec{\vec{D}}$ (*i.e.* in effect $\vec{\vec{E}}$) changes with time. Maxwell assumed that it is just as effective as $\vec{J}$ in producing magnetic field. The introduction of displacement current is one of Maxwell major contribution to the electromagnetic theory without which, the elecromagnetic theory would have been impossible.

In fact, relation (15.9) is called Maxwell's equation as derived from Ampere's law.

Significance of Displacement Current

The vacuum displacement current $\varepsilon_0\dfrac{\delta E}{\delta t}$ does not have the significance of a current in the sense of being the motion of charges, with the introduction of displacement of current the total current given in equation (15.5) is,

$$\vec{\vec{C}}=\vec{J}_{true}+\frac{\delta\vec{\vec{P}}}{\delta t}+\nabla\times\vec{\vec{M}}+\varepsilon_0\frac{\delta\vec{\vec{M}}}{\delta t}$$

$$\vec{\vec{C}}=\vec{J}_{total}+\varepsilon_0\frac{\delta\vec{\vec{E}}}{\delta t} \quad \ldots(15.10)$$

The current C obtained by adding the term $\varepsilon_0 \frac{\delta \vec{E}}{\delta t}$ to the total current, is a solenoidal current, of which divergence is zero. The introduction of displacement current is necessary in order to apply the Ampere's law for magnetic field due to non-stationary currents.

The geometrical significance of the solenoidal current C is that at points where there is an accumulation of charge, the current is assumed to be continuous across the discontinuity in the form of the rate of change of the field resulting from the accumulation of charge on the boundaries of the discontinuity. As an example charging a capacitor produces a closed current loop in terms of C.

Example of Displacement

The idea of displacement current orginated from the study of discharge of condenser. Before Maxwell the discharge current was supposed to flow in the open circuit formed by the condenser plates and the wires. Maxwell interpreted that all electric currents are divergenceless *i.e.* electric currents must be closed and in the discharge of condenser the conduction current is closed by the dispalcement current (arising due to change of electric displacement in the dielectric) flowing in the dielectric between the plates.

Consider a voltage applied to a resistor and a capacitor in parallel as shown in Fig. 15.1.

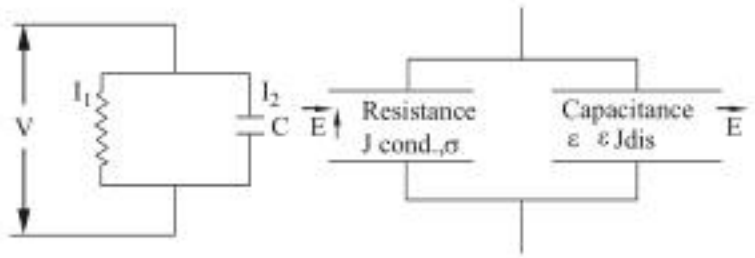


Fig. 15.1

The nature of current flowing through the resistor is different from that through the capacitor. A constant voltage across a resistor produces a continuous flow of current of constant value, on the other hand, the current through a capacitor will be constant only when the voltagge is changing.

The current I_1 through the resistor is given by,

$$I_1 = \frac{V}{R}$$

and the current through the capacitor is given by,

$$I_2 = \frac{dq}{dt} = C\frac{dV}{dt}$$

Since the instantaneous charge in the capacitor is given by $q = CV$.

The current through the resistor is a conduction current, while the current through the capacitor is displacement current. Although the current does not flow through the capacitor, the external effect is as though it did. Since as much current flows out of one plate as flows into the opposite one.

The electric field $\vec{E}$ equal the voltage V across the element divided by its length d, that is

$$E = \frac{V}{d}$$

The current density J_1 inside the resistor equals the product of the electric field E and the conductitity of the medium inside the resistor element and also I_1 divided by the cross-sectional area A or

$$J_1 = \sigma E = \frac{I_1}{A}$$

The capacitance of parallel plate capacitor

$$C = \frac{\varepsilon A}{d}, \text{ where } \varepsilon \text{ is permittivity of the medium, then}$$

$$I_2 = \frac{\varepsilon A}{d} \cdot \frac{d}{dt}(Ed) = \varepsilon A \frac{dE}{dt}$$

and current density

$$J_2 = \frac{I_2}{A} = \frac{\varepsilon d E}{dt}$$

Redcalling that $D = \varepsilon E$

$$J_2 = \frac{dD}{dt}$$

In this example J_1 is a conduction current density $J_{cond.}$ while J_2 is a displacement current density $J_{disp.}$ Since the current density $\vec{J}$, the electric displacement $\vec{D}$ and the electric field intensity $\vec{E}$ are actually space vectors which all have the same direction in isotropic media, hence we can write,

$$\vec{J}_{cond} = \sigma \vec{E}$$

and $$\vec{J}_{disp} = \frac{\varepsilon d \vec{E}}{dt} = \frac{d \vec{D}}{dt}$$

The total current density $\vec{J}_{total}$

$$\vec{J}_{total} = \vec{J}_{cond} + \vec{J}_{total} \qquad \ldots(15.11)$$

The there is a displacement current density in the region of condenser plagtes, the *concept of displacement current has very important consequences for dielectrics and for free space*. The displacement current is responsible for the production of magnetic fields in empty space since the conduction current is zero and the magnetic fields are entirely due to displacement currents. The dispalcement current cannot be detected directly but the confirmation of its existence is provided indirectly by the electromagnetic theory.

15.4 MAXWELL'S EQUATIONS

The complete equations of electromagnetic theory are listed below:

$$\left.\begin{aligned} &\nabla \cdot \vec{D} = \rho && (a)\\ &\nabla \cdot \vec{B} = 0 && (b)\\ &\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} && (c)\\ &\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\delta \vec{D}}{\delta t} \right) && (d) \end{aligned}\right\} \qquad \ldots(15.12)$$

Maxwell equations were not generally accepted for many years after they were postulated (1873). *His curl equations involving* $\nabla \times \vec{E}$ *and* $\nabla \times \vec{B}$ *implied that time-varying electric and magnetic fields in free space* (empty space) *were inter-dependent, a chaging electric field being able to generate a magnetic field, and vice versa*. The radio waves were unknown at the time and it was after 15 years (1888) when Hertz demonstrated that electromagnetic waves were possible as predicated by Maxwell.

Along with Maxwell's equations certain other fundamental relations are of importance in dealing with electromagnetic problems. Among these may be mentioned Ohm's law at a point.

$$\vec{J} = \sigma \vec{E}, \quad \text{(See eq. 8.25)} \qquad \ldots(15.13)$$

The continuity relation

$$\nabla \cdot \vec{J} = -\frac{\delta \rho}{\delta t}, \quad \text{(See eq. 8.22)} \qquad \ldots(15.14)$$

The force relations,

$$\vec{F} = q\vec{E} \qquad \text{(See eq. 4.10)} \qquad \ldots(15.15)$$

$$d\vec{F} = I d\vec{l} \times \vec{B} \qquad \text{(See eq. 9.5)} \qquad \ldots(15.16)$$

and Lorentz equation,

$$\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B}), \quad \text{(\textit{See} eq. 9.4)} \qquad \ldots(15.17)$$

and the relation between $\vec{E}$ and $\vec{D}$ between $\vec{B}$ and $\vec{H}$ as given by

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}, \quad \text{(\textit{See} eq. 6.47)} \qquad \ldots(15.18)$$

$$\vec{B} = \mu \vec{H} = \mu_0 (\vec{H} + \vec{M}) \quad \text{(See eq. 14.32)} \qquad \ldots(15.19)$$

The equation (15.12) to (15.19) together with Newton's laws of motion are sufficients to explain any electromagnetic phenomena.

15.5 MAXWELL'S EQUATIONS IN FREE SPACE

In the preceding section, Maxwell's equations are stated in their general form. For free space, where the current density $\vec{j}$ and charge density ρ are zero well as $\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$, the equations reduce to a similar differential form,

$$\left.\begin{aligned} &\nabla \cdot \vec{E} = 0 && (a)\\ &\nabla \cdot \vec{B} = 0 && (b)\\ &\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} && (c)\\ &\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\delta \vec{E}}{\delta t} && (d) \end{aligned}\right\} \qquad \ldots(15.20)$$

Any electromagnetic field must satisfy all of Maxwell equations. Let us assume that there is a changing charge and current distirbution within a certain region of space, and let us consider the fields produced by this source of radiation. In the free space outside the region, the charge and current densities are everywhere zero.

In integral form the equations are

$$\left.\begin{aligned} &\oint_S \vec{E} \cdot d\vec{S} = 0 && (a)\\ &\oint \vec{B} \cdot d\vec{S} = 0 && (b)\\ &\oint \vec{E} \cdot d\vec{l} = -\int_S \frac{\delta \vec{B}}{\delta t} \cdot d\vec{S} && (c)\\ &\oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \int_S \frac{\delta \vec{E}}{dt} \cdot d\vec{S} && (d) \end{aligned}\right\} \qquad \ldots(15.21)$$

First two equations are obtained by Gauss's divergence theorem $\left(\oint \vec{E} \cdot d\vec{S} = \int_v \text{div } E dv\right)$ and last two equations by Stoke's theorem $\left(\oint \vec{E} \cdot d\vec{l} = \int_s \text{curl } \vec{B} \cdot d\vec{S}\right)$.

15.6 ELECTROMAGNETIC WAVES IN FREE SPACE

For free space the charge density $\rho = 0$ and the current density $\vec{j} = 0$ and K (relative permittivity) = 1 and K_m (or μ_r) relative permeability = 1. Therefore, the constitutive equations are,

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{B} = \mu_0 \vec{H}$$

$$J = 0$$

$$\rho = 0$$

Maxwell's equations are given by Eq. (15.20) *i.e.*

$$\left.\begin{aligned} \nabla \cdot \vec{E} &= 0 && (a) \\ \nabla \cdot \vec{B} &= 0 && (b) \\ \nabla \times \vec{E} &= -\frac{\delta \vec{B}}{\delta t} && (c) \\ \nabla \times \vec{B} &= \varepsilon_0 \mu_0 \frac{\delta \vec{E}}{\delta t} && (d) \end{aligned}\right\}$$

To obtain the differential equations of electromagnetic wave propagation, we eliminate $\vec{B}$ from equations (15.20 c) and (15.20 d).

Taking the curl of equation (15.20 c), *i.e.*

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{\delta}{\delta t}(\nabla \times \vec{B})$$

Substituting $\nabla \times \vec{B}$ from eqn. (15.20 d), we get,

$$\nabla \times \nabla \times \vec{E} = -\frac{\delta}{\delta t}\left(\varepsilon_0 \mu_0 \frac{\delta \vec{E}}{\delta t}\right)$$

$$= \mu_0 \varepsilon_0 \frac{\delta^2 E}{\delta t^2}$$

According to vector product rule,

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

Since, $\nabla \cdot \vec{E} = 0$, [by eq. (15.20 (a)] hence above relation reduces,

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\delta^2 \vec{E}}{\delta t^2}$$

or $$\nabla^2 \overline{E} = -\mu_0 \varepsilon_0 \frac{\delta^2 E}{\delta t^2} \qquad \ldots(15.22)$$

Similarly $\overline{E}$ can be eliminated from Eqs. (15.20c) and (15.20d) yielding,

$$\nabla^2 \overline{B} = -\mu_0 z_0 \frac{\delta^2 B}{\delta t^2} \qquad \ldots(15.23)$$

Equations (15.22) and (15.23) representing the relation between the space and time variation of $\overline{E}$ *and* $\overline{B}$ *are called "three dimensional wave equation" for* $\overline{E}$ *and* $\overline{B}$ *respectively. If we substitute,*

$$\nabla^2 = \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right)$$

and $$\overline{E} = \hat{i}\, E_x + \hat{j}\, E_y + \hat{k}\, E_z$$

Then equation (15.20) becomes,

$$\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right) \left(\hat{i}\, E_x + \hat{j}\, E_y + \hat{k}\, E_z \right)$$

$$= \mu_0 \varepsilon_0 \frac{\delta^2}{\delta t^2} \left[\hat{i} E_x + \hat{j}\, E_y + \hat{k}\, E_z \right]$$

Similar expression for $\overline{B}$ is obtained if we analyse Eq. (15.23)

The wave equation given by Eqs. (15.22) and (15.23) is some times called "D" *Alembert's equation", have been integrated by him.*

Each component of E and B is identical with the familiar wave equations,

$$\frac{\delta^2 y}{\delta x^2} = \frac{1}{v^2} \frac{\delta^2 y}{\delta t^2}$$

The factor $\mu_0 \varepsilon_0$ *has the same significance as the* $1/v^2$. Dimensionally eq. (15.22),

$$\frac{\text{volts}}{\text{met}^3} = \mu_0 \varepsilon_0 \frac{\text{volt}}{\text{met.sec}^2}$$

since $$E = \text{volt/metre}$$

or $$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \text{met/sec.}$$

Same result is obtained if we analyse Eq. (15.23).

Thus it appears that $\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$ *has the dimensions of velocity and is in fact the velocity with which the wave propagates in the free space.* Hence the velocity of wave in free-space $\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$$= \frac{1}{\sqrt{[4\pi \times 10^{-7} \text{ weber/amp.met} . 8.85 \times 10^{-22} \text{ Coul}^2 / \text{Nwt met}^2}}$$

Since 1 Weber $= \dfrac{1 \text{ Newton.metre}}{\text{Amp.}}$

hence $$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-19} \times 8.85 \dfrac{\text{Newton.metre}}{\text{Amp}^2\text{met.}} \cdot \dfrac{\text{Coul}^2}{\text{Newton-metre}^2}}}$$

$$= \frac{1}{\sqrt{4 \times 3.14 \times 8.85 \times 10^{-19} \dfrac{1}{(\text{Coul/sec})^2} \cdot \dfrac{\text{Coul}^2}{\text{met}^2}}} \qquad \text{(Since Amp = Coul/Sec.)}$$

$$= \frac{1}{\sqrt{4 \times 3.14 \times 8.85 \times 10^{-19}}} \text{ met./sec.}$$

$$= 3.00 \times 10^8 \text{ met/sec.} \qquad \ldots(15.24)$$

which is same as the velocity of light in vacuum. Thus we can conclude that the field vectors $\vec{E}$ and $\vec{B}$ propagate as waves in vacuum (free space) with the velocity of light c. This result suggests that light waves are electromagnetic in nature.

15.7 ELECTROMAGNETIC WAVES IN ISOTROPIC NON-CONDUCTING MEDIA (I.E. DIELECTRICS)

Maxwell's equation

Let us first write the basic laws with medium of permittivity ε and permeability μ.

Gauss's law for electrostatics is given by,

$$\nabla \cdot \vec{D} = \rho$$

Gauss's law for magnetic field

$$\nabla \cdot \vec{B} = 0$$

Faraday's law, $\nabla \times \vec{E} = -\dfrac{\delta \vec{B}}{\delta t}$

Ampere's law in presence of medium is,

$$\oint \vec{H} \cdot d\vec{l} = I$$

by Stokes' theorem is

$$\nabla \times \vec{H} = \vec{J}$$

But $$\vec{B} = \mu \vec{H}$$

With this linear relation for $\vec{B}$ and $\vec{H}$, we can write Ampere's law in the presence of medium as,

$$\oint \frac{\vec{B}}{\mu} \cdot d\vec{l} = I$$

or $$\oint \vec{B} \cdot d\vec{l} = \mu I$$

Then by Stokes' theorem,

$$\nabla \times \vec{B} = \mu \vec{J}$$

Now with the introduction of displacement current $\dfrac{\delta \vec{D}}{dt}$ the Ampere's law is expressed by replacing J by $\vec{J} + \dfrac{\delta \vec{D}}{\delta t}$. Therefore, we can write,

$$\nabla \times \vec{B} = \mu \left(\vec{J} + \frac{\delta \vec{D}}{\delta t} \right)$$

Thus the complete equations of electro-magnetic theory are as follows:

$$\left.\begin{aligned} &\nabla \cdot \vec{D} = 0 && (a) \\ &\nabla \cdot \vec{B} = 0 && (b) \\ &\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} && (c) \\ &\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\delta \vec{D}}{\delta t} \right) && (d) \end{aligned}\right\} \quad \ldots(15.25)$$

with $\vec{D} = \varepsilon \vec{E}$

and $\vec{B} = \mu \vec{H}$

As dielectrics are non-conducting medium, therefore current $\vec{J} = 0$ and in a homogeneous isotropic medium there is no volume distribution of charge, thus, the charge density $\rho = 0$. Hence the constitutive equations are

$$\vec{D} = k\varepsilon_0 \vec{E} = \varepsilon \vec{E}$$

$$\vec{B} = \mu_0 \mu_r \vec{H} = \mu \vec{H}$$

$$\vec{J} = 0$$

$$\rho = 0$$

Hence the Maxwell's equations for dielectrics medium from equation (15.25).

$$\left.\begin{aligned} &\nabla \cdot \vec{E} = 0 && (a) \\ &\nabla \cdot \vec{B} = 0 && (b) \\ &\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} && (c) \\ &\nabla \times \vec{B} = \mu \varepsilon \frac{\delta \vec{E}}{\delta t} && (d) \end{aligned}\right\} \quad \ldots(15.26)$$

We can obtain the equation of propagation of wave in dielectrics medium by eliminating $\vec{E}$ from Eqns. (15.26 c) and (15.26d) in the same manner as for free space.

Taking the curl of equation (15.26d) we have,

$$\nabla \times \nabla \times \vec{B} = \nabla \times \mu\varepsilon \frac{\delta \vec{E}}{\delta t}$$

As the medium is isotropic μ and ε remain constant throughout the material of the medium hence can be taken out of the ∇ operation. Thus,

$$\nabla \times \nabla \times \vec{B} = \mu\varepsilon \nabla \times \frac{\delta \vec{E}}{\delta t}$$

$$= \mu\varepsilon \frac{\delta}{\delta t} (\nabla \times \vec{E})$$

Substituting, $\nabla \times \vec{E}$ from eqn. (15.26 c), we get,

$$\nabla \times \nabla \times \vec{B} = \mu \varepsilon \frac{\delta}{\delta t}\left(-\frac{\delta \vec{B}}{\delta t}\right)$$

$$= -\mu \varepsilon \frac{\delta^2 \vec{B}}{\delta t^2}$$

But, $\nabla \times \nabla \times \vec{B} = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$ (see eqn. 2.52)

Since $\nabla \cdot \vec{B} = 0$

Therefore, we get,

$$\nabla \times \nabla \times \vec{B} = -\nabla^2 \vec{B}$$

Substituting in the above equation, we get,

$$-\nabla^2 \vec{B} = -\mu\varepsilon \frac{\delta^2 B}{\delta t^2}$$

or $$\nabla^2 \vec{B} = -\mu\varepsilon \frac{\delta^2 \vec{B}}{\delta t^2} \quad \ldots(15.27)$$

Similarly $\vec{B}$ can be eliminated from the equations (15.26 c) and (15.26 d) and we get,

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\delta^2 \vec{E}}{\delta t^2} \quad \ldots(15.28)$$

Clearly equations (15.27) and (15.28) representing the relation between the space and time variation of magntic field $\vec{B}$ and electric field $\vec{E}$ are called wave equations for $\vec{B}$ and $\vec{E}$ respectively in a three dimensional case. If we substitute,

$$\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$$

$$\vec{B} = \hat{i}\, Bx + \hat{j}\, By + \hat{k}\, Bz$$

Then, $$\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)(\hat{i}\, Bx + \hat{j}\, By + \hat{k}\, Bz)$$

$$= \mu\varepsilon \frac{\delta^2}{\delta t^2}\ (\hat{i}\, B_x + \hat{j}\, B_y + \hat{k}\, B_z)$$

Similar expression for $\vec{E}$ is obtained if we analyse eq. (15.28). Each component of $\vec{B}$ and $\vec{E}$ is identical with the familiar wave equation

$$\frac{\delta^2 y}{\delta x^2} = \frac{1}{v^2}\frac{\delta^2 y}{\delta t^2}$$

Here, the factor $\mu\varepsilon$ has the same significance as the $\frac{1}{v^2}$.

Dimensionally equation (15.27) is,

$$\frac{1}{\text{metre}^2} \cdot \frac{\text{weber}}{\text{metre}^2} = \mu\varepsilon \frac{1}{\text{sec}^2} \cdot \frac{\text{weber}}{\text{metre}^2}, \text{ since } B = \frac{\text{Weber}}{\text{metre}^2}$$

so that $\frac{1}{\sqrt{\mu\varepsilon}} = \frac{\text{meter}}{\text{sec.}}$

Same result is obtained if we analyse equation (15.28).

Thus, it appears that $1/\sqrt{\mu\varepsilon}$ *has the dimensions of velocity and is in fact the velocity with which the wave propagates in the dielectrics,* if represented by v, then

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \ldots(15.29)$$

Thus we can conclude that the field vectors $\vec{B}$ and $\vec{E}$ propagates as waves with velocity $1/\sqrt{\mu\varepsilon}$. The velocity is a characteristic of the medium depending on the constants μ and ε.

15.8 INDEX OF REFRACTON

Since $\varepsilon = \varepsilon_0 k$

and $\mu = \mu_0 \mu_r$

where k is relative permittivety and μ_0 is relative permeability. Then,

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 k}}$$

$$= \frac{1}{\sqrt{\mu_0 \varepsilon_0} \cdot \sqrt{\mu_r k}}$$

by equation (15.24), $\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$ (velocity of light), hence,

$$v = \frac{C}{\sqrt{\mu_r k}}$$

$$= \frac{V}{\eta} \qquad \ldots(15.30)$$

where η is the refractive index of the dielectric which is always greater than unity. *Thus velocity of a wave in a dielectric is less than in free space.* For non-magnetic (or non-ferrous) medium, permeability $\mu_r = 1$ refractive index,

$$\gamma = \frac{C}{v} = \sqrt{\mu_r k} = \sqrt{k} \qquad \ldots(15.31)$$

15.9 SOLUTIONS OF THE WAVE EQUATIONS

Let us now discuss the solutions of wave equations (15.27) and (15.28). There are many solutions corresponding to different kinds of electro-magnetic waves such as plane, spherical and cylindrical waves. We shall consider only uniform plane wave. Therefore, we shall assume that $\vec{E}$ and $\vec{B}$ are functions of only one space coordinate along which the wave travels and of the time t and there is no variation in the component of $\vec{E}$ and $\vec{B}$ in any other direction. If $\vec{E}$ and $\vec{B}$ vary along x-axis, then

$$\vec{E} = \vec{E}(x, t) \text{ and } \vec{B} = \vec{B}(x, t)$$

and all derivatives of the components of $\vec{E}$ and $\vec{B}$ with respect to y and z are zero. Thus,

$$\frac{\delta E_y}{\delta y}=0,\ \frac{\delta E_z}{\delta z}=0,\ \frac{\delta B_y}{\delta y}=0 \text{ and } \frac{\delta B_z}{\delta z}=0 \qquad \ldots(15.32)$$

Then eqs. (15.26 a) and (15.26 b), *i.e.*

$$\nabla\cdot\vec{E}=\text{ and }\nabla\cdot\vec{B}=0$$

written in cartesian coordinates are,

$$\nabla\cdot\vec{E}=\frac{\delta E_x}{\delta x}+\frac{\delta E_y}{\delta y}+\frac{\delta E_z}{\delta z}=0$$

$$\nabla\cdot\vec{B}=\frac{\delta B_x}{\delta x}+\frac{\delta B_y}{\delta y}+\frac{\delta B_z}{\delta z}=0$$

For plane wave by equation (15.32),

$$\frac{\delta E_x}{\delta x}=0 \qquad \ldots(15.33a)$$

$$\frac{\delta B_x}{\delta x}=0 \qquad \ldots(15.33b)$$

Equations (15.33a) and (15.33 b) indicate that E_x and B_x cannot be a function of x which is the direction of propagation. This means E_x and B_x are constant in x-direction. However, we are not interested in uniform fields sinec there are not the part of any wave-equation, therefore, we shall set,

$$E_x=0 \text{ and } B_x=0 \qquad \ldots(15.33c)$$

Hence the vector $\vec{E}$ and $\vec{B}$ can have only y and z component *i.e. it has no "longitudinal" component but has only "transverse" components. Thus,*

$$\left.\begin{aligned}\vec{E}&=\hat{j}E_y+\hat{k}E_z\\ \vec{B}&=\hat{j}B_y+\hat{k}B_z\end{aligned}\right\} \qquad \ldots(15.33d)$$

To be more clear, the Maxwell's electromagnetic waves are purely transverse and never logitudinal. Also from equation (15.26c), we have,

$$\nabla\times\vec{E}=-\frac{\delta\vec{B}}{\delta t}$$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z}\\ E_x & E_y & E_z\end{vmatrix}=-\left(\hat{i}\frac{\delta B_x}{\delta t}+\hat{k}\frac{\delta B_y}{\delta t}+\hat{k}\frac{\delta B_z}{\delta t}\right)$$

On equating the components separately, we get,

$$\frac{\delta E_z}{\delta y}-\frac{\delta E_y}{\delta z}=-\frac{\delta B_x}{\delta t}$$

$$\frac{\delta E_x}{\delta z}-\frac{\delta E_z}{\delta x}=-\frac{\delta B_y}{\delta t}$$

$$\frac{\delta E_y}{\delta x}-\frac{\delta E_x}{\delta y}=-\frac{\delta B_z}{\delta t}$$

Now remembering that $E_x = 0$ and $\frac{\delta}{\delta y} = \frac{\delta}{\delta z} = 0$, we get,

$$\frac{\delta B_x}{\delta t} = 0 \qquad \ldots(15.33e)$$

$$\frac{\delta E_z}{\delta x} = \frac{\delta B_y}{\delta t} \qquad \ldots(15.33f)$$

$$\frac{\delta E_y}{\delta x} = -\frac{\delta B_z}{\delta t} \qquad \ldots(15.33g)$$

Similarly from eq. (15.26*d*), *i.e.*

$$\nabla \times \bar{B} = \mu\varepsilon \frac{\delta \bar{E}}{\delta t}$$

can be written in components, *i.e.*

$$\frac{\delta B_z}{\delta y} - \frac{\delta B_y}{\delta z} = \mu\varepsilon \frac{\delta E_x}{\delta t}$$

$$\frac{\delta B_x}{\delta z} - \frac{\delta B_z}{\delta x} = \mu\varepsilon \frac{\delta E_y}{\delta t}$$

$$\frac{\delta B_y}{\delta x} - \frac{\delta B_x}{\delta y} = \mu\varepsilon \frac{\delta E_z}{\delta t}$$

Again remebering that $B_x = 0$ and $\frac{\delta}{\delta y} = \frac{\delta}{\delta z} = 0$, we get,

$$\frac{\delta E_x}{\delta t} = 0 \qquad \ldots(15.33h)$$

$$\frac{\delta B_z}{\delta x} = -\mu\varepsilon \frac{\delta E_y}{\delta t} \qquad \ldots(15.33i)$$

$$\frac{\delta B_y}{\delta x} = \mu\varepsilon \frac{\delta E_z}{\delta t} \qquad \ldots(15.33j)$$

The equations [15.33 (*f, g, i, j*)] clearly represent that E_z is associated with B_y and E_y is associated with B_z. Thus E_z and B_y are quite indpendent of E_y and B_z. Thus we can write,

$$\frac{\delta E_y}{\delta x} = -\frac{\delta B_z}{\delta t} \qquad \ldots(15.33k)$$

and $$\frac{\delta B_z}{\delta x} = -\frac{\delta E_y}{\delta t} \qquad \ldots(15.33l)$$

Differentiating equation (15.33*k*) with respect to *x* and eqn. (15.33*l*) with respect to *t*, we get

$$\frac{\delta^2 E_y}{\delta x^2} = -\frac{\delta^2 B_z}{\delta x\, \delta t}$$

and $$\frac{\delta^2 B_z}{\delta x \delta t} = -\mu\varepsilon \frac{\delta^2 E_y}{\delta t^2}$$

hence, $$\frac{\delta^2 E_y}{\delta x^2} = \mu\varepsilon \frac{\delta^2 E_y}{\delta t^2} \qquad \ldots(15.34a)$$

or $$\frac{\delta^2 E_y}{\delta t^2} = \frac{1}{\mu\varepsilon}\frac{\delta^2 E_y}{\delta x^2} \qquad \ldots(15.34b)$$

The velocity of wave $v = \frac{1}{\sqrt{\mu\varepsilon}}$

The wave equation (15.34*a*) is a linear partial differential equation of the second order describing the propagation of the *y*-component of the electric field $\vec{E}$ along the *x*-axis with constant velocity *v*.

For simplifying the calcuations let us choose our coordinate axes such that y-axis is parallel to the vector $\vec{E}$. We can do this since E_y and B_z are quite independent of E_z and B_y. Thus we suppose that $E_z = 0$ and $B_y = 0$, then

$$\vec{E} = \hat{j}\,E_y \quad \text{and} \quad \vec{B} = \hat{k}\,B_z$$

Here, as we are interested in electromagnetic wave in which the field is a simply periodic function of time *i.e.*, in general the time dependence $\vec{E}$ and $\vec{B}$ is $e^{j\omega t}$ for single fixed frequency.

Thus we write in equation (15.34*a*),

$$E_y(x,t) = f(x)\,e^{j\omega t} \qquad \ldots(15.35)$$

After substitution in (15.34*a*), we get,

$$\frac{d^2 f}{dx^2} = \frac{\omega^2}{v^2} f \qquad \ldots(15.36a)$$

This equation has two solutions,

$$f(x) = e^{j\omega x/v}$$

and $$f(x) = e^{-j\omega x/v}$$

Hence a general solution for (15.34) is

$$E_y = E_{oy}\left[e^{j\omega(t+x/v)} + e^{j\omega(t-x/v)}\right] \qquad \ldots(15.36b)$$

Therefore, either term alone is a solution or the sum as in eqn. (15.36*b*) is a solution. We can choose 1st term of equation (15.36*b*) and take the imaginary part as solution then we get,

$$E_y = E_{oy}\sin\omega\left(t + \frac{x}{v}\right) \qquad \ldots(15.37)$$

The significance of eqn. (15.37) can be illustrated by evaluating E_y as a function of *x* for several values of time *t*.

At $t = 0$, $E_y = \sin\omega\frac{x}{v}$, hence at $x = 0$, $E_y = 0$

At $t = \frac{T}{4}$, $\omega t = \frac{2\pi}{T}\frac{T}{4} = \frac{\pi}{2}$ and $E_y = \sin\left(\frac{\pi}{2} + \frac{\omega x}{v}\right)$

Hence at $x = 0$, $E_y = +1$, The maximum value of E_y is unity.

At $t = \frac{T}{2}$, $\omega t = \frac{2\pi}{T}\cdot\frac{T}{2} = \pi$ and $E_y = \sin\left(\pi + \frac{\omega x}{v}\right)$

Hence at $x = 0$, $E_y = -1$

The curve areashown in Fig. 15.2 (*a*).

Focussing our attention on the crest of one of the waves as indicated by the point P, we note that as time progresses, P moves to the left. Thus we can conclude that equation (15.37) is representing a wave travelling to the left *i.e.* neagive *x*-axis. The point P is a point of constant phase and is characterised by the condition $x + vt$ = const., taking the time derivatives $\frac{dx}{dt} + v = 0$

or $$\frac{dx}{dt} = -v$$

If we take, $E_y = E_{oy} \sin \omega\left(t - \frac{x}{v}\right)$

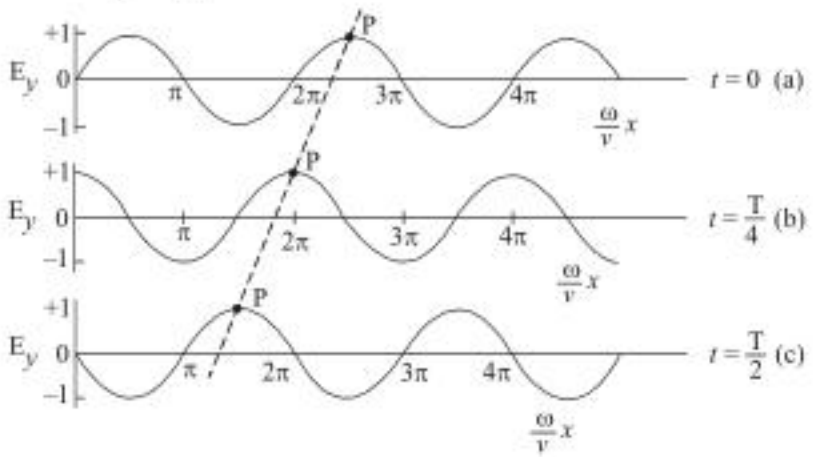


Fig. 15.2 (a)

Then proceeding in the same manner we found that it represents the wave travelling to the right as shown in Fig. 15.2(*b*) positive *x*-axis. To "summarise" the negative sign in $t \mp x/v$ is associated with a wave to the right, while a positive sign is associated with a wave to the left. Other equivalent solution can also be used such as,

$$E_y = E_{oy} \cos \omega\left(t - \frac{x}{v}\right) \text{ or } E_y = E_{oy} \cos\left(\frac{\omega}{v}x - \omega t\right)$$

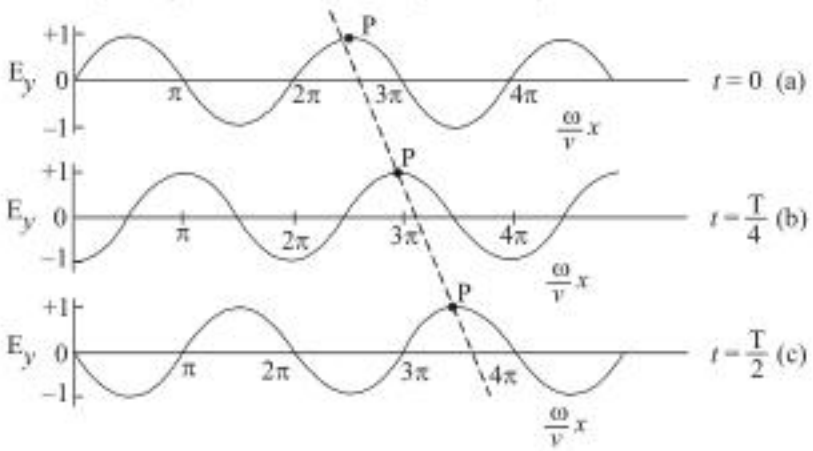


Fig. 15.2 (b)

Again for simplifying the calculations, let us consider only the wave travelling in the positive direction of the *x*-axis, therefore, we have,

$$E_y = E_{oy} \sin \omega\left(t - \frac{x}{v}\right) \qquad \ldots(15.38a)$$

Now we can calculate B_z by substituting E_y in eq. (15.33*k*)

$$\frac{\delta}{\delta x}\left[E_{oy} \sin \omega\left(t - \frac{x}{v}\right)\right] = -\frac{\delta}{\delta t} B_z \; -E_{oy} \cos \omega\left(t - \frac{x}{v}\right) \cdot \frac{\omega}{v} = -\frac{\delta}{\delta t} B_z$$

Integrating with respect to t, we get,

$$E_{oy} \sin \omega\left(t - \frac{x}{v}\right) \cdot \frac{1}{v} = B_z \qquad \ldots(15.38b)$$

or $$B_z = B_{oz} \sin \omega\left(t - \frac{x}{v}\right)$$

where $$B_{oz} = \frac{E_{oy}}{v} \qquad \ldots(15.39)$$

Hence ratio of eqns. (15.38 *a*) and (15.38*b*)

$$\frac{E_y}{B_z} = \frac{E_{oy}}{B_{oz}} = v$$

or $$\frac{E_y}{B_z} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(by eqn. 15.29)} \qquad \ldots(15.40a)$$

If the medium is free space then,

$$\frac{E_y}{B_z} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \ldots(15.40b)$$

The value of E_y/B_z is real and depends on the medium in which wave is travelling. Equations (15.38*a*) and (15.38*b*) represent linearly polarised plane harmonic wave travelling along the $+x$-axis in which $\vec{E}$ is everywhere parallel to y-axis and $\vec{B}$ is parallel to z-axis. As the ratio E_y/B_z is a real positive constant, hence the two vectors $\vec{E}$ and $\vec{B}$ are in phase when $\vec{E}$ has its maximum value directed along $+y$-direction , $\vec{B}$ has its maximum value directed along $+z$-axis.

By equation (15.40*a*)

$$B_z = \sqrt{\mu\varepsilon}\, E_y$$

Now multiplying the left side by $\hat{k}$ and right side by its equivalent $\hat{i} \times \hat{j}$, we get,

$$\hat{k}\, B_z = \sqrt{\mu\varepsilon}\, \hat{i} \times \hat{j}\, E_y$$

or $$\vec{B} = \sqrt{\mu\varepsilon}\, \hat{i} \times \vec{E} \qquad \ldots(15.41a)$$

Similarly, $$E_y = \frac{1}{\sqrt{\mu\varepsilon}} B_z$$

Multiplying left side by $\hat{j}$ and right side its equivalent $\hat{k} \times \hat{i}$, we get,

$$\hat{j}\, E_y = \frac{1}{\sqrt{\mu\varepsilon}}\, \hat{k} \times \hat{i}\, B_z$$

$$= -\frac{1}{\sqrt{\mu\varepsilon}}\, \hat{i} \times \hat{k}\, B_z$$

$$\vec{E} = -\frac{1}{\sqrt{\mu\varepsilon}}\hat{i} \times \vec{B} \qquad \ldots(15.41b)$$

The vectors $\vec{E}$ and $\vec{B}$ are perpendicular to one another, the direction of $\vec{E}$ and $\vec{B}$ are related to the direction of propagation by the rule of "right hand screw". The vector $\vec{E}$ and $\vec{B}$ are such oriented that their vector product $\vec{E} \times \vec{B}$ points in the direction of propagation of the wave as shown in Fig. 15.3.

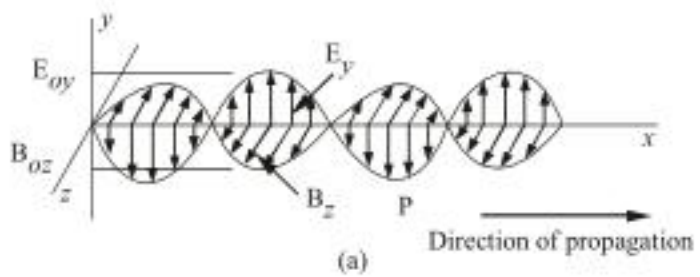


Fig. 15.3

15.10 IMPEDANCE OF DIELECTRIC MEDIA

We can also write equation (15.40a) in a more appropriate form as follows. Since $B_z = \mu H_z$ where μ is permeability of the medium,

$$\frac{E_y}{H_z} = \frac{\mu}{\sqrt{\mu\varepsilon}}$$

or
$$\frac{E_y}{H_z} = \sqrt{\frac{\mu}{\varepsilon}} \qquad \ldots(15.42a)$$

However, for free space,

$$\frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad \ldots(15.42b)$$

Dimensions of equation (15.42) expressed in MKS units are,

$$\frac{\text{volt/metre}}{\text{Amps./metre}} = \frac{\text{volts}}{\text{Amps}} = \text{ohms}$$

Thus $\sqrt{\mu/\varepsilon}$ has the dimensions of an impedance and the ratio E_y/H_z is called the impedance of the medium. For free space, E_y/H_z = 376 ohms.

15.11 ENERGY DENSITY OF ELECTROMAGNETIC WAVE AND POYNTING THEOREM

One important characteristic of an elecromatnetic wave is that it can transport energy from point to point. The direction of electromagnetic wave at a given point is the direction in which energy is being transmitted. We may describe a wave in terms of the new vector $\vec{S}$ which is defined as follows:

$$\vec{S} = \frac{1}{\mu_0 (\vec{E} \times \vec{B})} \text{ or } (\vec{E} \times \vec{H})$$

By this definition the direction of a wave is the direction of the vector $\vec{S}$ called the "poynting vector" after John Henry Poynting who first pointed out its properties. Let us now calculate the divergence of Poynting vector $\vec{S}$ for electromagnetic field in free space.

$$\nabla \cdot \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\mu_0} \left(\vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B}) \right) \qquad \text{[See eq. 3.9]}$$

Substituting $\nabla \times \vec{E}$ and $\nabla \times \vec{B}$ from eq. (15.20), we get,

$$\nabla \cdot \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\mu_0} \left(-\vec{B} \cdot \frac{\delta \vec{B}}{\delta t} - \vec{E}_0 \cdot \mu_0 \frac{\delta \vec{D}}{\delta t} \right)$$

$$= -\frac{1}{\mu_0} \left[\mu_0 \varepsilon_0 \vec{E} \cdot \frac{\delta \vec{E}}{\delta t} + \vec{B} \cdot \frac{\delta \vec{B}}{\delta t} \right] [\text{since } \vec{D} = \varepsilon_0 \vec{E}]$$

$$= -\frac{\delta}{\delta t} \left[\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right]$$

or $$\nabla \cdot (\vec{E} \times \vec{H}) = -\frac{\delta}{\delta t} \left[\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right] \quad [\text{since } B = \mu_0 H]$$

On integrating this over the volume v enclosed by the surface S, we get

$$\int_v \nabla \cdot (\vec{E} \times \vec{H}) \, dv = -\frac{\delta}{\delta t} \int_v \left(\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) dv \qquad \ldots(15.43)$$

The left hand side can be transferred to the surface integral by divergence theorem, we get,

$$\int_v (\vec{E} \times \vec{H}) \cdot d\vec{A} = -\frac{\delta}{\delta t} \int_v \left(\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) dv \qquad \ldots(15.44a)$$

where $d\vec{A}$ is area.

For any other medium of permittivity ε and permeability μ, we have,

$$\int_s (\vec{E} \times \vec{H}) \cdot d\vec{A} = -\frac{\delta}{\delta t} \int_v \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv \qquad \ldots(15.44b)$$

As we know that electrostatic field energy per unit volume is $\frac{1}{2} \varepsilon E^2$ (or $\frac{1}{2} \varepsilon_0 E^2$ for free space) and the magnetostatic field energy per unit volume is $\frac{1}{2} \mu H^2$ (or $\frac{1}{2} \mu_0 H^2$ for free space). We shall assume that these expressions are valid even though the electric and magnetic fields vary rapidly. Thus the total electromagnetic energy in a given volume in free space is given by,

$$W = \frac{1}{2} \int_v (\varepsilon_0 E^2 + \mu_0 H^2) \, dv \qquad \ldots(15.45)$$

therefore from eq. (15.44*a*),

$$\int_s (\vec{E} \times \vec{H}) \cdot dA = -\frac{dW}{dt}$$

The term $-dW/dt$ represents the rate at which energy is decreasing within the given volume. *The rate of decrease of energy must be equal to the net rate at which energy is flowing out of the volume. Therefore the left hand side represents the total outward flux of energy in watts over the surface enclosing the volume.* The $\vec{S}$ represents the amount of power unit area. The integral of $\vec{S}$ over the closed surface clearly represents the rate at which energy leaves the enclosed volume.

Thus, for any arbitrary closed surface, the amount of power flowing out through the surface is

$$\oint \vec{S} \cdot d\vec{A}$$

where $d\vec{A}$ is an element of area, therefore,

$$-\frac{dW}{dt} = \oint \vec{S} \cdot d\vec{A} \qquad \ldots(15.46)$$

Equation (15.46) is known as "Poynting's theorem". It states an important property of $\vec{S}$ (the Poynting vector).

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \text{ or } (\vec{E} \times \vec{H}) \text{ watts/metre}^2$$

represents the power flowing through a unit area normal to the direction of propagation is known as Poynting vector.

Be careful to note that in some situation $\vec{E} \times \vec{H}$ does not represent energy flow, as for example, in a static magntic field superimposed on a static electric field. Hence *the interpretation that the Poynting vector is representing the flow of energy is not absolute or regorous rule but usually correct.*

15.12 ENERGY PER UNIT VOLUME

In Fig. 15.4 an area A is normal to the x-axis. If a plane wave is travelling in the positive x-direction with speed v, then all the energy contained in the volume A $v\,dt$ will pass through the area A in time dt.

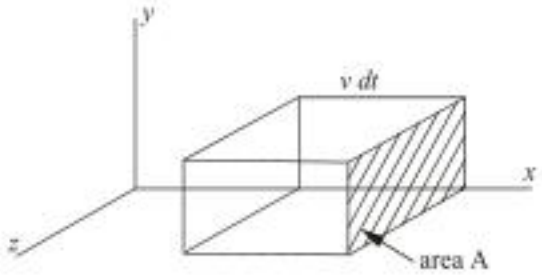


Fig. 15.4

Thus, if U is energy per unit volume, the total energy contained in the volume A $v\,dt$ is U A $v\,dt$. Therefore, the energy passing through unit area per unit time is,

$$S = \frac{UA\,v\,dt}{A\,dt} = U\,v$$

Thus, $$U = \frac{S}{v} \qquad \ldots(15.47)$$

Example. 15.1 *Paraffin has relative permittivity $K = 2.1$. Find the index of refraction for paraffin and also the velocity of the wave in paraffin.*

Solution. As paraffin is a non-ferrous material, hence by eq. (15.31)

$$\eta = \sqrt{K} = \sqrt{2.1} = 1.45$$

By Eq. (15.30) $$v = \frac{C}{\eta} = \frac{C}{\sqrt{K}}$$

$$= \frac{C}{\sqrt{2.1}} = 2.07 \times 10^6 \text{ metre/sec.}$$

15.13 THE RETARDED POTENTIALS

In the article 15.9 we have shown that Maxwell's equations have wave type solutions and that the velocity of propagation is C in free space. The plane wavs have been treated without inquiring how these waves were produced. We have not shown, however, when these solutions are physically relevant or meaningful, nor how they are realted to the charges and the currents that must be their sources.

Hence our next step is to consider prescrbied charge and current distributions, $\rho(r, t)$ and $\vec{J}(r, t)$ and find the fields produced by them that answer above questions. There are several ways of appraoching the problem, of which the msot fruitful is the potential approach, which is developed analogously to the procedure used in electrostatics and magnetostatics. Therefore if we wish to relate the fields of the changing charge and current distribution which produce them it is far easier to work with the potentials than with the fields directly. To be more specific, we shall discuss the generation of electromagnetic waves.

We will determine the potentials in terms of given charge and current distribution. Once the potentials have been determined the fields can be obtained from the relation given in Eqs. (10.35) and (11.11).

Let the point P on Fig. 15.5 have a position vector r and the point Q have position vector r'. if the charge distribution near Q changes with time, the information that it has changed only be appreciated at the point P after the time $r - \frac{r'}{C}$. This is the time taken by electromagnetic radiation to travel the distance $r - r'$. It means that at a time t the contribution to the potential V (r, t) at P from the charge within the volume element dv' and Q depends not on what the charge is at time t but on what is was at the time $t - \frac{|r - r'|}{C}$. This is true for all the volume elements within the considered volume. Note that the time lapse $\frac{r - r'}{C}$ depends on the position vector r' and so varies over the total volume. The potential is given by,*

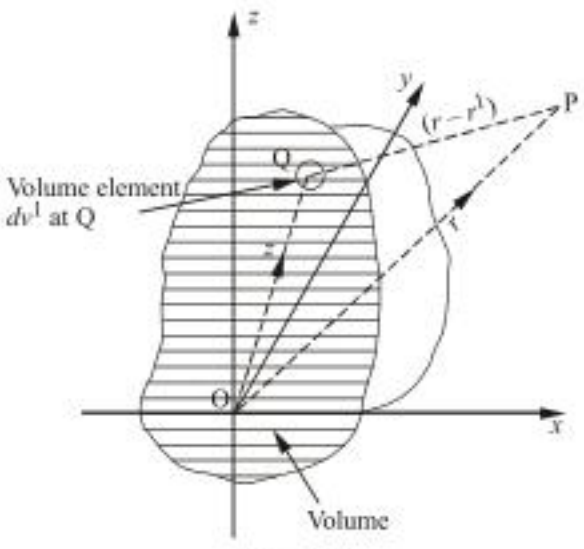


Fig. 15.5

$$V(r,t) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{all \\ volume}} \frac{\rho\left(r', t - \frac{|r - r'|}{C}\right)}{|r - r'|} dv' \qquad \ldots(15.48)$$

which is known as the "retarded scalar potential".

*For details see "Electromagnetism" by I.S. Grant and W.R. Phillips.

15.14 THE GENERAL WAVE EQUATION

The complete equations of electromagnetic theory and other fundamental relations are listed in equations 15.12 to 15.19

$$\left.\begin{aligned} &\nabla\cdot\bar{D}=\rho && (a)\\ &\nabla\cdot\bar{B}=0 && (b)\\ &\nabla\times\bar{E}=-\frac{\partial\bar{B}}{\partial t} && (c)\\ &\nabla\times\bar{B}=\mu\left(\bar{j}+\frac{\partial\bar{D}}{\partial t}\right) && (d)\\ &\text{with } \bar{B}=\mu\bar{H}\\ &\qquad \bar{j}=\sigma\bar{E} \end{aligned}\right\} \qquad \ldots(15.46)$$

i.e. One of the most important application of Maxwell's equations is in the derivation of equation for electromagnetic waves. We can obtain two equations of propagation of wave by eliminating $\bar{E}$ and $\bar{B}$ from equation 15.49 (*c*) and (*d*).

Taking the curl of equation 15.49 *d* we have

$$\nabla\times\nabla\times\bar{B}=\mu\left[\nabla\times\bar{j}+\nabla\times\varepsilon\frac{\partial\bar{E}}{\partial t}\right]$$

$$=\mu\sigma\,(\nabla\times\bar{E})\,\mu\varepsilon\left(\nabla\times\frac{\partial E}{\partial t}\right)$$

$$=\mu\sigma\,(\nabla\times\bar{E})\,\mu\varepsilon\frac{\partial}{\partial t}\,(\nabla\times\bar{E})$$

The order to time and space differentiation can be interchanged if $\bar{E}$ is a sufificiently well behaved function which we assume to be the case substituting $\nabla\times\bar{E}$ from eq. (15.49*c*)

$$\nabla\times\nabla\times\bar{B}=-\mu\sigma\frac{\partial\bar{B}}{\partial t}+\mu\varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial\bar{B}}{\partial t}\right)$$

$$=-\mu\sigma\frac{\partial\bar{B}}{\partial t}-\mu\varepsilon\frac{\partial^2\bar{B}}{\partial t^2}$$

The vector identity

$$\nabla\times\nabla\times\bar{B}=\nabla\,(\nabla\cdot\bar{B})-\nabla^2\,B \qquad \text{see eq. (2.52)}$$

but $\nabla\cdot\bar{B}=0$

$$-\nabla^2\cdot\bar{B}=-\mu\sigma\frac{\partial\bar{B}}{\partial t}-\mu\varepsilon\frac{\partial^2\bar{B}}{\partial t^2}$$

$$\nabla^2\cdot\bar{B}=\mu\varepsilon\frac{\partial^2\bar{B}}{\partial t^2}+\mu\sigma\frac{\partial\bar{B}}{\partial t}$$

or $$\nabla^2\cdot\bar{B}-\mu\varepsilon\frac{\partial^2\bar{B}}{\partial t^2}-\mu\sigma\frac{\partial\bar{B}}{\partial t}=0 \qquad \ldots(15.50)$$

However the vector $\vec{E}$ *satisfied the same wave equation which is obtained by elminating* $\vec{B}$ *form eqn. (15.49 c and d).*

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

Substituting $\nabla \times \vec{B}$ from (15.49*d*) we get

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\mu j + \mu \frac{\partial \vec{D}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}\left(\mu\sigma \vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Using vector identity

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

And resticting the application of equation to change free space $\nabla \cdot E = \nabla \cdot D = 0$. We get

$$-\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} \qquad \ldots(15.51)$$

or $$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0$$

The wave equations 15.50 and 15.51 derived above govern the electromagnetic field in a homogenous, linear medium in which the charge density is zero, whether the medium is conducting or non-conducting. Both the equations contain damping terms proportional to the conductivity of the medium.

However it is not enough that these equations be satisfied; Maxwell equations must be satisfied. It is clear that *e.g.* 15.50 and 15.51 are a necessary consequence of Maxwell equation, but the converse is not true. In solving the wave equation special care must be used to obtain solutions to Maxwell's equation. One method which works very well for monochromatic waves is to obtain a solution for $\vec{E}$. The curl of $\vec{E}$ then gives the time derivation of $\vec{B}$, which for monochromatic waves is sufficiently related to $\vec{B}$ so that $\vec{B}$ can be easily found.

Monochromatic waves may be described as waves which are characterized by a single frequency. The method of complex variables analysis offered a convenient way of treating such waves. The time dependent of the field $\vec{E}$ is taken to be as $e^{-i\omega t}$ so that

$$\vec{E}(r,t) = \vec{E}_s(r)\, e^{-i\omega t} \qquad \ldots(15.52)$$

It must be remembered that the physical electric field is obtained by taking the real part of the equation 15.52 furthermore $\vec{E}_s(r)$ is in general complex, so that the actual electric field is proportional

to $\cos(\omega r + \varphi)$ where φ is the phase of $\vec{E}_s(r)$; using eq. 15.52 in eq. 15.51 we get

$$e^{-i\omega t}\left[\nabla^2 \vec{E}_s + \omega^2 \varepsilon\mu \vec{E}_s + i\omega\sigma\mu \vec{E}_s\right] = 0 \qquad \ldots(15.53)$$

for the equation governing the spatial variation of the electric field (the common factor $e^{-i\omega t}$ can, of course be dropped). The nex task is to solve the equation 15.53 in various special cases of interest determine the spatial variationof the electromagnetic fields.

15.15 PROPAGATION OF ELECTROMAGNETIC WAVES IN DIELECTRICS

Plate Wages: An electromagnetic waves is called a plane wave if the field vectors of the waves have the same magntiude at all points of any plane perpendicular to the direction of wave propagation. Naturally these vectors vary from plane to plane. In other words the constant phase surfaces in a plane wave are planes perpendicular to the direction of propagation. A wave is called monochromatic if the time variation of the field vectors of the wave obeys a harmonic law with a certain constant frequency. For example if a plane electromagnetic wave propagates along the z-axis, the field vectors of the wave have the form

$$\vec{E}(z,t) = \vec{E}(z)\, e^{i\omega t};\ \vec{B}(z,t) = \vec{B}(z)\, e^{i\omega t} \qquad \ldots(15.54)$$

If the constant-phase surfaces coincide with the surfaces of constant amplitude the wave is called homogeneous.

Field vectors of a wave: Let us direct the z-axis along the velocity of propagation of an electromagnetic wave. In this case, the field vectors are defined by the eq. 15.54 putting in eq. 15.51 and taking $\sigma = 0$ for dielectrics we get:

$$\frac{d^2 \vec{E}(z)}{dt^2} + \omega^2 \varepsilon\mu\, \vec{E}(z) = 0$$

or
$$\frac{d^2 \vec{E}(z)}{dt^2} + k^2 \vec{E}(z) = 0 \qquad \ldots(15.55)$$

where $k = \omega\sqrt{\varepsilon\mu}$

The general solution of this equation is given by

$$\vec{E}(z) = \vec{E}_{01}\, e^{-ikz} + \vec{E}_{02}\, e^{ikz} \qquad \ldots(15.56)$$

Where E_{01} and are E_{02} are constants, substituting eq.15.56 in 15.54 we get

$$\vec{E}(z,t) = \vec{E}_{01}\, e^{i(\omega t - kz)} + \vec{E}_{02}\, e^{i(\omega t + kz)} \qquad \ldots(15.57)$$

The first term of the R.H.S. of the above equation describes a wave propagating in the direction of positive z-axis while the second term to the negative z-axis.

The solution for $\vec{B}$ is obtained in a similar way. Suppose that a wave propagates in the positive direction of the z-axis then:

$$\vec{E}(z,t) = \vec{E}_0\, e^{i(\omega t - kz)};\ \vec{B}(z,t) = B_0\, e^{i(\omega t - kz)} \qquad \ldots(15.58)$$

Phase velocity. The equation 15.58 indicates that plane wave propagates in a homogeneous dielectric without a change in their amplitude *i.e.*, without absorption. The velocity of motion of an equiphase surface is called the phase velocity. It can be found be differentiating the conditon of constant phase.

$$\omega t - kz = \text{const}.$$

differentiation w.r.t. gives

$$\omega - k\frac{dz}{dt} = 0$$

where $v = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}}$ since $k = \omega\sqrt{\varepsilon\mu}$ ...(15.59)

Equation 15.58 are written for a special choice of the coordinate system when the z-axis coincide with the direction of wave propagation. This restriction can be eliminated by introducing the wave vector k directed along the vector of velocity of the wave. By definitionof the plane wave propagating in the direction of the vector k. Vector $\overline{\mathrm{E}}$ and $\overline{\mathrm{B}}$ are the same at any point of a plane perpendicular of this direction. Let $\vec{r}$ be radius vector of a point on such an equiphase plane we get following equation instead of eq. 15.58:

$$\vec{\mathrm{E}}(\vec{r},t) = \vec{\mathrm{E}}_0 e^{i(\omega t - \vec{k}\vec{r})};\ \vec{\mathrm{B}}(\vec{r},t) = \vec{\mathrm{B}}_0\, e^{i(\omega t - \vec{k}\vec{r})} \quad \ldots(15.60)$$

Wave length: By definition, it is the distance over which a point on an equi-phase surface moves during a period of oscillations.

$$\lambda = v\mathrm{T} = \frac{\omega \mathrm{T}}{k} = \frac{2\pi}{k} \quad \ldots(15.61)$$

hence $k = \frac{2\pi}{\lambda}$ ...(15.62)

is the wave number.

Properties of waves: In order to investigate the properties of a plane wave, we substitute expression 15.58 into 15.26*a* and 1526*b*. However the Maxwell equation div E = 0 after substitute eq. 15.58 we get:

$$\nabla\cdot\vec{\mathrm{E}} = \nabla\,(\mathrm{E}_0\, e^{i(\omega t - \vec{k}\cdot r)} = \mathrm{E}_0\, e^{i(\omega t - \vec{k}\vec{r})}(-ik)$$

$$= -i\,\vec{k}\cdot\vec{\mathrm{E}} = 0 \quad \ldots(15.63)$$

This means electric field vector, $\vec{\mathrm{E}}$ of the wave is perpendicular to $\vec{k}$ i.e. perpendicular to the direction of propagation. Similarly Maxwell equation

$$div\ \vec{\mathrm{B}} = \nabla\cdot\vec{\mathrm{B}} = -i\vec{k}\cdot\vec{\mathrm{B}} = 0 \quad \ldots(15.64)$$

Show that vector $\vec{\mathrm{B}}$ is also perpendicular to the direction of wave propagation.

Substituting eq. 15.60 into relation

$$\nabla\times\vec{\mathrm{B}} = \mu\varepsilon\frac{\partial\vec{\mathrm{E}}}{\partial t} \text{ and } \nabla\times\vec{\mathrm{E}} = -\frac{\partial\vec{\mathrm{B}}}{\partial t}$$

we obtain

$$-\vec{k}\times\vec{\mathrm{B}} = \varepsilon\mu\omega\vec{\mathrm{E}} \quad \ldots(15.65)$$

$$\vec{k}\times\vec{\mathrm{E}} = \omega\vec{\mathrm{B}} \quad \ldots(15.66)$$

Let $\hat{n}$ be a unit vector in the direction of wave propagation then

$$\vec{\mathrm{K}} = \hat{n}\,\omega\sqrt{\varepsilon\mu}$$

$$= \hat{n} \cdot \frac{\omega}{v} \qquad \ldots(15.67)$$

the equation 15.66

$$\hat{n} \times \vec{E} = v \vec{B} \qquad \ldots(15.68)$$

It was shown by using eq. 15.63 and 15.64 that vector E and B perpendicular to vector n the equation 15.65, 15.66 and eq. 15.68 indicate that these vectors are also mutually perpendicualr, magnitudes by eq. 15.68

$$E = vB \qquad \ldots(15.69)$$

We may conclude from relation 15.68 that in a homogeneous dielectric vectors $\vec{E}$ and $\vec{B}$ vary in phase. All formulas of this section are valid for vacuum as well if we put $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and $v = c$. The spatial variation of the field vectors via a plant is shown in Fig. 15.6.

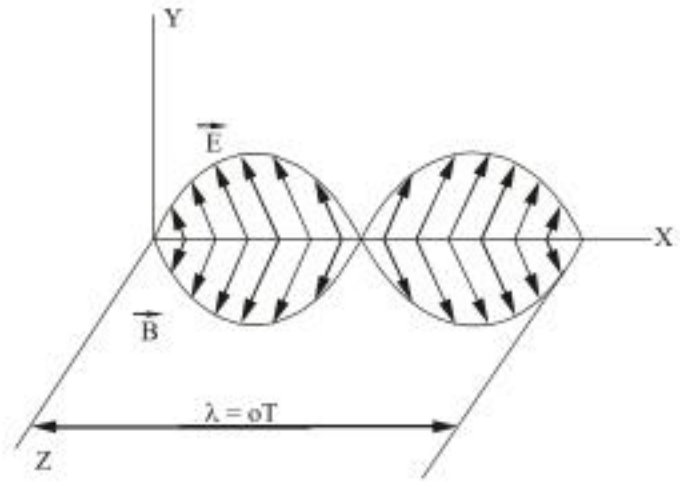


Fig. 15.6

15.16 PROPAGATIONOF ELECTROMAGNETIC WAVES IN CONDUCTING MEDIA

We consider a homogeneous medium $\mu = \text{const}$. $\varepsilon = \text{constant}$ and $\sigma = \text{constant}$ $(\sigma \neq 0)$ *i.e.* we have a conducting medium. In this case maxwell's equation have the form

$$\nabla \times \vec{B} = \mu \vec{j} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t} = \mu\sigma \vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \ldots(15.70)$$

Where $\vec{j} = \sigma \vec{E}$ $\qquad \ldots(15.71)$

Substituting into these equations expression for field vectors from 15.54 *i.e.*

$$\vec{E}(r,t) = E_0 e^{i(\omega t - \vec{k} \cdot \vec{r})} \quad \vec{B}(r,t) = B_0 e^{i(\omega t - \vec{k} \cdot \vec{r})}$$

for the $\vec{B}$ and $\vec{E}$ we obtain

$$-\vec{K}_\omega \times \vec{B} = \omega\mu \left(\omega + \frac{\sigma}{i\omega} \right) \vec{E} \qquad \ldots(15.72)$$

$$\vec{K}_\omega \times \vec{E} = \omega \vec{B} \qquad \ldots(15.73)$$

Where for k from 15.54 we use the notation $k_\omega = k^{(0)} \cdot k_\omega$, $\hat{k}_0$ being the unit vector.

For $\sigma = 0$ the eq. 15.72 coincides with the equation for dielectrics. Thus mathematically a conducting medium differs from a dielectric only in that the corresponding expression contains the complex permittivity

$$\varepsilon_\omega = \varepsilon + \frac{\sigma}{i\omega} - = \varepsilon - \frac{i\sigma}{\omega} \qquad \ldots(15.74)$$

All subsequent calculations are identical to those for dielectric where ε_ω is used instead of ε. Thus, instead of real wave number of the complex values quality k_ω appeared

$$k_\omega{}^2 = \omega^2 \varepsilon_\omega \mu = \omega^2 \varepsilon\mu - i\omega\sigma\mu \qquad \ldots(15.75)$$

representing $k\omega$ in the form of a complex number

$$k_\omega = k - is \qquad \ldots(15.76)$$

we can write eq. 15.75 in the form

$$k^2 - 2iks - s^2 = \omega^2 \varepsilon\mu - i\omega\sigma\mu \qquad \ldots(15.77)$$

equating the real and imaginary parts of eq. 15.77 we get

$$k^2 - s^2 = \omega^2 \varepsilon\mu \equiv a \qquad \ldots(15.78)$$

$$2ks = \omega\sigma\mu \equiv b \qquad \ldots(15.79)$$

The solution of this algebraic system of equations has the form

$$k^2 = \frac{a}{2}\left(\sqrt{1+\frac{b^2}{a^2}}+1\right) = \frac{\omega^2\varepsilon\mu}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right) \qquad \ldots(15.80)$$

$$s^2 = \frac{a}{2}\left(\sqrt{1+\frac{b^2}{a^2}}+1\right) = \frac{\omega^2\varepsilon\mu}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right) \qquad \ldots(15.81)$$

Penetration depth: Let us investigate the amplitude of a plane wave propagating in the direction of positive z-axis.

$$E = E_0 e^{i(\omega t - k_\omega z)}$$

$$= E_0 e^{-sz} e^{i(\omega t - kz)} \quad \text{see eq. (15.76)} \qquad \ldots(15.82)$$

This equation indicates that the amplitude of the wave decreases during its propagation *i.e.*, the electromagnetic wave propagating in a conducting medium is damped along the path

$$\Delta = 1/s. \qquad \ldots(15.83)$$

The amplitude of the electric field of the wave decreaes to $1/e$ of its initial value the quantity Δ is called the penetration depth oif a plane wave into a conducting medium. The wave length of visible light is $\lambda = 0.533 \times 10^{-6} m$.

Which corresponds to the frequency of the order of $5 \times 10^{-15} s^{-1}$. The electric conductivity of metals is of the order to $10^7 \Omega^{-1} m^{-1}$ and the value of ε can be assumed to be equal ε_0 thus

$$\frac{\sigma}{\varepsilon\omega} = 2 \times 10^2 \gg 1$$

for waves longer than light waves this inequality becomes even more stringent consequently we may ignore unity in equation 15.81 incomparison with $\sigma/\varepsilon\omega$ and the expression for s becomes

$$s = \sqrt{\frac{\omega\sigma\mu}{2}} \qquad \ldots(15.84)$$

thus the penetration depth is given by

$$\nabla = \frac{1}{s} = \sqrt{\frac{2}{\omega\sigma\mu}} \qquad \ldots(15.85)$$

Since the wave lengths λ is connected with frequency ω through the relation

$$\omega = \frac{2\pi}{\lambda\sqrt{\varepsilon\mu}}$$

$$\nabla = \sqrt{\frac{\lambda}{\pi\sigma}}\sqrt[4]{\frac{\varepsilon}{\mu}} \qquad \ldots(15.86)$$

Where $\sqrt{\frac{\mu}{\varepsilon}}$ has the dimensions of resistance and is the characterized resistance of the medium. For vacuum

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega \qquad \ldots(15.87)$$

For copper $\sigma = 5\times10^{7}\,\Omega^{-1}m^{-1}$, $\mu \cong \mu_0$, $\varepsilon = \varepsilon_0$ for $\lambda = 1\,\text{m}$ the penetration depth in $\Delta = 4\times10^{-6}\,\text{m}$ this means that essentially we cannot speak of any penetration of the wave not the conducting medium since absorption takes place only in very thin surface of layer. This conclusion is valid for very short wave for example for wave length of the order of that of light wave ($\lambda \cong 10^{-6}\,\text{m}$) the penetration depth ∇ is about $4\times10^{-9}\,\text{m}$.

Physical meaning of absorption. The physical reason is the conversion of the electromagneteic energy of the wave into Joule's heat. The electric field of the wave induces in a conducting medium currents which in accordance with joule's law heat the material of the medium.

Skin effect. The expression eq. 15.85 for penetration depth concides with the expression for the thickness of skin effect. This has deep physcial grounds. The energy transported by current propoagation in the space surrounding the conductors in the form of elecromagnetic energy. This energy partially penetrates the conductor surface to sustain the motion of electrons. In the conductor it is converted into the kinetic energy of electrons. Which in turn is transferred into joule's heat, for this reason, the current is maintained in the parts of the conductors which receive electromagnetic energy from the surrounding space. Since this enrage can penetrate the conductor only to the depth ∇ see eq. 15.85 current may exist near the surface of conductor only with in the limits of depth ∇ *i.e.*, ∇ is the thickness of the skin layer.

Phase velocity and the wave length in a conducting medium

We know $v = \omega/k$. Substituting from eqn. 15.80

$$v = \frac{1}{\sqrt{\omega\mu}}\left\{\frac{2}{\left[\left[1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{1/2}+1\right]}\right\}^{1/2} \qquad \ldots(15.88)$$

This velocity is lower than the velocity of waves in an non-conducting medium with the same values of ε and μ *i.e.*, the conductivity of the medium reduces the phase velocity in it. The wave length in a conducting medium is given by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\varepsilon\mu}} \left\{ \frac{2}{\left[1+\left(\frac{\sigma}{\varepsilon\omega} \right)^2 \right]^{1/2} + 1} \right\}^{1/2} \qquad \ldots(15.89)$$

which is smaller in comparison with the wave length in a non-conducting medium with the same values of ε and μ. Equation 15.89 clearly indicated that the phase velocity in a conducting medium depends on frequency *i.e.*, dispersion takes place in the case. Therefore a conducting medium is always dispersive. A most important feature of propagation of signals in a dispersive media is the variationof their form during propagation.

Relation between the Phases of field vectors

A complex quantity k_ω in eq. 15.76 can be represented conveniently in the exponential form

$$\vec{k}_\omega = \left| k_\omega \right| e^{i\varphi}$$

Thus the eq. 15.73 can be written as follows

$$\vec{B} = \frac{\left| k_\omega \right|}{\omega} e^{i\varphi}\, \hat{k}_0 \times \vec{E} \qquad \ldots(15.90)$$

where $\hat{k}_0$ is the unit vector in the direction of wave propagationin the present case z-axis. The vector $\vec{E}$ and $\vec{B}$ are perpendicular to this axis.

The electric field vector of the wave is given in eq. 15.82 $E = E_0 e^{-sz}\, E^{i(\omega t - kz)}$ on substitution the value from eq. 15.82 in the above eq.

$$B = \frac{|k_\omega|}{\omega} \hat{k}_0 \times \vec{E}_0\, e^{-sz}\, e^{i(\omega t - kz + \varphi)} \qquad \ldots(15.92)$$

the real parts of E and B for a actual oscillations of the field vectors in a plane wave perpendicular in conducting medium.

$$\vec{E} = \vec{E}_0\, e^{-sz} \cos^{(\omega t - kz)} \qquad \ldots(15.93)$$

$$B = \frac{|k_\omega|}{\omega} \vec{k}_0 \times \vec{E}_0\, e^{-sz} \cos^{(\omega t - kz + \varphi)} \qquad \ldots(15.94)$$

i.e., the electric and magnetic vectors oscillate in a plane wave in different phases by relation

$$k_\omega = k - is$$

$$\tan\varphi = -\frac{s}{k} \qquad \ldots(15.95)$$

i.e., the angle φ is negative. This means that B attains a certain value later than E. This is manifested in two ways.

If we consider the oscillations of the field vectors of a wave at fixed point which the wave travels by, $\overline{B}$ attains, say its maximum value later that $\overline{E}$ does. In other words as a function of time lags behind $\overline{E}$. If we consider a wave at a certain fixed instant of time, $\overline{B}$, attains its maximum value for smaller value of z than $\overline{E}$ does, since $\overline{B}$ as a function of z leads $\overline{E}$.

Relationship between the amplitude of field vectors.

Taking the ratio of eq. 15.92 and 15.93

$$\frac{|B|}{|E|} = \frac{|k_\omega|}{\omega} \cong \sqrt{\mu, \varepsilon} \left\{ 1 + \left(\frac{\sigma}{\mu \varepsilon} \right)^2 \right\}^{1/4}$$

comparison of this relation with the relation for non conducting medium see eq 15.69 we conclude that ratio between | B | and | E | in a conducting medium is larger than that in a non conducting medium with the same value of μ and ε.

Example 15.2 *The relative magnitude of $\overline{H}$ in a plane wave is 1 amp./metre. Find the magnitude of $\overline{E}$ for a plane wave in free space.*

Solution. By equation (15.42*b*)

$$E = \sqrt{\frac{\mu_0}{\varepsilon_0}} H$$

$= 376 \times 1$ ohms. Amp/metre

$= 376$ volts/metre

REVIEW QUESTIONS

15.1 Write the Maxwell's equations. Explain the physical significance of each equations.

15.2 Explain different types of currents you know. Explain fully the concept of displacement current.

15.3 Write Maxwell's equations. Solve them in free space.

15.4 Deduce the equation for the propagation of the plane electromagnetic waves in free space. Show that electric and magnetic vectors are normal to each other and to the direction of propagation and are in phase.

15.5 State Maxwell's equations and solve them to obtain the velocity of the electromagnetic waves in a homogeneous isotropic dielectric medium.

15.6 Deduce theoretically Poynting theorem for the flow of energy in an electromagnetic field.

15.7 Deduce Maxwell' equation for free space and prove that electromagnetic waves are transverse.

15.8 Write short notes on:

(1) Poynting vector

(2) Index of refraction

(3) Impedance of dielectric media

(4) Displacement current

15.9 What are retatded potentials?

SELF-TEST OBJECTIVE QUESTIONS

15.1 Current through a capacitive element is called

(*a*) conduction current (*b*) transient current

(*c*) displacement current (*d*) steady state current

15.2 The idea of displacment current is due to

(*a*) Ampere (*b*) Faraday
(*c*) Gauss (*d*) Maxwell

15.3 Which of the following is correct

(*a*) $\vec{D} = \varepsilon \vec{E}$ (*b*) $\nabla \vec{V} = -\vec{E}$

(*c*) $\vec{J} = \sigma \vec{E}$ (*d*) all are correct

15.4 Velocity of palne electromagnetic wave in vacuum is given by

(*a*) $C = \sqrt{\mu_0 / \varepsilon_0}$ (*b*) $C = \sqrt{\mu_0 \varepsilon_0}$

(*c*) $C = \dfrac{1}{\sqrt{(\mu_0 \varepsilon_0)}}$ (*d*) none of these

15.5 According to wave equation for electric field $\vec{E}$

(*a*) $\nabla \cdot \vec{E} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$ (*b*) $\nabla \times \vec{E} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$

(*c*) $\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$ (*d*) $\nabla^2 \vec{E} = \dfrac{1}{\varepsilon_0 \mu_0} \dfrac{\partial^2 E}{\partial t^2}$

15.6 The direction of propagation of electromagnetic waves is given by the direction of

(*a*) Vector $\vec{E}$ (*b*) Vector $\vec{H}$

(*c*) Vector $\dfrac{1}{\mu_0 (\vec{E} \times \vec{B})}$ (*d*) none of these

15.7 Wave equation for magnetic field B states that

(*a*) $\nabla^2 \vec{B} = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} \dfrac{\partial^2 \vec{B}}{\partial t^2}$ (*b*) $\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 B}{\partial t^2}$

(*c*) $\nabla^2 \cdot \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 B}{\partial t^2}$ (d) none of these

15.8 Maxwell's electromagnetic equations are valid under all conditions except one and that is

(*a*) they do not apply to non-isotropic media
(*b*) they do not apply to non-homogeneous media
(*c*) they do not apply to media which move with respect to system of coordinates
(*d*) they do not apply to non-linear media

15.9 The Maxwell's electromagnetic wave are purely transverse and never longitudianl. The statement is

(*a*) True (*b*) Fase

15.10 $\sqrt{\mu / \varepsilon}$ has the dimensions of

(*a*) an inductance (*b*) a capacitance
(*c*) an impedance (*d*) an electric field

15.11 Plane electromagnetic waves follow the laws of reflection just like light radiations. This statement is

(*a*) False (*b*) True

15.12 Retarded potential is due to the fact that the wave propagation is not an intantaneous phenomenon

(*a*) True (*b*) False

ANSWERS

1. (*c*)	**2.** (*d*)	**3.** (*d*)	**4.** (*c*)	**5.** (*c*)	**6.** (*c*)
7. (*b*)	**8.** (*a*)	**9.** (*a*)	**10.** (*c*)	**11.** (*b*)	**12.** (*a*)

CHAPTER 16

ALTERNATING CURRENTS

16.1 DESCRIPTION OF AN ALTERNATING ELECTRIC CURRENT

An alternating current, in general, is a current, that is a periodic function of time. In other words, an *alternating current is one which passes through a cycle of changes at regular intervals*. Each cycle consists of two half cycles during one of which the current in entirely positive whereas during the following half-cycle the current is entirely negative.

In an alternating current circuit we are concerned with a steady state current and voltage which are oscillating sinusoidally without change in amplitude.

The variation of emf is written as,

$\xi = \xi_0 \sin \omega t$ [as discussed in article (12.6)] and the corresponding current is

$I = I_0 \sin \omega t$

where ξ_0 is the amplitude or peak value of voltage ξ represents the voltage at any time t and $\omega = 2\pi f$ is the angular frequency of the voltage applied. The periodic time of the oscillation is $T = 2\pi/\omega$. The term ωt is called the phase angle. The emf and the current first rise to a maximum in one direction and fall to zero. The direction then quickly reverses so that emf and the current rise to maximum in the reverse direction and again fall to zero. This completes one cycle of A.C. voltage or current as shown in Fig. 16.1. *The number of such cycles per second is called the frequency (f).*

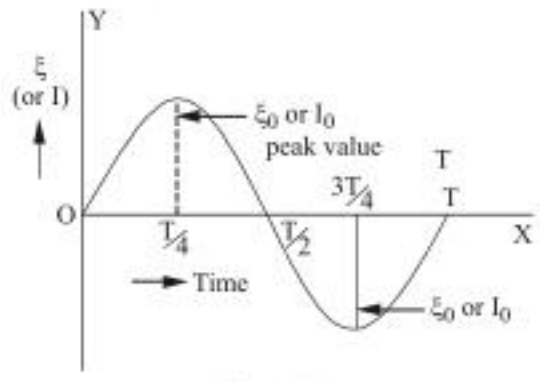


Fig. 16.1

16.2 (A) AVERAGE OR MEAN VALUE OF ALTERNATING CURRENT OR VOLTAGE WAVE OVER ONE COMPLETE CYCLE

The value of current at any instant t is given by,

$I = I_0 \sin \omega t$

The average value of a sinusoidal wave over one complete cycle is given by,

$$I_{av} = \frac{\int_0^T I_0 \sin \omega t \, dt}{\int_0^T dt} = \frac{I_0}{\omega} \frac{|\cos \omega|_0^T}{T}$$

$$= -\frac{I_0}{\omega T}\left|\cos \frac{2\pi t}{T}\right|_0^T, \quad \left(\text{since } \omega = \frac{2\pi}{T}\right)$$

$$= -\frac{I_0}{\omega T}[\cos 2\pi - \cos 0]$$

$$= -\frac{I_0}{\omega T}[1-1] = 0 \qquad \ldots(16.1)$$

Thus, *the average value of alternating current over one complete cycle is zero. The same is also true for alternating emf.*

(b) Average Value of Alternating Current or Voltage During Half Cycle

However, the average over one half-cycle is finite. It can be obtained very easily as follows:

$$I_{av} = \frac{\int_0^{T/2} I_0 \sin \omega t \, dt}{\int_0^{T/2} dt}$$

$$= -\frac{I_0}{\omega \frac{T}{2}}\left[\cos \omega t \right]_0^{T/2}$$

$$= -\frac{I_0}{\frac{2\pi}{T} \cdot \frac{T}{2}}\left[\cos \frac{2\pi}{T} t\right]_0^{T/2}$$

$$= -\frac{I_0}{\pi}[\cos \pi - \cos 0]$$

$$= \frac{2I_0}{\pi} \qquad \ldots(16.2)$$

$$= 0.637\, I_0 \qquad \ldots(16.3)$$

Similarly, for sinuscidal alternative voltage, average value over one half cycle is,

$$\xi_{av} = \frac{2\xi_0}{\pi} = 0.637\, \xi_0 \qquad \ldots(16.4)$$

The average value during the second half cycle is given by,

$$I_{av} = \frac{\int_{T/2}^{T} I_0 \sin \omega t \, dt}{\int_{T/2}^{T} dt}$$

After evaluating the above equation,

$$I_{av} = -\frac{2I_0}{\pi} = -0.637 I_0 \qquad \ldots(16.5)$$

The average value of alternating current (or voltage) during the first and second half cycles are equal but opposite in sign i.e. they are alternately positive and negative so that the average over one cycle is zero.

(c) Root mean Square or Virtual or Effective Value of sinusoidal Voltage or Current

The effecitve or virtual value of an alternating current is that steady current which, when passed through a given resistance for a certain time will develop the same amount of heat as the actual alternating current shall develop when passed for the same time.

The rate of heat generation in a current carrying resistance is given by I^2R. As we are interested in the average or mean value of this quantity, which we shall write as $\bar{I}^2$ where $\bar{I}^2$ represents the mean value of $\bar{I}^2$. Thus,

$$I_{eff}^{\ 2} = \bar{I}^2$$

or
$$I_{eff} = \sqrt{\bar{I}^2} \qquad \ldots(16.6)$$

Thus, the effective value of an alternating current is the square root of the mean of the squares of the instantaneous values over one complete cycle. Due to this reason the effective current is also called the root mean square (RMS) current.

Let us calculate the average of I^2. If the alternating current is represented by,

$$I = I_0 \sin \omega t$$

$$I^2 = \frac{\int_0^T I^2\, dt}{\int_0^T dt} = \frac{\int_0^T I_0^{\ 2} \sin^2 \omega t\, dt}{T}$$

$$= \frac{I_0^{\ 2}}{2T}\int_0^T (1 - \cos 2\omega)\, dt$$

$$= \frac{I_0^{\ 2}}{2T}\left[t - \frac{\sin 2\omega}{2\omega}\right]_0^T$$

$$= \frac{I_0^{\ 2}}{2}$$

Hence, $$I_{eff} = \sqrt{\frac{I_0^{\ 2}}{2}} = \frac{I_0}{\sqrt{2}} = 0.707\, I_0 \qquad \ldots(16.7)$$

Thus, the effective or RMS value of an alternating current is $\frac{1}{\sqrt{2}}$ *times its peak value of maximum value.* Similarly, the effective or virtual value of alternating emf is given by,

$$\xi_{eff} = \frac{\xi_0}{\sqrt{2}} = 0.707\, \xi_0 \qquad \ldots(16.8)$$

Form Factor

It is defined as the ratio of the virtual or R.M.S. value to the average value of an alternating current or voltge. Thus,

$$\text{Form factor} = \frac{I_{rms}}{I_{av}} = \frac{\xi_{rms}}{\xi_{av}} \qquad \ldots(16.9)$$

Substituting values from eqs. (16.2) and (16.7), we get

$$\frac{I_0}{\sqrt{2}} \frac{\pi}{2I_0} = \frac{\pi}{2\sqrt{2}} = 1.11 \qquad \ldots(16.10)$$

The form factor gives an indication of the wave shape of an alternating voltage or current.

A more general form of a alternating current or voltage may be written as,

$$\xi = \xi_0 \sin(\omega t + \alpha) \qquad \ldots(16.11)$$

and the current $I = I_0 \sin(\omega t + \alpha)$

where α is called phase angle and added to the phase ωt. This addition does not affect tghe virtual or effective value (ξ_{eff}). ξ_{eff} is still given by eq. (16.8) which you can verify.

16.3 A.C. CIRCUITS

In Chapter 11, we considered the transient currents in a circuit after a switch is opened or closed. The problem of determining the steady state condition was relatively simple since the circuits were energised by direct current sources (batteries). The currents after a long time were either zero or else could easily be calculated by means of Ohm's law.

An interesting and perhaps more important problem is to determine what happens when an alternating current generator is used in place of battery. Just as in the direct current case, we have to solve the differential equation for a particular circuit. The solution of the differential equation for a particular circuit may be written as the sum of complimentary function and a particular solution. The complementary function which represents the transient solution has the same form when an alternating current source is used as when a battery is used. If you review the results of Chapter 11, the complementary function may be expressed in terms of exponential functions of time. For all circuits having some resistance, the exponential terms have negative real parts. Consequently, after a relatively short time the value of the transient (current or voltage) is practically zero.

The new problem in the alternating current case is to determine the steady-state solution, which is, of course, the particular solution of the differential equation. For example, the circuit in Fig. 16.2 shows an alternating current generator to a resistance R and a capacitance C in series. Here, *the steady state current is not zero as in the case of direct current. In one cycle, this capacitor is charged, discharged, recharged oppositely, and then discharged once more*. This happens again and again.

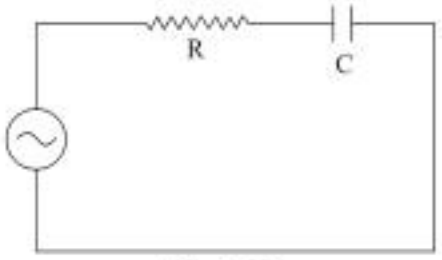


Fig. 16.2

The objective of this Chapter is to determine the current in such circuit. Bascially there are two methods, namely,

1. The vector method; and
2. The complex number method.

We shall develop the complex number method.

16.4 CIRCUIT ANALYSIS USING COMPLEX NUMBER REPRESENTATION

The A.C. network analysis becomes simple and convenient by the complex number representation. *A combination of real and imaginary quantity is called a complex quantity*. A complex number can be written as,

$$\vec{Z} = x + jy \qquad \ldots(16.12)$$

where $\vec{Z}$ is a complex number having a real part x and an imaginary part jy, where $j = \sqrt{-1}$. Equation (16.12) expresses the complex number in algebraic (or rectangular form).

The complex numbers are represented in cartesian coordinates by a vector in a complex plane by choosing real axis along the x-axis and imaginary axis along the y-axis as shown in Fig. 16.3. The complex number is completely defined by point P since the projection along the real axis of line $O\vec{P}$ gives the real part, and its projection along y-axis gives the imaginary part of the number.

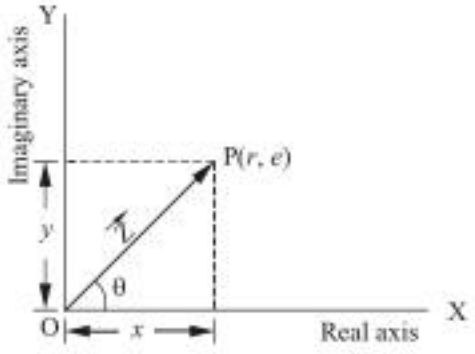


Fig. 16.3

The length of the line $(O\vec{P} = \vec{Z})$ represents the absolute value or modulus (magnitude) of the complex number and angle θ is the phase of this number. Thus, the vector properties of complex quantities suggest that it may be convenient to express them in terms amplitude and direction as well as in real and imaginary parts. Hence the complex number can be expressed in the following trigonometrical (polar) form as seen from Fig. 16.3. If point P has polar coordinates (r, θ) then we have,

$$x = r \cos \theta$$

$$y = r \sin \theta$$

After substituting in Eq. (16.12), we get,

$$\vec{Z} = r \cos \theta + j\, r \sin \theta$$

$$= r\,(\cos \theta + j \sin \theta)$$

$$= r\, e^{j\theta} \qquad \ldots(16.13)$$

where $r = \sqrt{x^2 + y^2}$ ...(16.14)

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ ...(16.15)

Important points about complex numbers

1. Equality

Two complex numbers are said to be equal if both their real parts are equal and their imaginary parts are equal. Thus if,

$$Z_1 = x_1 + jy_1$$

$$Z_2 = x_2 + jy_2$$

then the equality, $Z_1 = Z_2$

gives the two real relations,

$$x_1 = x_2 \text{ and } y_1 = y_2$$

2. Complex Conjugate

We define the complex conjugate to $Z_1 = x_1 + j\,y_1$ as,

$$Z_1^* = x_1 - j\,y_1 \qquad \ldots(16.16)$$

Z_1^* is a complex number such that its product with Z_1 is,

$$Z_1\,Z_1^* = x_1^2 + y_1^2$$

3. Addition and Subtraction

Complex numbers are added and subtracted like vector quantities. Let the complex numbers be represented by,

$$\vec{Z}_1 = x_1 + j\,y_1$$

and $\vec{Z}_2 = x_2 + j\,y_2$

Then the resultant complex number will be (Fig. 16.4)

$$\vec{Z}_{Ra} = \vec{Z}_1 + \vec{Z}_2 = (x_1 + x_2) + j\,(y_1 + y_2) \text{ addition} \qquad \ldots(16.17)$$

and $Z_{Rs} = Z_1 - Z_2(x_1 - x_2) + \vec{J}\,(y_1 - y_2)$ subtraction $\qquad \ldots(16.18)$

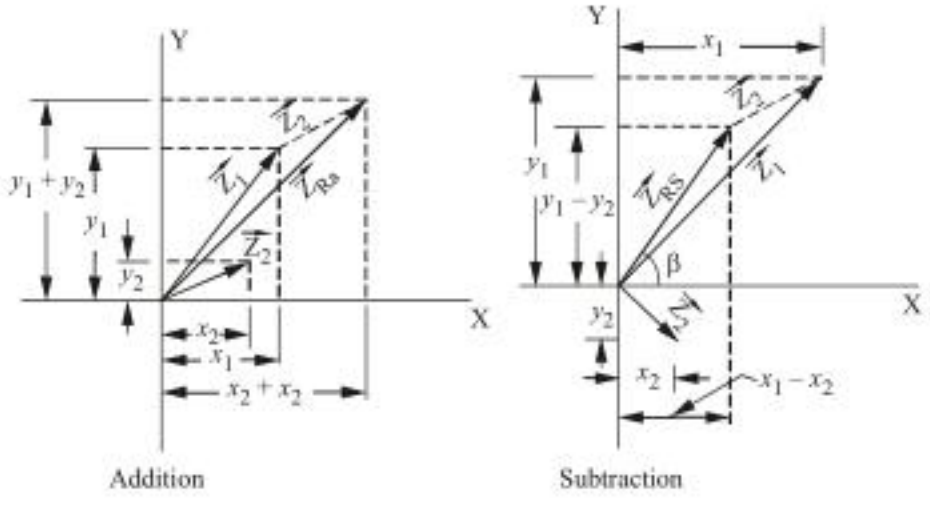


Fig. 16.4

The magnitude of the resultant complex number is given by

$$\left.\begin{aligned} Z_{Ra} &= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \text{ addition} \\ Z_{Rs} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \text{ substraction} \end{aligned}\right\} \qquad \ldots(16.19)$$

and the phase by

$$\left.\begin{aligned} \alpha &= \tan^{-1}\frac{y_1 + y_2}{x_1 + x_2} \text{ addition} \\ \beta &= \tan^{-1}\frac{y_1 - y_2}{x_1 - x_2} \text{ subtraction} \end{aligned}\right\} \qquad \ldots(16.20)$$

We can generalise the above rule that the *sum of several complex number is a complex number whose real part is the algebraic sum of the real part of the added numbers and its imaginary part is the algebraic sum of the imaginary parts of the added numbers.* The same is also true for substraction.

4. Multiplication and Division

Products of complex numbers are defined according to the usual algebraic rules. For example, the product of two complex numbers Z_1 and Z_2 if expressed by Z_m is given by,

$$\bar{Z}_m = \bar{Z}_1 \bar{Z}_2 = (x_1 + j y_1)(x_2 + j y_2)$$

$$= (x_1 x_2 - y_1 y_2) + j (x_1 y_2 + y_1 x_2) \qquad \ldots(16.21)$$

It can be seen that sums and products of complex numbers are also complex numbers.

Division

Division can be defined as the inverse of multiplication. When complex numbers are divided, it is first necessary to eliminate the imaginary unit contained in the denominator and then to divide the complex numbers termwise. Thus,

$$\bar{Z}_d = \frac{\bar{Z}_1}{\bar{Z}_2} = \frac{x_1 + j y_1}{x_2 + j y_2} \qquad \ldots(16.22)$$

To eliminate the imaginary unit contained in the denominator, multiply the numerator and denominator by the complex conjugate of the denominator,

$$\bar{Z}_d = \frac{\bar{Z}_1}{\bar{Z}_2} \cdot \frac{\bar{Z}_1}{\bar{Z}^*_2} = \frac{x_1 + j y_1}{x_2 + j y_2} \cdot \frac{x_2 + j y_2}{x_2 + j y_2}$$

$$= \frac{x_1 x_2 + y_1 y_2 + j (x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \qquad \ldots(16.23)$$

However, *the complex numbers can be multiplied and divided conveniently with less effort by transforming them from algebraic form to polar form (i.e. exponential form). By equation (16.13),*

$$\bar{Z}_1 = r_1 e^{j\theta_1} \text{ and } \bar{Z}_2 = r_2 e^{j\theta_2}$$

Then,

$$\bar{Z}_m = \bar{Z}_1 , \bar{Z}_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)} \qquad \ldots(16.24)$$

and $$Z_d = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \qquad \ldots(16.25)$$

Thus, the product of two complex numbers is a complex number whose magnitude is the product of the magntidues of the given complex numbers and the phase is the algebraic sum of the cofactors. The division (quotient) of two complex numbers is a complex number, the magnitude of which is the quotient of the magnitudes of the dividend and diviser, whole its phase is the algebraic difference of the phases of the dividend and divisor. Fig. 16.5 shows the magnitudes and phases of the two complex numbers after multiplication and division.

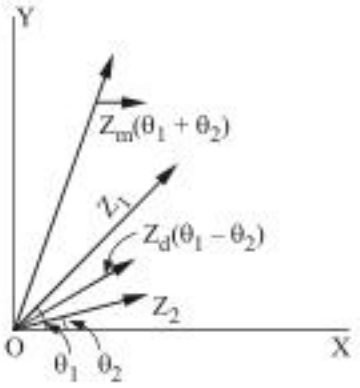


Fig. 16.5

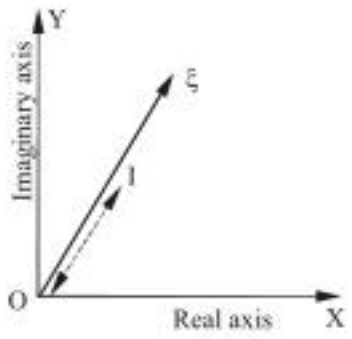


Fig. 16.6

16.5 COMPLEX REPRESENTATION OF A.C. CIRCUIT QUANTITIES

We know that any vector in a complex plane can be completely specified by complex number. This suggests that the A.C. network analysis can be simplified by expressing sinusoidal voltages and currents in complex number representation rather than in vector form. *Hence the sinusoidal voltage* $\xi = \xi_0 \sin(\omega t + \alpha)$ *and current* $I = I_0 \sin(\omega t + \alpha)$ *as expressed in eq. (16.11) that is general forms are nothing but the imaginary parts of* $\xi_0 j^{(\omega t + \alpha)}$ *and* $I_0 e^{j(\omega t + \alpha)}$ *and* $I_0 e^{j(g\omega t + \alpha)}$ *respectively.* Thus, alternating voltage and current can be represented conveniently by complex expression. The complex representation of voltge and current in shown in Fig. 16.6.

16.6 CIRCUIT CONTAINING PURE RESISTANCE ONLY

Fig. 16.7 (*a*) shows a circuit containing a pure resistance R with an alternating voltage $\xi = \xi_0 \sin \omega t$ applied across it. This voltage is the imaginary part of the complex number $\xi_0 e^{j\omega t}$. Thus, in complex number representation we can write,

$$\xi = \xi_0 e^{j\omega t} \qquad \ldots(16.26)$$

If I is the instantaneous current in the circuit, at any time *t*, then emf equation of the circuit is,

$$RI = \xi_0 e^{j\omega t} \qquad \ldots(16.27)$$

or $$I = \frac{\xi_0 e^{j\omega t}}{R} \qquad \ldots(16.28)$$

or $$I = I_0 e^{j\omega t}$$

where $$I_0 = \frac{\xi_0}{R}$$

is the *maximum value of the current.*

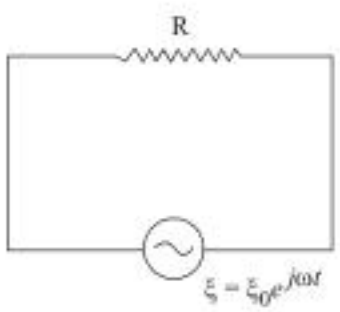


Fig. 16.7 (*a*)

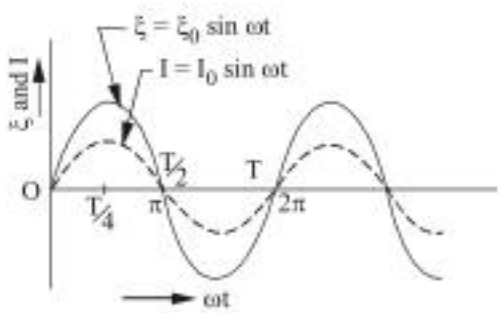


Fig. 16.7 (*b*)

Equation (16.28) represents the variation of current with time t in the circuit. Comparison of Eqs. (16.28) and (16.26) shows that the current has the same form and frequency as the applied voltage. *Thus the current is in phase with applied voltage as shown in* Fig. 16.7 (*b*). Fig. 16.7 (*c*) shows the complex number representation of voltage and current.

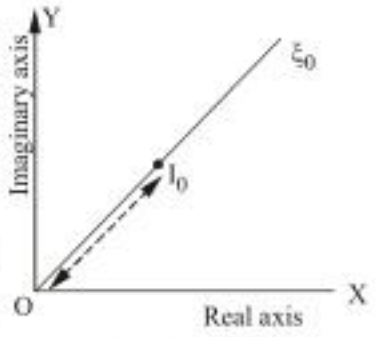


Fig. 16.7 (*c*)

Here R is the impedance of the resistance, which when multiplied by the current through the element gives potential difference across that element. *The inverse of the impedance denoted by Y is usually called admittance.*
Thus,

$$\text{Admittance,} \quad = Y = \frac{1}{Z} = \frac{1}{R} \qquad \ldots(16.29)$$

16.7 CIRCUIT CONTAINING PURE INDUCTANCE ONLY

Fig. 16.8 (*a*) shows a circuit containing an inductor of self-inductance L with an alternating voltage $\xi = \xi_0 \sin \omega t$ applied across it. This voltage is the imaginary part of the complex number $\xi_0 e^{j\omega t}$. Thus in complex number representation, we can write,

$$\xi = \xi_0 e^{j\omega t}$$

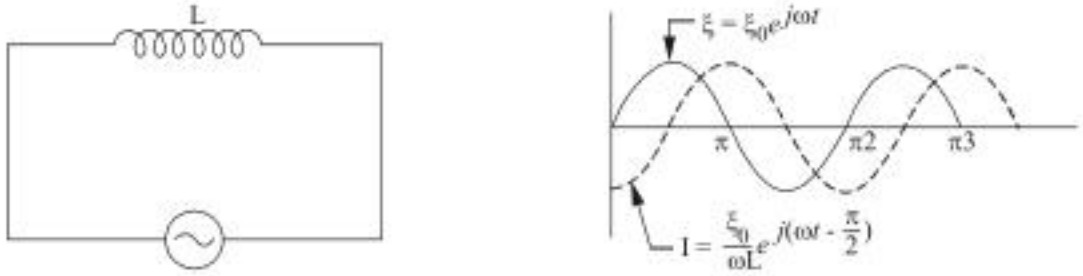


Fig. 16.8 (*a*) & (*b*)

Let I be the instantaneous current in the circuit. Further let dI/dt be the rate of change of current at any instant. Then emf induced in the inductor is given by,

$$\xi_{ind} = -Ld\frac{I}{dt}$$

The negative sign indicates that the induced emf opposes the change of current. To maintain the current flow through the inductor, the applied voltage must be equal and opposite to the induced voltage. Thus, the emf equation of the circuit is,

$$\xi = \xi_{ind} = L\frac{dI}{dt} \text{ or } \xi - L\frac{dI}{dt} = 0$$

or $$\xi_0 e^{j\omega t} - L\frac{dI}{dt} = 0$$

or $$\frac{dI}{dt} = \frac{\xi_0 e^{j\omega t}}{L}$$

On integration, we get,

$$\int dI = \int \frac{\xi_0 e^{j\omega t}}{L} dt$$

$$I = \frac{\xi_0}{L} - \frac{1}{j\omega} - e^{j\omega t} \qquad \ldots(16.30)$$

Now, $\frac{1}{j} = -j = e^{-j\pi/2}$ $\left[\text{since } e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 0 - j = -j\right]$

therefore,

$$I = \frac{\xi_0}{\omega L} e^{j(\omega t - \pi/2)}$$

The quantity ωL is called *inductive reactance* of the inductance and denoted by X_L. Thus,

$$X_L = \omega L$$

Hence,

$$I = \frac{\xi_0}{X_L} e^{j(\omega t - \pi/2)}$$

$$= I_0 e^{j(\omega t - \pi/2)} \qquad \ldots(16.31)$$

where $I_0 = \frac{\xi_0}{X_L}$ is the *peak value of the current.*

Thus,

$$X_L = \omega L = \frac{\xi_0}{I_0} \qquad \ldots(16.32)$$

Thus, the inductive reactance is obtained by dividing peak value of voltage by peak value of current. Again, $X_L = \omega L = 2\pi f_L$ shows that the *inductive reactance is directly proportional to the frequency of the A.C. supply voltage. Its value for D.C. supply is zero.* Further, since,

$$X_L = \frac{\xi_0}{I_0} = \frac{\frac{\xi_0}{\sqrt{2}}}{\frac{I_0}{\sqrt{2}}} = \frac{\xi_{eff}}{I_{eff}} \qquad \ldots(16.33)$$

Thus, the inductive reactance may be defined as the ratio of root mean square (effective or virtual) emf across the inductance to the root mean square current flowing through it. The effect of inductive reactance $X_L = \omega L$ in an A.C. circuit is the same as the resistance R in D.C. circuit to reduce the current in it, but *without loss of power*. The current I in the inductance L is obtained by dividing voltage across it by $j\omega L$. The impedance of the inductance is given by Eq. (16.30).

Equation (16.31) shows that the current lags behind the voltage by $\pi/2$ (or quarter of cycle) as shown in Fig. 16.8 (*b*). Fig. 16.8 (*c*) shows the complex number representation.

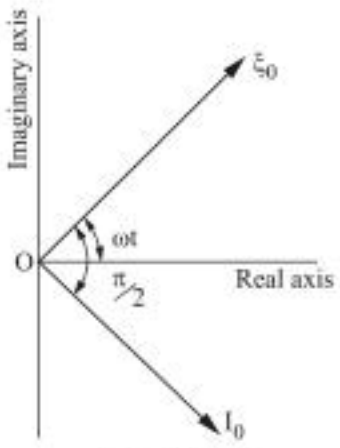


Fig. 16.8 (*c*)

16.8 A.C. CIRCUIT CONTAINING CAPACITOR ONLY

Fig. 16.9 (*a*) shows a circuit containing a capacitor of capacitance C with an alternating voltage $\xi = \xi_0 \sin\omega t$ applied across it. This voltage

is the imaginary part of the complex number $\xi_0\, e^{j\omega t}$. Thus, in complex number representation, we can write,

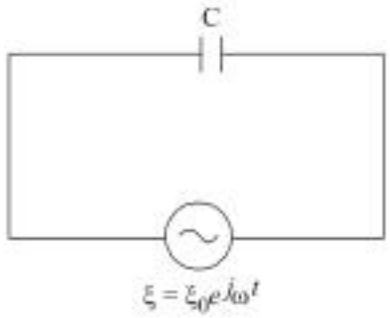


Fig. 16.9 (*a*)

The instantaneous charge on the capacitor at any time t is,

$$q = C\xi = C\xi\, e^{j\omega t} \qquad \ldots(16.34)$$

Corresponding to this charge q, the instantaneous current which is in fact the displacement current due to the variation of voltage across the capacitor plates in the circuit is given by,

$$I = \frac{dq}{dt} = C\,\xi_0 \cdot j\omega e^{j\omega t} = \frac{\xi_0}{\frac{1}{\omega C}}\, j\, e^{j\omega t}$$

$$= \frac{\xi_0}{\frac{1}{j\omega C}} e^{j\omega t} \qquad \ldots(16.35)$$

But $e^{j\pi/2} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$

Hence, $I = \frac{\xi_0}{1/\omega C}\, e^{j(\omega t + \pi/2)}$

or, $$I = \frac{\xi_0}{X_C}\, e^{j(g\omega t + \pi/2)} = I_0\, e^{j(\omega t + \pi/2)} \qquad \ldots(16.36)$$

where $X_C = \frac{1}{\omega C}$ is known as *capacitative reactance* and $I_0 = \frac{\xi_0}{X_C}$ is the *peak value of the current.*

Clearly equation (16.36) shows that the *current leads the voltage by* $\frac{\pi}{2}$ *as shown in Fig. 16.9 (b).*

Futher, the complex number representation is shown in Fig. 16.9 (*c*). The currents I in the circuit is obtained by dividing emf applied by impedance of the capacitor $\left(\frac{1}{j\omega C}\right)$ as shown in eq. (16.35).

Since the capacitance reactance $X_C = \frac{1}{\omega C}$ is

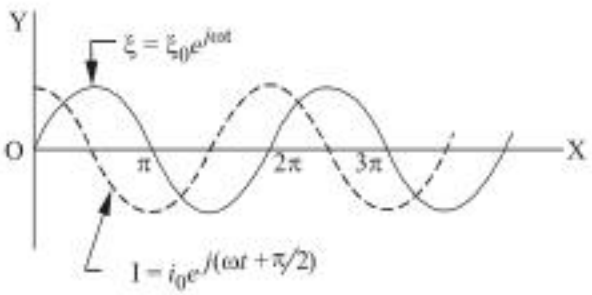


Fig. 16.9 (*b*)

inversely proportional to the frequency of applied voltage. Hence a condenser *offers negligible reactance at high frequency and works are "by-pass capacitor" and very high reactance at low frequency and works as "blocking condenser".*

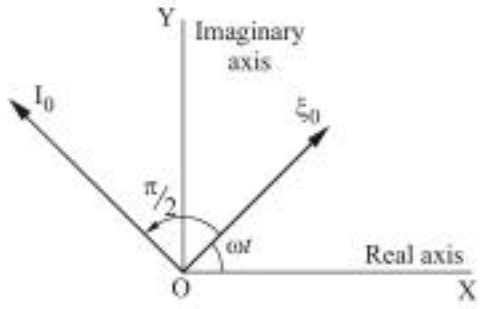


Fig. 16.9 (*c*)

16.9 A.C. CIRCUIT CONTAINING INDUCTANCE AND RESISTANCE IN SERIES

Let us consider a circuit as shown in Fig. 16.10 (a) containing a series combination of an inductance L and resistance R with an alternating emf $\xi = \xi_0 e^{j\omega t}$ applied across the combination. The instantaneous current in the circuit is given by,

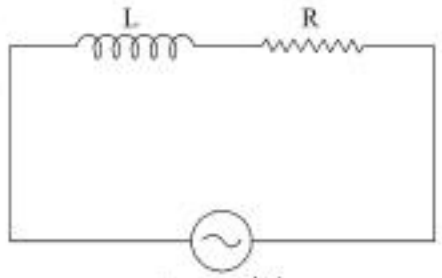


Fig. 16. 10 (*a*)

$$I = \frac{\text{Applied emf}}{\text{Vector impedance of the circuit}} \qquad \ldots(16.37)$$

In the present case,

Vector impedance $Z = R + j\omega L$

where R is impedance due to resistance and $j\omega L$ is the impedance due to inductance L. Z is also known as impedance operator. However, impedance |Z| is given by,

$$|z| = \sqrt{R^2 + \omega^2 L^2}$$

Thus the current,

$$I = \frac{\xi_0 e^{j\omega t}}{R + j\omega L} \qquad \ldots(16.38)$$

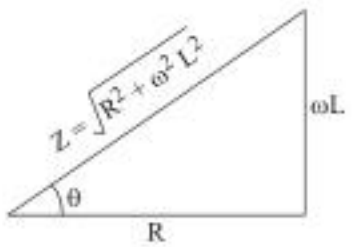


Fig. 16.10 (*b*)

We can express $\frac{1}{R + j\omega L}$ as follows:

$$\frac{1}{R + j\omega L} = \frac{R - j\omega L}{(R + j\omega L)(R - j\omega L)}$$

$$= \frac{R - j\omega L}{R + \omega^2 L^2}$$

$$= \frac{R}{R^2 + \omega^2 L^2} - \frac{j\omega L}{R^2 + \omega^2 L^2} \qquad \ldots(16.39)$$

Let, $\alpha = \frac{R}{R^2 + \omega^2 L^2}$ and $\frac{\omega L}{R^2 + \omega^2 L^2}$

Then, $\dfrac{1}{R + j\omega L} = \alpha - j\beta$

Further put $\tan\theta = \dfrac{\omega L}{R}$, as shown in Fig. 16.10 (*b*).

Then, $\cos\theta = \dfrac{R}{\sqrt{R^2 + \omega^2 L^2}}$

$\sin\theta = \dfrac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$

Then, $\dfrac{1}{R + j\omega L} = \alpha - j\beta = \dfrac{1}{\sqrt{R^2 + \omega^2 L^2}}\left[\cos\theta - j\sin\theta\right]$

Hence, eq. 19.38 becomes, $I = \dfrac{\xi_0 e^{j(\omega t - \theta)}}{\sqrt{R^2 + \omega^2 L^2}} = I_0 e^{j(\omega - \theta)}$ ...(16.40)

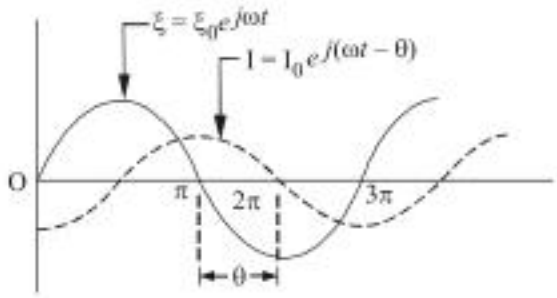
where impedance $Z = \sqrt{R^2 + \omega^2 L^2}$ and $I_0 = \dfrac{\xi_0}{\sqrt{R^2 + \omega^2 L^2}}$

Equation (16.40) represents the variation of current in the circuit with time as shown in Fig. 16.10 (*c*). Equation (16.40) also indicates that

the *current lags behind the applied emf in phase by an angle* θ where $\theta = \tan^{-1}\dfrac{\omega L}{R}$ as shown in Fig. 16.10 (*d*). In other words, we can say that emf (or voltage) leads the current by angle θ. Fig. 16.10 (*d*) represents the complex number representation. However, the voltage ξ_R across the resistance is in phase with the current while voltage ξ_L across the inductance leading the current I by an angle $\pi/2$ and the total voltage, $\xi_0 = I_0\sqrt{R^2 + \omega^2 L^2}$ leading the current by an angle $\theta = \tan^{-1}\dfrac{\omega L}{R}$. The impedance

$$Z = \frac{\xi_0}{I_0} = \frac{\frac{\xi_0}{\sqrt{2}}}{\frac{I_0}{\sqrt{2}}} = \frac{\xi_{eff}}{I_{eff}}$$

Thus impedance may also be defined as the ratio of the r.m.s. (or virtual) value of the emf applied across the L – R circuit to the r.m.s. (or virtual) value of current flowing through the circuit.



Fg. 16. 10 (*c*)

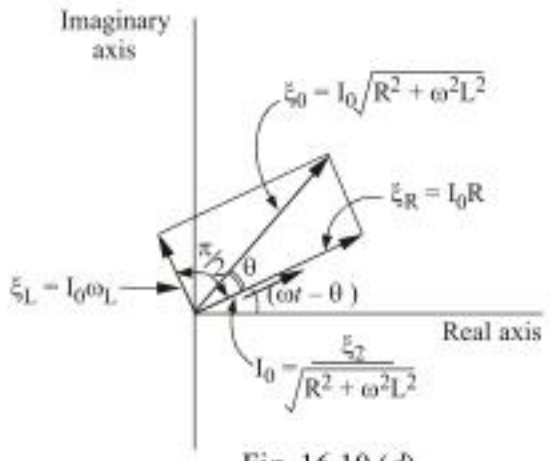


Fig. 16.10 (*d*)

Special Cases

1. If $R \to 0$, *i.e.*, when inductive reactance ωL is very large than to ohmic resistance R which can be neglected. Then $Z = \omega L$ and $\theta = \dfrac{\pi}{2}$.

In *practice, it occurs when the frequency is very high.*

Then current is given by,

$$I = I_0 e^{j(\omega t - \pi/2)}$$

and $$I_0 = \frac{\xi_0}{\omega L}$$

Thus the peak value of current *i.e.* I_0 is inversely proportional to both self inductance and frequency. Therefore, when the frequency is high, a high voltage can produce only small current.

Further since phase lag $\theta = \tan^{-1} \frac{\omega L}{R}$

hence greater the reactance compared with the ohmic resistance the phase lag increases. Thus the current and the emf can be brought into phase by introduction of a non-inductive resistance. Practically this is adopted in A.C. voltmeters and wattmeters.

2. If $\omega L \to 0$ *i.e.* when inductance or the frequency $(f = \omega/2\pi)$ is so small as to be negligible, then ωL will be very small compared to R and we get,

$$\theta \simeq 0 \text{ and } Z = R$$

and $$I = I_0 e^{j\omega t} = \frac{\xi_0 e^{j\omega t}}{R} = \frac{\xi}{R}$$

16.10 A.C. CIRCUIT CONTAINING A RESISTANCE AND A CAPACITANCE IN SERIES

Fig. 16.11 (*a*) shows a circuit consisting of a series combination of a resistance R and a capacitor C with an alternating emf $\xi = \xi_0 j\omega t$ applied across the combination. The instantaneous current I is given by,

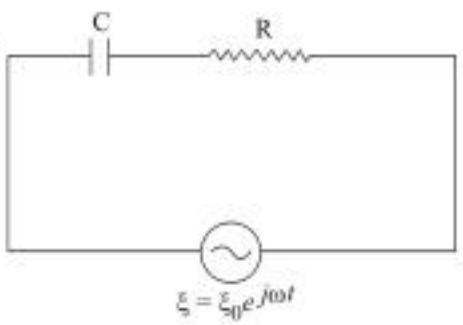


Fig. 16.11 (*a*)

$$I = \frac{\text{Applied emf}}{\text{Vector impedance of the circuit}}$$

$$= \frac{\xi_0 e^{j\omega t}}{R + \frac{1}{j\omega C}} = \frac{\xi_0 e^{j\omega t}}{R - \frac{j}{\omega C}} \qquad \ldots(16.41)$$

where R is impedance due to resistance and $\frac{1}{j\omega C}$ is the impedance due to capacitance C. As in the previous case, we can express,

$$\frac{1}{R - \frac{j}{\omega C}} = \frac{R + \frac{j}{\omega C}}{R^2 + \frac{1}{\omega^2 C^2}}$$

$$= \frac{R}{R^2 + \frac{1}{\omega^2 C^2}} + \frac{j\omega C}{R^2 + \frac{1}{\omega^2 C^2}}$$

Put $$\alpha = \frac{R}{R^2 + \frac{1}{\omega^2 C^2}} \text{ and } \beta = \frac{1/\omega C}{R^2 + \frac{1}{\omega^2 C^2}}$$

and $$\tan\theta = \frac{1}{\omega C R}$$

We get,

$$\frac{I}{R - \frac{j}{\omega C}} = a + j\beta$$

$$= \frac{1}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \left[\cos\theta + j\sin\theta\right]$$

$$= \frac{1}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} e^{j\theta}$$

Hence Eq. (16.41) reduces to

$$I = \frac{\xi_0 e^{j(\omega t + \theta)}}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}$$

or $$I = I_0 e^{j(\omega t + \theta)} \qquad \ldots(16.42)$$

or $$I_0 = \frac{\xi_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}$$

indicates the *peak value of the current*. Equation (16.42) represents the variation of current with time and shows that *current leads the applied voltage in phase by angle* θ as shown in Fig. 16.11 (*b*) given by,

$$\theta = \tan^{-1}\frac{1/\omega C}{R}$$

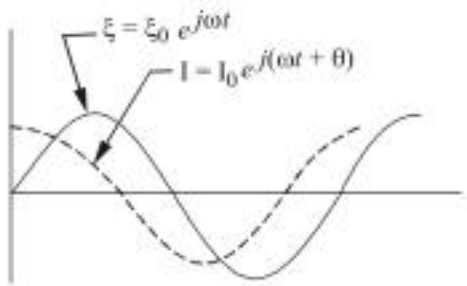


Fig. 16.11 (*b*)

The impedance,

$$Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$$

$$= \sqrt{(\text{Resistance})^2 + (\text{Capacitative reactance})^2}$$

Fig. 16.11 (*c*) shows the complex number representation of current and voltage. *The current leads the applied voltage by phase is the same as the voltage lags behind the current in phase.* The voltage ξ_R across the resistance is in phase with current, while the voltage ξ_C across the condenser lagging the current I by an angle $\pi/2$. Finally the total voltage,

$$\xi_0 = I_0 \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$$

lagging the current I by an angle $\theta = \tan^{-1} \frac{\frac{1}{\omega C}}{R}$.

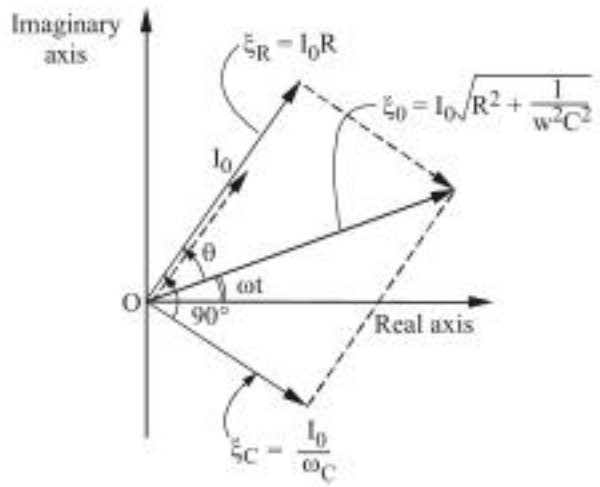


Fig. 16.11 (*c*)

Special Cases

1. If $C \to 0$, then $I_0 \to 0$, this means no current will flow in the circuit with zero capacity and this causes a break in the circuit. The same result will be obtained with very low frequenceis.
2. If $C \to \infty$, then $I_0 = \frac{\xi_0}{R}$. Thus a condenser of infinite capacity does not offer any resistance to A.C. and $\tan\theta = 0$ or $\theta \to 0$. This means the introduction of such ε condenser causes no phase change. The same result follows if the frequency is very high.
3. If $R \to 0$, then $I = \xi_0\, \omega C$ and $\tan\theta \to \infty$ or $= \frac{\pi}{2}$. Thus the current is leading by $\frac{\pi}{2}$ in phase with respect to emf.

16.11 A.C. CIRCUIT WITH RESISTANCE, CAPACITANCE AND INDUCTANCE IN SERIES

Let a sinusoidal alternating emf $\xi = \xi_0 e^{j\omega t}$ be applied to the series combination of L.C.R. as shown in Fig. 16.12 (*a*). The vector impedance of the combination is sum of three parts.

1. R due to resistance R
2. $j\omega L$ due to inductance L
3. $\frac{1}{j\omega C} = -j\omega C$ due to capacitance C.

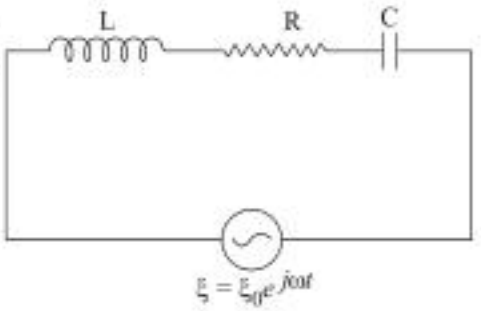


Fig. 16.12 (*a*)

Thus in vector form the emf of eqn. of the circuit is

$$\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I = \xi_0\, e^{j\omega t} \qquad \ldots(16.43)$$

where I represents the instantaneous current in circuit, is given by,

$$I = \frac{\xi_0\, e^{j\omega t}}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \qquad \ldots(16.44)$$

Adopting the same procedure as in the previous article, we can write,

$$\frac{1}{R+j\left(\omega L-\frac{1}{\omega C}\right)}=\frac{R-j\left(\omega L-\frac{1}{\omega C}\right)}{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$$

$$=\alpha-j\beta$$

where $$\alpha=\frac{R}{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$$

and $$\beta=\frac{\omega L\frac{1}{\omega C}}{R^2+\left[\omega L-\frac{1}{\omega C}\right]^2}$$

and if we put, $$\tan\theta=\frac{\omega L-\frac{1}{\omega C}}{R}$$

Then, $$\cos\theta=\frac{R}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$$

and $$\sin\theta=\frac{\omega L-1/\omega C}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$$

Hence, $$\frac{1}{R+j\left(\omega L-\frac{1}{\omega C}\right)}=\frac{1}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}(\cos\theta-j\sin\theta)$$

$$=\frac{1}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}e^{-j\theta}$$

Then the expression for current *i.e.* eq. (16.44) reduces,

$$I=\frac{\xi_0}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}j(\omega t-\theta)$$

or $$I=I_0\,e^{j(\omega t-\theta)} \qquad \ldots(16.45)$$

where, $$I_0=\frac{\xi_0}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}} \qquad \ldots(16.46)$$

I_0 is the *peak value of the current*. And,

$$\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}=Z$$

where Z is called the impedance of the circuit (in magnitude), which is the ratio of peak emf to the peak current. Again impedance, from eq. (16.46), is given by

$$Z=\frac{\xi_0}{I_0}=\frac{\xi_0/\sqrt{2}}{I_0/\sqrt{2}}=\frac{\xi_{eff}}{I_{eff}}$$

Thus the impedance of a L.C.R. circuit can be defined as the ratio of the r.m.s. value of emf applied to the r.m.s. current flowing through it.

Eq. (16.45) shows that the *current logs the applied voltage in phase by an angle* θ *given by*

$$\theta=\tan^{-1}\frac{\omega L-\frac{1}{\omega C}}{R}$$

The variation of current and voltage with time is shown in Fig. 16.12 (*b*). In other words, the emf and current have a phase difference of θ. This phase difference is shown in Fig. 16.12 (*c*) in the complex plane. The voltage across the resistance ξ_R is shown in phase with the current while,

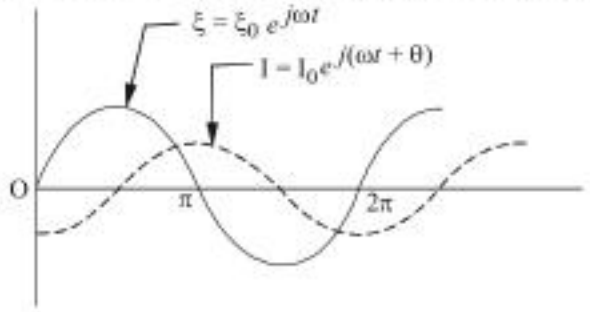


Fig. 16.12 (*b*)

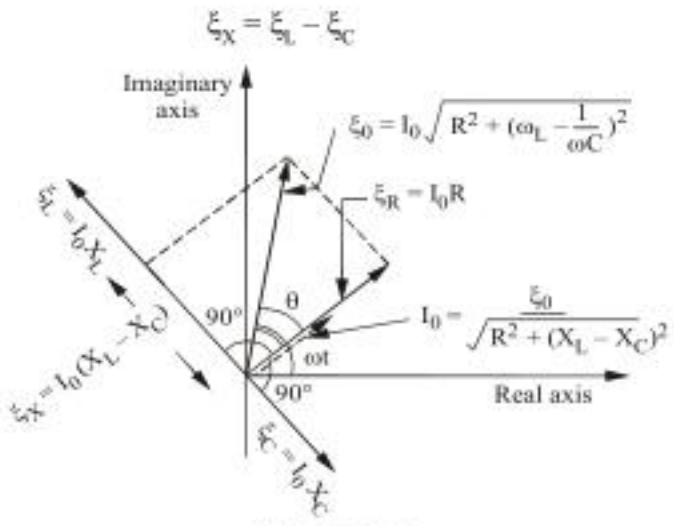


Fig. 16.12 (*c*)

which is the net voltage across the combination of L and C and leads the current by an angle $\pi/2$. The total voltage ξ_0 is shown leading the current by an angle,

$$\theta=\tan^{-1}\frac{X_L-X_C}{R}$$

Special Cases

The phase difference (i.e. leading or lagging) in fact depends on the value of θ which depends on values of X_L and X_C. Depending on the values of X_L and X_C there are three important cases.

1. If $\omega L > \frac{1}{\omega C}$ *i.e.* when impressed frequency $\left(f = \frac{\omega}{2\pi}\right)$ is very large, the phase angle θ will be positive and the current will lag behind the applied emf. The potential difference $I_0 X_L = I_0 \omega L$ across the inductance is greater than potential difference $I_0/\omega C$ across the capacitance and the circuit behaves as an *inductive circuit*. The vector voltage in complex plane is shown in Fig. 16.12 (*c*).

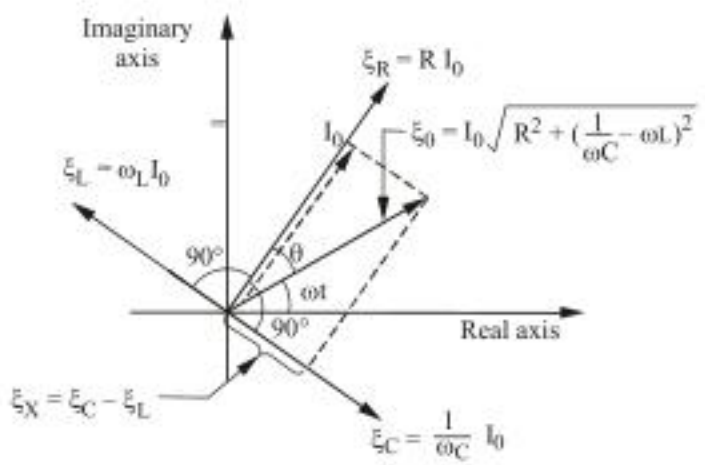


Fig. 16.12 (*d*)

2. If $\omega L < \frac{1}{\omega C}$, the phase angle θ will be negative and the current will lead the applied emf. The circuit behaves as *capacitative circuit*. The vector diagram for the case is shown in Fig. 16.12 (*d*) in complex plane.

3. If $\omega L = \frac{1}{\omega C}$ *a very important and interesting case arises*. In this case,

$$\theta = \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R} = 0$$

i.e. the phase angle θ becomes zero and the emf and current will be in phase. *The potential difference across the inductance and capacitance are equal in magnitude but opposite in phase and therefore cancel out* and the whole voltage is dropped across the resistance. Therefore, the circuit. The vector diagram in complex plane is shown in Fig. 16.12 (*e*).

Under this condition, the peak current will be maximum given by I_0 max

$$I_{0\max} = \frac{\xi_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{\xi_0}{R} \qquad \ldots(16.47)$$

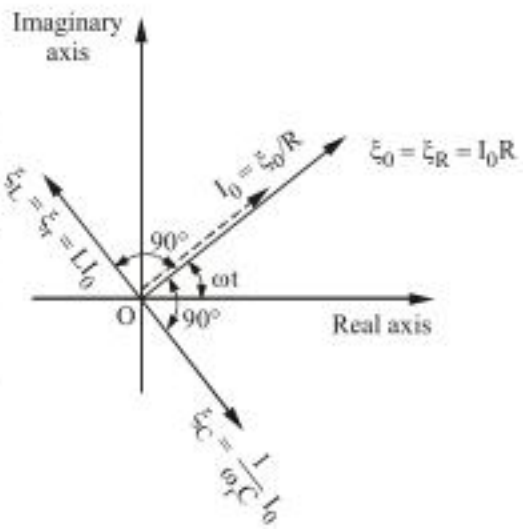


Fig. 16.12 (*e*)

Thus, in this particular case when $\omega L = \frac{1}{\omega C}$ the emf and current will be in phase and *current is maximum*. The circuit is said to be a *series resonant circuit* and the phenomenon of maximum current is called *resonance*. Thus at resonance,

$$\omega_r L = \frac{1}{\omega_r C}$$

or $$\omega_r = \frac{1}{LC}$$

or $$f_r = \frac{1}{2\pi\sqrt{LC}} \qquad \ldots(16.48)$$

Here subscript r corresponds to resonance.

The frequency at which resonance occurs is known as *resonance frequency*. Evidently from Eq. (16.48) the resonant frequency is the same as the frequency of the oscillatory discharge of a series LCR circuit when the resistance is low, as we have discussed in article 11.30. Thus the series LCR circuit is in resonance with applied voltage. The *frequency of the applied voltage coincides with the natural frequency of the circuit*. Fig. 16.12 (*e*) gives the vector diagram of voltage and current for this circuit on the complex plane under this condition. Fig. 16.12 (*f*) shows the variation of peak current with frequency for resonant circuit.

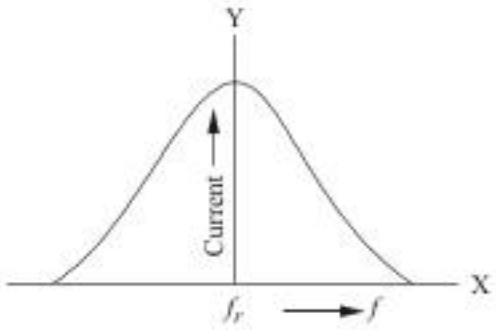


Fig. 16.12 (*f*)

16.12 SHARPNESS OF RESONANCE OF A SERIES LCR CIRCUIT

We have discussed in the previous article that the current amplitude is,

$$I_0 = \frac{\xi_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

At resonance $\omega L = \frac{1}{\omega C}$

therefore, $I_{0max} = \frac{\xi_0}{R}$

Let us examine the effect of circuit resistance onthe current when the supply frequency is varied continuously from a value below the resonant frequency to above it, when the resistance of the circuit is (1) low, (2) medium and (3) high. The curves obtained by plotting amplitude of current (or voltage drop across a fixed resistance) against ω (or f) are known as *resonance curves*.

Fig. 16.13 (*a*) shows the variation of current amplitude as function of frequency for three different values of resistance R.

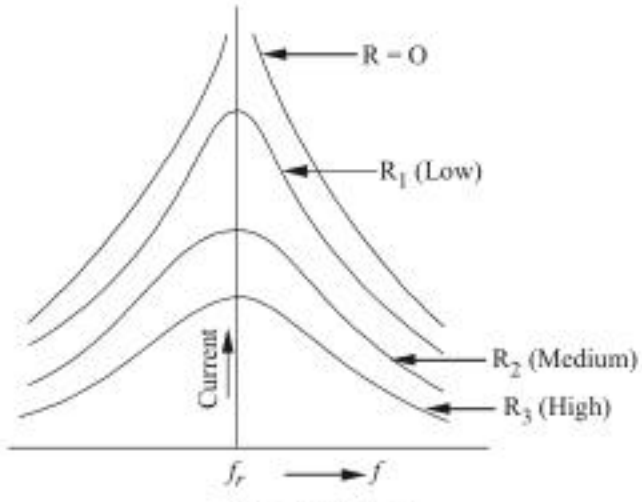


Fig. 16.13 (*a*)

Clearly, as the resistance in the circuit is reduced the resonance curve becomes sharper. The peak value of I_0 shows that the circuit responds only to the frequency exactly equal to its natural frequency of the circuit $\omega_r \left(\omega_r = \frac{1}{\sqrt{LC}} \right)$ for low resistance in LCR circuit) and to none others. The resonance here is, therefore, said to be *sharp.*

Thus, *sharpness of resonance is, in a way, a measure* of the rate of *fall of amplitude from its maximum value at resonant frequency on either side of it*. The more quickly the current amplitude falls for changes of frequency on both sides of the resonant frequency f_r the sharper is said to be resonance.

On the other hand, if the amplitude remains more or less at it peak value over an appreciable range of frequency on either side $\omega_r = \frac{1}{\sqrt{LC}}$ resonant frequency i.e. even when $\omega_r \neq \frac{1}{\sqrt{LC}}$. Thus, the *circuit responds to a number of freequencies* near about f_r on either side of it. The resonance in this case, is therefore, said to be *flat*.

Variation of Voltage across Inductance and Capacitance in a Series LCR Resonant Circuit

The voltage drop across the inductance coil is

$$\xi_L = I_0 X_L = \frac{\xi_0}{Z} X_L = \frac{\xi_0}{\sqrt{R^2 + \omega^2 L^2}} \omega L$$

For frequency considerably above resonant frequency $\omega L \gg R$, we have,

$$\xi_L = \frac{\xi_0 \omega L}{\omega L} = \xi_0$$

equal to *amplitude of impressed voltage*.

Similarly, the voltage drop across the capacitor is,

$$\xi_C = I_0 X_L = \frac{\xi_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \frac{1}{\omega C}$$

For very low frequencies approaching zero, R^2 can be neglected in comparison with $\frac{1}{\omega^2 C^2}$, we have,

$$\xi_C = \frac{\xi_0}{1/\omega C} \frac{1}{\omega C} = \xi_0$$

equal to the impressed voltage.

Fig. 16.13 (*b*) shows the variation of ξ_L and ξ_C with ω. Both of these curves are similar in shape in the vicinity of the resonance to the curves of Fig. 16.13 (a). This is becuase the resonance phenomenon takes place in a very limited range. The reactance across which the voltage is developed remains substantially the same in the region round about the resonance. The product of current and reactance, therefore, varies with frequency practically in the same way as current.

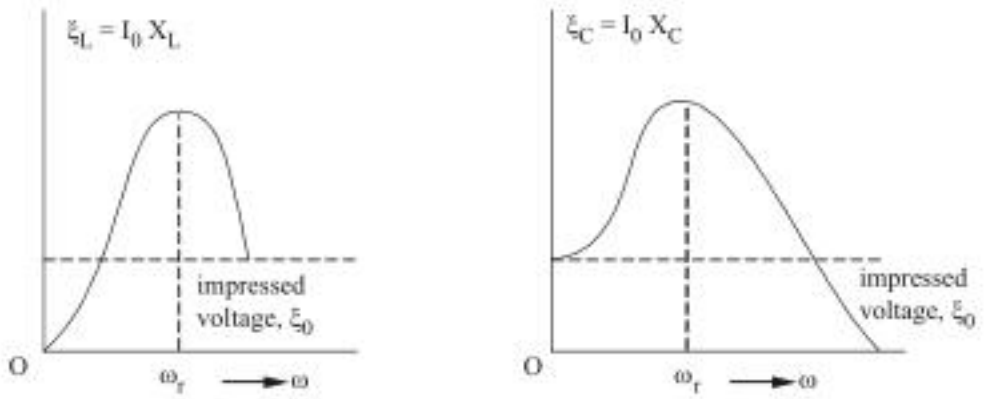


Fig. 16.13 (*b*)

Q of a Circuit

Qualitatively the sharpness of resonance curve is determined by a quality factor called "Q" of the circuit. It is *defined as the ratio of reactance of either the inductance or capacitance at the resonant freequency to the total resistance of the circuit*. Thus, mathematically,

$$Q = \frac{X_L}{R} = \frac{\omega_r L}{R}$$

or
$$Q = \frac{X_C}{R} = \frac{1}{\omega_r C R} \qquad \ldots(16.49)$$

Since at resonance $X_L = X_C$, *i.e.*, $\omega_r L = \dfrac{1}{\omega_r C}$, we get the same value of Q from both expressions.

How the sharpness depends on "Q" can be understood as follows. Since at resonance the current is inversely proportional to the resistance $\left(I_0 = \dfrac{\xi_0}{R} \right)$. So directly proportional to Q (see Eq. 16.49). The value of the current for frequencies appreciably different from resonant frequency is mainly determined by the reactance and not by the resistance. Thus the current is nearly independent of the resistance of the circuit. The result of increasing the resistance of the circuit (*i.e.* lowering Q) reduces the response at the resonant frequency but leaves the other frequencies nearly unaffected. Therefore, if Q is large the resonance curve is sharp.

"Q" is also defined in terms of *lower and upper half-power frequencies*. The half-power frequencies are the frequencies at which the power dissipationin the circuit drops to one half its resonance value. If f_1 and f_2 are the lower and upper half power frequencies and f_r is the resonant frequency.

The power at resonant frequency $= I_{0\max}^2$

$$= \left(\frac{\xi_0}{R} \right)^2 R$$

The power dissipation at f_1 is given by,

$$I_1^2 R = \frac{1}{2} \left(\frac{\xi_0}{R} \right)^2 R$$

where I_1 is current at lower half power frequency

or $$I_1 = \frac{1}{\sqrt{2}} \cdot \frac{\xi_0}{R} = \frac{1}{\sqrt{2}} \text{ current at resonance.}$$

Simlarly I_2 the current at upper half power frequency is

$$I_2 = \frac{1}{\sqrt{2}} \frac{\xi_0}{R} = \frac{1}{\sqrt{2}} \text{ current at resonance}$$

Thus at lower or upper half power frequencies the current amplitude becomes $\frac{1}{\sqrt{2}}$ *times the current amplitude at resonance.*

Hence from eq. (16.47) at lowe or upper half power frequency, we have,

$$\frac{\xi_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{1}{\sqrt{2}} \cdot \frac{\xi_0}{R}$$

On squaring and rearranging, we get

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$$

or $$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

This equation has two solutions one for lower half and other for upper half power frequencies. Thus,

$$\omega_1 L - \frac{1}{\omega_1 C} = -R \qquad \ldots(16.50)$$

$$\omega_2 L - \frac{1}{\omega_2 C} = +R \qquad \ldots(16.51)$$

Adding these two equations, we get,

$$(\omega_1 + \omega_2) L - \frac{1}{C}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) = 0$$

or $$(\omega_1 + \omega_2) L - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \frac{1}{C} = 0$$

or $$\omega_1 \omega_2 = \frac{1}{LC} \qquad \ldots(16.52)$$

or $$\omega_1 \omega_2 = \omega_r^2 \qquad \text{(by eq. 16.48)} \qquad \ldots(16.53)$$

Thus, the resonant frequency f_r is the geometric mean of the upper and lower half power frequencies.

Subtracting Eq. we get,

$$(\omega_2 - \omega_1) L + \frac{1}{C}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right) = 2R$$

$$(\omega_1 + \omega_2)\,L - \frac{\omega_1 + \omega_2}{\omega_1\,\omega_2} - \frac{1}{C} = 2R$$

Substituting $\omega_1\,\omega_2$ from Eq. (16.52)

$$(\omega_2 - \omega_1)\left[L + \frac{1}{1/LC}\,\frac{1}{C}\right] = 2R$$

$$(\omega_2 - \omega_1) = \frac{R}{L}$$

Therefore,

$$\frac{\omega_r}{\omega_2 - \omega_1} = \frac{f_r}{f_2 - f_1} = \frac{\omega_r L}{R} = Q \qquad \ldots(16.54)$$

Q is terms of *band width* is expressed as,

$$Q = \frac{\omega_r}{2\Delta\omega} \qquad \ldots(16.55)$$

where $\Delta\omega$ is the change in ω on either side of ω_r as a result of which the power dissipation becomes half its value at the resonant frequency. Thus,

$$\omega_1 = \omega_r - \Delta\omega$$

and $$\omega_2 = \omega_r + \Delta\omega$$

The frequency difference,

$$\omega_2 - \omega_1 = 2\Delta\omega$$

$2\Delta\omega$ is called the *band width.*

Thus, greater value of Q corresponds to smaller $\omega_1 - \omega_1$ or $2\Delta\omega$ band width, the sharper is the resonance.

Selectivity

If the applied alternating voltage has a number of frequency components (for example, the signal received by a radio antenna has wide range of frequencies). Then a series L.C.R. circuit will give a maximum response (*i.e.*, it will pass a maximum current and have a maximum potential difference a across its inductance) for only that frequency component which has the frequency.

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

Thus out of number of available frequencies it selects one frequency for which the current is maximum and for other frequencies the current is comparatively very small, in other words it shows selectivity. The resonant circuit is consequently known as the "acceptor circuit" as it practically accepts one frequency component of the applied voltage and rejects other. It may be noted that sharper the response (*i.e.* higher the Q value) the more highly selective the acceptor circuit is.

Voltage Magnification

The circuit Q is also a masure of voltage magnification in the series L.C.R. circuit which can be understood as follows.

At resonance the potential difference across inductance and capacitance are equal and 180° out of phase and hence cancel. Therefore, the only potential difference at resonance is across the resistance.

As resonant current is maximum and given by,

$$I_{0\max} = \frac{\xi_0}{R}$$

potential difference across resistance $R = I_{0\max} R = \xi_0$.

Thus, at resonance the potential difference across resistance is equal to the applied emf. The voltage magnification in L.C.R. circuit is defined as,

$$\text{Voltage magnification} = \frac{\text{p.d. across the inductance or capacitance}}{\text{Applied voltage}}$$

$$m = \frac{I_{0\max}\,\omega_r\,L}{\xi_0}$$

$$= \frac{\xi_0}{R}\,\frac{\omega_r L}{\xi_0} = \frac{\omega_r L}{R} = Q = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad \ldots(16.56\,a)$$

Similarly for capacitance,

$$m = \frac{I_{0\max}}{\xi_0}\,\frac{1}{\omega_r C}$$

$$= \frac{\xi_0}{R\,\xi_0}\,\frac{1}{\omega_r C}$$

$$= \frac{1}{\omega_r C R} = Q = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad \ldots(16.56\,b)$$

Thus, the voltage magnification of L.C.R. circuit is equal to its quality factor Q at resonance.

Example 16.1 *An alternating voltage of 110 V 50 cycles is applied to a circuit which contains an inductance of 0.02 henery and a resistance of 10 ohms in series. Determine the current and the phase lag.*

Solution. By Eq. (16.39)

$$I_0 = \frac{\xi_0}{\sqrt{R^2 + \omega^2 L^2}} = \frac{110}{\sqrt{10^2 + (2\times 3.1416\times 50\times 0.02)^2}}$$

$$= 10.5 \text{ amp}$$

$$\theta = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}\frac{2\times\pi\times 50\times 0.02}{10}$$

$$= \tan^{-1}\frac{\pi}{5}$$

$$= 32°8' \text{ lagging}.$$

Example 16.2 *A circuit having a pure inductance of 0.01 henry in series with a resistance of 5 ohms is fed by 200 V. 50 c/sec. a.c. supply. Calculate the current and its phase, the drop of potential across the inductor and resistor, and the power.*

Solution. $I_0 = \frac{\xi_0}{\sqrt{R^2 + \omega^2 L^2}} = \frac{200}{\sqrt{25 + (2\pi f L)^2}}$ on substitution

$$= \frac{200}{\sqrt{25 + (3.14)^2}} = \frac{200}{5.91} = 33.9 \text{ amps}$$

The angle of lag of current is given by,

$$\theta = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}\left(\frac{3.14}{5}\right) = 32^\circ$$

Now the potential difference across the resistance and the inductance are,

$$\xi_R = I_0 R = 33.9 \times 5 = 169.5 \text{ volts}$$

and $$\xi_L = I_0 \omega L = 33.9 \times 3.14 = 106 \text{ volts}$$

Power dissipation is $P = I_0^2 R = (33.9)^2 \times 5 = 5746$ volts

Example 16.3 *An electric lamp marked 100 volts d.c. consumes a current of 10 amperes. It is connected to a 200 volts 50 cycles a.c. mains. Calculate the inductance of the required choke.*

Solution. Given, $I_0 = 10$ amp $R = \frac{V}{I} = \frac{100}{10} = 10$ ohms

$\xi_0 = 200$ volts, $f = 50$ c/sec.

$\omega = 2\pi f = 2\pi \times 50 = 100\pi$

Now, $$I_0 = \frac{\xi_0}{\sqrt{R^2 + \omega^2 L^2}}$$

or $$10 = \frac{200}{\sqrt{(10)^2 + (100\pi L)^2}}$$

or $$(100\pi L)^2 = 300$$

therefore,

$$L = \frac{\sqrt{300}}{100\pi} = 0.055 \text{ henry}$$

Example 16.4 *Find the capacity of a condenser to run a 30 volt, 10 watt lamp when connected in series with an alternating e.m.f. of 220 volts and frequency 50 cycles per second.*

Solution. Here, power of the lamp $= V \times I = 10$ watts, $V = 30$ volts, therefore,

$$I = \frac{W}{V} = \frac{10}{30} = \frac{1}{3} \text{ amp.}$$

and $$R = \frac{V}{I} = \frac{30}{1/3} = 90 \text{ ohms}$$

Now, $$I_0 = \frac{\xi_0}{\sqrt{R^2 + \left(\frac{1}{\omega c}\right)^2}}$$

or $$\frac{1}{3} = \frac{220}{\sqrt{(90)^2 + \left(\frac{1}{100\pi C}\right)^2}}$$

On solving, we get, $C = 4.866 \times 10^{-6}$ F $= 4.866\ \mu$F

Example 16.5 *An alternating emf of 200 volts 50 cycles per sec. is applied to a condenser in series with a 100 volts 50 watt lamp. Find the capacityof the condenser.*

Solution. $\xi I = 50$

$$I = \frac{50}{\xi} = \frac{50}{100} = \frac{1}{2} \text{ amp.}$$

$$R = \frac{\xi}{I} = \frac{100}{1/2} = 200 \text{ ohms}$$

and $$R^2 + \frac{1}{\omega^2 C^2} = \left(\frac{\xi}{I}\right)^2 = \left[\frac{200}{1/2}\right]^2 = 400^2$$

$$\frac{1}{\omega^2 C^2} = 400^2 - 200^2 = 12 \times 10^4$$

$$\omega^2 C^2 = \frac{1}{12 \times 10^4}$$

$$\omega = 2\pi f = 2\pi \times 50 = 100\pi$$

$$C = \frac{1}{100\pi \times 100 \times \sqrt{12}}$$

$$= 9.16 \times 10^{-6} \text{ farads}$$

$$= 9.16 \text{ microfarads}$$

Example 16.6 *A circuit consists of a resistance and capacitance in series. If an alternating emf of 180 volts and 100 c/sec frequency is applied to it, calculate the value of resistance and capacitance when the maximum current is 6 amps. and active power is 360 watts.*

Solution. The circuit resistance is given by power relation

$$R = \frac{P}{I^2} = \frac{360}{6^2} = 10 \text{ ohms.}$$

and the circuit impedance is

$$Z = \frac{V}{I} = \frac{180}{6} = 30 \text{ ohms}$$

The capacitance reactance is

$$X_c = \sqrt{Z^2 - R^2} = \sqrt{30^2 - 10^2} = 28.3 \text{ ohms}$$

therefore the capacitance is

$$C = \frac{1}{\omega X_v} = \frac{1}{2 \times 3.14 \times 100 \times 28.3}$$

$$= 26.2 \text{ micro farad.}$$

Example 16.7 *A resistance of 10 ohms is joined in series with an inductance of 0.5 henry. What capacitance must be put in series with the combination to obtain maximum current? What will be the potential drop across each element of circuit, if it is connected to 200 V 50 Hz mains?*

Solution. Let the capacity be C, then $\left(\frac{1}{\omega C} = \omega L\right)$ for maximum current which corresponds to resonance

or $$C = \frac{1}{\omega^2 L} = \frac{10^6}{0.5 \times (2 \times 3.14 \times 50)^2} \text{ microfarad}$$

$$= 20.25 \text{ microfarad}$$

Current $$I = \frac{V}{R} = \frac{200}{10} = 20 \text{ ampere}$$

Potential difference across $R = RI = 10 \times 20 = 200\,V$

Potential difference across L or $C = \omega LI$

$$= 0.5 \times 2 \times 3.14 \times 50 \times 20 = 3142 \text{ volts}$$

Example 16.8 *A 48 ohms resistor is connected in series with an inductance of 450 mH and a capacitance of 9 microfard (i) What is the resonant frequency? (ii) What will be the current in the circuit when it is supplied from a 120 V source operating at resonant frequency? (iii) What is the voltage across the inductance under these conditions?*

Solution. $\omega L = \frac{1}{\omega C}$ (condition of resonant)

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{450 \times 10^{-3} \times 9 \times 10^{-6}}} = 2\pi f$$

(*i*) $$f = \frac{1}{2\pi\sqrt{450 \times 9 \times 10^{-9}}} = 79 \text{ cycles/sec.}$$

(ii) $$I_{max} = \frac{120}{48} = 2.5 \text{ amp.}$$

Voltage across inductance $= \omega L\, I_{max} = 2\pi \times 79 \times 450 \times 10^{-3} \times 2.5$

$$= 557.0 \text{ volts}$$

Example 16.9 *In a series L.C.R. circuit the values of the various quantities are as follows:*

$\xi = 100$ *volts,* $f_0 = 60$ *c/sec.* $R = 1$ *ohm,* $L = 1$ *henry and* $C = 7.04$ *microfarad.*

Prove that the voltage magnification is equal to quality factor Q.

Solution. Inductive reactance $X_L = \omega_0 L = 2\pi f_0 L$

$$= 2 \times 3.14 \times 60$$

$$= 377 \text{ ohms (approx.)}$$

Capacitive reactance $= X_C = \frac{1}{\omega_0 C} = \frac{1}{2\pi f_0 C} = 377$ ohms (approx.)

At resonance maximum current,

$$I_{max} = \frac{E}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{E}{R} = 100 \text{ amp.}$$

Therefore potential difference across resonance resistance

$$\xi_R = I_{max} R = 100 \text{ volts.}$$

Potential difference across inductance

= potential difference across capacitance

$$= \omega_0 \, L \, I_{max} = \frac{I_{max}}{\omega C} = 377 \times 100 = 37700 \text{ volts}$$

$$= Q\xi \left(\text{as } Q = \frac{\omega_0 L}{} = \frac{1}{\omega_0 CR} = 377 \right)$$

therefore,

Voltage magnification is equal to quality factor Q.

16.13 A PARALLEL (OR ANTI) RESONANT CIRCUIT

Let an alternating emf $\xi = \xi_0 \, e^{j\omega t}$ be applied across a coil of inductance L of negligible resistance in parallel with a capacitor of capacitance C as shown in Fig. 16.14 (*a*). The current through the inductance lags behind the applied emf in phase by $\pi/2$ and is given by equation (16.30),

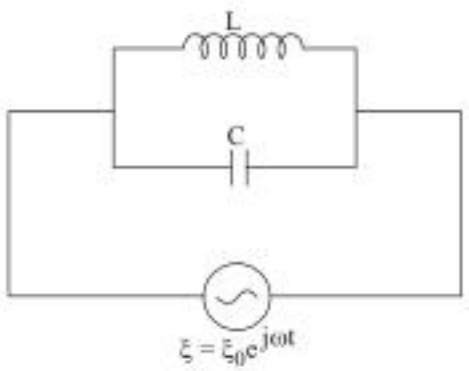


Fig. 16.14 (*a*)

$$I_L = \frac{\xi_0}{j\omega L} e^{j\omega t} = \frac{\xi_0}{\omega L} e^{j(\omega t - \pi/2)}$$

as $$\frac{1}{j} = -j = e^{-j\pi/2}$$

The current through the capacitor leads the applied emf in phase by $\pi/2$ and is given by Eq. (16.35),

$$I_C = \frac{\xi_0}{\frac{1}{j\omega C}} e^{j\omega t} = \frac{\xi_0}{\frac{1}{\omega C}} e^{j(\omega t + \pi/2)}$$

$$(j = e^{-j\pi/2})$$

Therefore the current in the two braches differ in phase by π. The total current I_T will be the vector sum I_L and I_C and will be equal to their numerical difference, since they are $180°$ out of phase. Thus,

$$I_T = I_L + I_C$$

$$= \frac{\xi_0}{j\omega L} e^{j\omega t} + j\xi_0 \, \omega C \, e^{j\omega t}$$

$$= j\left(\omega C - \frac{1}{\omega L} \right) \xi_0 \, e^{j\omega t} \qquad \ldots(16.57)$$

The phase relationship between applied voltage and current is shown in Fig. 16.14 (*b*) in complex plane.

The total impedance of the parallel circuit is given by,

$$\frac{1}{Z_T} = \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{Z_L + Z_C}{Z_L \, Z_C}$$

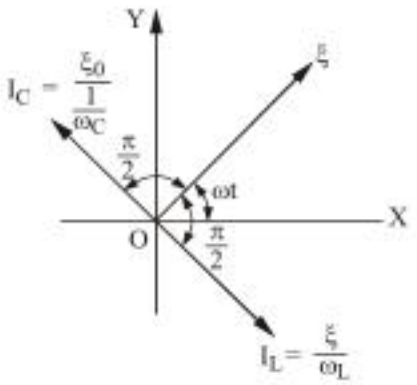


Fig. 16. 14 (*b*)

or $$Z_T = \frac{Z_L Z_C}{Z_L + Z_C}$$

$$Z_T = \frac{j\omega L \frac{1}{j\omega C}}{j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$= \frac{L/C}{j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$= \frac{L/C}{\frac{jL}{C}\left(\omega C - \frac{1}{\omega L}\right)}$$

$$= \frac{1}{j\left(\omega C - \frac{1}{\omega L}\right)}$$

by equation (16.57),

$$Z_T = \frac{\xi_0 e^{j\omega t}}{I_T} \qquad \ldots(16.58)$$

The resonance is obtained when the current in two branches is equal. Since current is out of phase in two branches, the total current will be zero. Hence by Eq. (16.57), at resonance

$$\omega_r C - \frac{1}{\omega_r L} = 0$$

or $$\omega_r^2 = \frac{1}{LC}$$

The resonant frequency $f_r = \frac{1}{2\pi\sqrt{LC}}$

This is identical with the series resonant case.

When I_T is zero. $Z_T = \infty$

Thus at the frequency f_r of the supply voltage, the circuit offeres infinite impedance to the flow of the current so that no current is drawn from the supply. Such a circuit offering infinite impedance to the flow of current is called *parallel resonant circuit* and the particular frequency f_r is called *resonant frequency.*

Thus the current through the parallel resonant circuit is zero and it works as a perfect choke for A.C. and circuit is righly called the *rejector circuit*. Such circuits are used in radios as *filter circuits.*

The paradox of *current flowing* in the coil and the condenser while no current is being supplied by the source is explained by the fact that circuit is now oscillating and the condenser is alternately changing and discharging through the coil in synchronised manner with the impressed frequency. As the coil and condenser have no resistance, the amount of energy taken by the condenser for initial charge from the source is not dissipated. The system is analogous to a frictionaless pendulum which will oscillate continuously upon receiving an initial displacement without further energy being supplied.

Futher at resonance,

$$Z_T = \frac{Z_L\, Z_C}{Z_L + Z_C}$$

$Z_L + Z_C$ is the total impedance when L and C are connected in series. Let us denote it by Z_s, hence we get,

$$Z_T = \frac{j\omega_r L \dfrac{1}{j\omega_r C}}{Z_s}$$

but at resonance $\omega_r L = \dfrac{1}{\omega_r C}$

Therefore,

$$Z_T = \frac{(\omega_r L)^2}{Z_s}$$

multiplying numerator and denominator by $\xi_0 e^{j\omega t}$, we get

$$Z_T = \frac{(\omega_r L)^2}{\xi_0 e^{j\omega t}} \cdot \frac{\xi_0 e^{j\omega t}}{Z_s}$$

For given supply voltage and circuit,

$\dfrac{\omega_r L}{\xi_0 e^{j\omega t}}$ is constant. Hence,

$$Z_T = \text{constant } I_S \qquad \ldots(16.59)$$

where $I_S = \dfrac{\xi_0 e^{j\omega t}}{Z_S}$ is the current when L and C are connected in series with the same emf applied.

Equation (16.59) shows that the impedance variation with frequency in parallel resonance case has exactly the same shape as the current variation with frequency in series resonance case.

16.14 PARALLEL RESONANT CIRCUIT WHEN INDUCTANCE L HAVE SOME RESISTANCE

In actual practice some resistance is always associated with inductance used. Suppose an alternating emf ξ is applied to a capacitance C and inductance L having a resistance R connected in parallel as shown in Fig. 16.15 (*a*). *Because of resistance, the current through the inductive coil will lag behind the applied emf by an angle less than* $\pi/2$ *and so the phase difference between* I_L *and* I_C *will be less than* $180°$. Due to this the resultant impedance of the combination will be maximum at resonance instead of being infinite because of the presence of resistance in the circuit. Clearly the problem of defining parallel resonance becomes more complicated. There are two criterion for solving such problems.

1. The condition of maximum impedance and
2. The condition of unity power factor.

In order to avoid confusion parallel resonance is usually taken to mean the condition for maximum impedance is called anti-resonance. *Anit in the sense that in series circuit at resonance the impedance is minimum.*

Let the combination shown in Fig. 16.15 (*a*) be connected to A.C. emf given by,

$$\xi = \xi_0 e^{j\omega t}$$

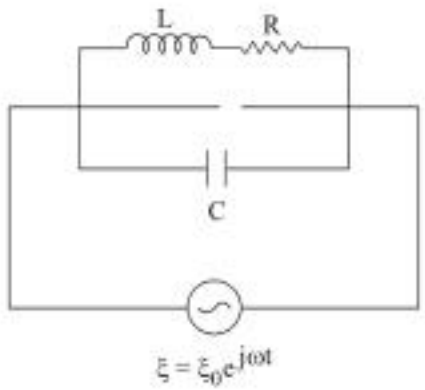


Fig. 16.15 (*a*)

The total impedance Z_T for the combination is given by,

$$\frac{1}{Z_T} = \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{R + j\omega L} + j\omega C$$

$$\frac{1}{Z_T} = \frac{1 + j\omega CR - \omega^2 LC}{R + j\omega L}$$

or $$Z_T = \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega CR}$$

$$= \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega CR} \cdot \frac{(1 - \omega^2 LC) - j\omega CR}{(1 - \omega^2 LC) - j\omega CR}$$

$$= \frac{R + j\omega \left[L - C(\omega^2 L^2 + R^2) \right]}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

$$= R_0 + j\omega L_0 \qquad \text{(say)} \qquad \ldots(16.60)$$

where, $$R_0 = \frac{R}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2} \qquad \ldots(16.61)$$

represents the *effective or apparent resistance of the coil* and,

$$L_0 = \frac{L - C(\omega^2 L^2 + R^2)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2} \qquad \ldots(16.62)$$

represents the *effective inductance of the coil*.

The magnitude of the impedance of the combination is given by,

$$Z = |Z_T| = \sqrt{R_0^2 + \omega^2 L_0^2}$$

or $$Z^2 = R_0^2 + \omega^2 L_0^2$$

After substituting R_0 and L_0, we get,

$$Z^2 \left[(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2 \right]^2$$

$$= R^2 + \omega^2 \left[L - C(\omega^2 L^2 + R^2) \right]^2$$

After simplification,

$$Z^2\left[(1-\omega^2 LC)^2+\omega^2 C^2 R^2\right]^2$$

$$=R^2\left[(1-\omega^2 LC)^2+\omega^2 C^2 R^2+\omega^2 L^2\right]\left[(1-\omega^2 LC)^2+\omega^2 C^2 R^2\right]$$

therefore,

$$Z=\left[\frac{R^2+\omega^2 L^2}{(1-\omega^2 LC)^2+\omega^2 C^2 R^2}\right]^{1/2} \qquad \ldots(16.63)$$

Condition for Unity Power Factor

We know that the current in the circuit will be in phase with the applied voltage *if the impedance of the combination is purely resistive* which corresponds when the *power factor is unity*. Then by eq. (16.60), the appartent reactance L_0 must be zero for impedance to be purely resistive, thus from eq. (16.62),

$$L-C(\omega^2 L^2+R^2)=0$$

or $$R^2+\omega^2 L^2=\frac{L}{C} \qquad \ldots(16.64)$$

And the value of angular frequency which makes the apparent reactance zero is denoted by ω_r given by,

$$\omega_r=2\pi f_r=\sqrt{\frac{1}{LC}-\frac{R^2}{L^2}} \qquad \ldots(16.65)$$

or $$f_r=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{L^2}} \qquad \ldots(16.66)$$

The frequency f_r for which power factor is unity is called the resonant frequency for the given parallel circuit *corresponding to this frequency the impedance is maximum* (*but not infinite*). The current will be minimum corresponding to this paraticular frequency f_r. Thus the circuit will work as *rejector*. The variation of current with frequency in parallel resonant is shown in Fig. 16.15 (*b*). Clearly the current is minimum at resonant frequency. The frequency at parallel resonance given by Eq. (16.66) reduces to the frequency at series resonance given by eq. (16.48) if R is very small so that R^2/L^2 becomes ngligible as compared to 1/LC.

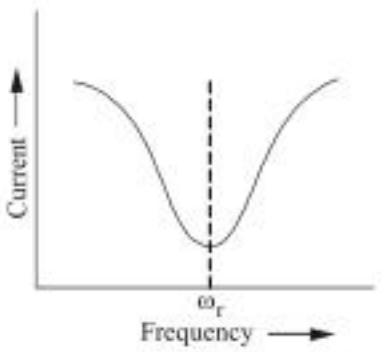


Fig. 16.15 (*b*)

At resonance the impedance of the circuit is obtained by putting the Eqs. (16.64) and (16.65) in (16.63),

$$Z_{Tr}=\sqrt{\frac{LC}{\left[1-\left(\frac{1}{LC}-\frac{R^2}{L^2}\right)LC\right]^2+\left(\frac{1}{LC}-\frac{R^2}{L^2}\right)C^2R^2}}$$

$$=\sqrt{\frac{LC}{\left[1-1+\frac{R^2C}{L}\right]^2+\left(\frac{1}{LC}-\frac{R^2}{L^2}\right)C^2R^2}}$$

$$= \sqrt{\frac{LC}{\frac{R^4C^2}{L^2} + \frac{C^2R^2}{LC} - \frac{R^4C^2}{L^2}}}$$

$$= \sqrt{\frac{L^2}{C^2R^2}}$$

$$= \frac{L}{CR} \qquad \ldots(16.67)$$

Dimensionally $\left|\frac{L}{R}\right| = [T]$, time constant

and similarly [CR] = [T]. Therefore $\left[\frac{L}{CR}\right] = [R]$

is purely resistive and is known as the dynamic resistance of the combination.

Be careful to note that the *resonant frequency in parallel resonance is no longer independent of the resistance in the circuit as in the case with series resonance.*

For parallel resonance to occur,

$\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$ should be real

i.e., $\frac{1}{LC} - \frac{R^2}{L^2} > 0$

or $R^2 < \frac{L}{C}$

or $R < \sqrt{\frac{L}{C}}$

Hence the resistance R should be kept as low as possible.

Current Magnification

The A.C. supply current at resonance is obtained by dividing the supply voltage by impedance at resonance *i.e.*

$$I_r = \frac{\xi_0 e^{j\omega t}}{Z_{TR}} \text{ substituting } Z_{TR} \text{ from eqn. (16.67)}$$

$$= \frac{\xi_0 e^{j\omega t}}{\frac{L}{CR}} = \frac{\xi_0 CR}{L} e^{j\omega t}$$

$$= I_0 e^{j\omega t}, \text{ where } \frac{\xi_0 CR}{L} = I_{ro} \qquad \ldots(16.68)$$

while the oscillating current I_{Cr} in the capacitor is given by,

$$I_{Cr} = \frac{\xi_0 e^{j\omega t}}{\text{Capacitive reactance}}$$

$$= \frac{\xi_0 e^{j\omega t}}{\frac{1}{j\omega_r C}} = j\,\xi_0\,\omega_r\,C\,e^{j\omega t}$$

$$= \xi_0\,\omega_r\,C\,e^{j(\omega t+\pi/2)} \quad (\text{Since } j = e^{j\pi/2})$$

$$= I_{Cro}\,e^{j(\omega t+\pi/2)} \qquad \ldots(16.69)$$

where $I_{Cro} = \xi_0\,\omega_r\,C$

The current magnification is defined as follows:

$$m = \frac{\text{amplitude of current across capacitance}}{\text{amplitude of the supply current}}$$

$$= \frac{I_{Cro}}{I_{ro}}$$

$$= \frac{\xi_0\,\omega_r\,C}{\frac{\xi_0\,CR}{L}}$$

$$= \frac{\omega_r L}{R} = Q \text{ (Quality factor by Eq. (16.56 } a\text{)} \qquad \ldots(16.70)$$

Similarly the oscillatory current I_{Lr} in the inductor is given by,

$$I_{Lr} = \frac{\xi_0\,e^{j\omega t}}{\text{Inductive reactance}}$$

$$= \frac{\xi_0\,e^{j\omega t}}{j\omega_r L}$$

$$= -\frac{j\,\xi_0\,e^{j\omega t}}{\omega_r L}$$

$$= \frac{\xi_0}{\omega_r L}\,e^{j(\omega t-\pi/2)} \quad (\text{as } -j = e^{-j\pi/2})$$

$$= I_{Lro}\,e^{j(\omega t-\pi/2)}$$

Magnification,

$$m = \frac{I_{Lro}}{I_{ro}}$$

$$= \frac{\frac{\xi_0}{\omega_r L}}{\frac{\xi_0\,CR}{L}}$$

$$= \frac{1}{\omega_r\,CR} = Q \qquad [\text{See Eq. (16.56 } b\text{)}] \qquad \ldots(16.71)$$

Thus a parallel circuit (*i.e., rejector circuit), gives a current magnification equal to the quality factor at resonance similar to the voltage magnification by a series (i.e., acceptor circuit).*

Selectivity of a Parallel Resonant Circuit

We have interpreted eq. (16.59) and concluded that impedance variation with frequency in parallel resonance case has exactly the same shape as the current variation with frequency in series resonance case. Fig. 16.15 (*c*) shows such variation. At resonance the impedance is given by eq. (16.67), *i.e.*

$$Z_{Tr} = \frac{L}{CR}$$

By *selectivity of the circuit we mean its ability to present a high impedance at resonant frequency and much lower impedance at other frequencies*. Thus greater the sharpness (or peakness) of the curve (*i.e.* high Q) the greater is the selectivity. The impedance variation of the parallel circuit is shown in Fig. 16.15 (*c*),

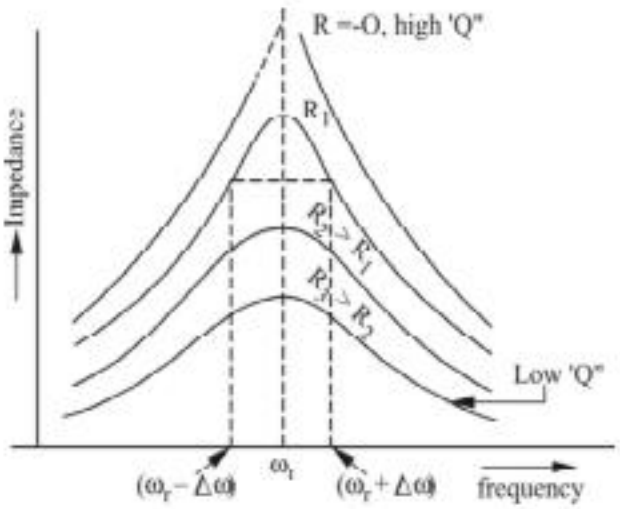


Fig. 16.15 (*c*)

$$\frac{1}{Z_T} = \frac{1}{Z_L} + \frac{1}{Z_C}$$

$$= \frac{1}{R + j\omega L} + j\omega C$$

$$= \frac{R + j\omega CR - \omega^2 LC}{R + j\omega L}$$

or $$Z_T = \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega CR}$$

In practice $R \to 0$ and we have,

$$Z_T = \frac{j\omega L}{(1 - \omega^2 LC) + j\omega CR}$$

And its magnitude directly

$$Z_T = \frac{\omega_r L}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}}$$

at $\omega = \omega_r$, $Z_T = Z_{Tr}$

Now if $\omega = \omega_r \pm \Delta\omega$, the terms ωL and $\omega^2 C^2 R^2$ will not differ appreciably from $\omega_r L$ and $\omega_r^2 C^2 R^2$ but the term $(1 - \omega^2 LC)$ will give a considerable change.

Thus when $\omega = \omega_r$ and $R \to 0$, we get,

$$Z_T = \frac{\omega_r L}{\sqrt{(1 - \omega^2 LC)^2 + \omega_r^2 C^2 R^2}}$$

If at $\omega = \omega_r + \Delta\omega$

the impedance Z_T is $\frac{1}{\sqrt{2}}$ times its maximum value. Thus,

$$Z_T = \frac{1}{\sqrt{2}} Z_{Tr} = \frac{1}{\sqrt{2}} = \frac{L}{CR}$$

At resonance and under condition $R \to 0$ by Eq. (16.65)

$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$\frac{1}{\sqrt{2}} \frac{L}{CR} = \frac{\omega_r L}{\sqrt{\left[1-(\omega_r + \Delta\omega)^2 LC\right]^2 + \omega_r^2 C^2 R^2}}$$

On squaring, we get,

$$\left[1-(\omega_r + \Delta\omega)^2 LC\right]^2 + \omega_r^2 C^2 R^2 = 2\omega_r^2 C^2 R^2$$

or $$\left[1-(\omega_r + \Delta\omega)^2 LC\right]^2 = \omega_r^2 C^2 R^2$$

or $$\left[1-(\omega_r + \Delta\omega^2 + 2\omega_r \Delta\omega)\, LC\right]^2 = \omega_r^2 C^2 R^2$$

The term $\Delta\omega^2$ is very small and neglected,

$$\left[1-\omega_r^2 LC - 2\omega_r \Delta\omega\, LC\right]^2 = \omega_r^2 C^2 R^2$$

but $\omega_r^2 = \frac{1}{LC}$

$$\left(-\frac{2\Delta\omega}{\omega_r}\right)^2 = \omega_r^2 C^2 R^2$$

or $$-\frac{2\Delta\omega}{\omega_r} = -\omega_r CR$$

or $$-\frac{\omega_r}{2\Delta\omega} = \frac{1}{\omega_r CR}$$

$$= Q$$

$$= \frac{\omega_r L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad \ldots(16.72)$$

$2\Delta\omega$ *is the band width of the resonance curve. Thus greater the value of Q corresponds to smaller band width the sharper is resonance and greater selectivity.*

16.15 COMPARATIVE STUDY OF A SERIES RESONANT AND A PARALLEL RESONANT

It is of interest to compare the two circuits. The Table below gives the result discussed in the last two articles.

Series resonant circuit	*Parallel resonant circuit*
1. Series resonant frequency is $f_r = \frac{1}{2\pi\sqrt{LC}}$	Here, $f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$
2. At resonance the power factor is unity and the impedance is purely resistive $Z_r = R$	In this case the power factor is also unity at resonance frquency but the impedance is given by, $Z_r = \frac{L}{CR}$
3. At resonance, the impedance of the circuit is *minimum* while the admittance is maximum.	Here at resonance the impedance of the circuit is *maximum* while the admittance is minimum.
4. Series circuit admits *maximum current* at resonant frequency.	On the other hand this circuit allows a *minimum current* at the resonant frequency.
5. Series circuit is called *acceptor circuit* because it accepts a particular frequency and rejects others.	This circuit is called a *rejector* because it rejects only one frequency and accepts others.
6. At resonance the circuit exhibits a *voltage magntification* equal to the quality factor.	This circuit shows a similar current *magnification* equal to the quality factor.

16.16 POWER IN A.C. CIRCUIT

Power is the rate of doing work. As in an A.C. circuit both the applied emf and the current vary continuosly with time, hence the power in an AC circuit is equal to the product of the instantaneous emf and instantaneous current averaged over a complex cycle.

The power cannot be calculated by complex number analysis *i.e.* by directly multiplying complex ξ and complex I becasue the real or imaginary parts of ξI is not equal to the product of the real or imaginary parts of ξ and I respectively. Therefore the power is calculated by actual emf and actual current that is by taking imaginary parts of the complex quantities.

Let us consider difference circuit:

1. Purely Resistance Circuit

In such circuit the current and the applied emf are in the same phase and can be expressed as,

$$\xi = \xi_0 \sin \omega t$$

$$I = I_0 \sin \omega t$$

The power $= \xi I = \xi_0 I_0 \sin^2 \omega t$

Average rate of doing work or power over one cycle.

$$= \frac{\xi_0 I_0 \int_0^{2\pi} \sin^2 \alpha \, dx}{\int_0^{2\pi} dx} \text{ where } \alpha = \omega t$$

$$= \frac{\xi_0 I_0}{2\pi} \frac{1}{2} \int_0^{2\pi} (1 - \cos 2\alpha)\, d\alpha$$

$$= \frac{\xi_0 I_0}{2} = \frac{\xi_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}}$$

$$= \xi_{rms} \times I_{rms} \qquad \ldots(16.73)$$

Unit of Power

When emf is expressed in volt and current in ampere then unit of power is watt.

2. Power in inductive circuit

As we have discussed that the current through a pure inductance lags behind the applied emf in phase by $\pi/2$, therefore,

$$\xi = \xi_0 \sin \omega t$$

and $$I = I_0 \sin\left(\omega t - \frac{\pi}{2}\right)$$

Instantaneous power $= \xi I$

$$= \xi_0 I_0 \sin \omega t \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= -\xi_0 I_0 \sin \omega t \cos \omega t$$

$$= -\frac{\xi_0 I_0}{2} \sin 2\omega t$$

Average power over one cycle,

$$= -\frac{\xi_0 I_0}{2} \frac{\int_0^{2\pi} \sin 2\alpha \, d\alpha}{\int_0^{2\pi} d\alpha}$$

$$= \frac{\xi_0 I_0}{4\pi} \frac{1}{2} \Big| \cos 2\alpha \Big|_0^{2\pi} = 0$$

Thus power dissipation in a pure inductance is zero and the current in such circuit is known as idle or wattless current. The choke coil works on this principle.

3. Purely Capacitative Circuit

As we have discussed that the current in a purely capacitative circuit leads the applied emf in phase by $\pi/2$. Thus,

$$\xi = \xi_0 \sin \omega t$$

then $$I = I_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

Instantaneous power $= \xi I = \xi_0 I_0 \sin \omega t \cos \omega t$

$$= \frac{\xi_0 I_0}{2} \sin 2\omega t$$

As the average of $\sin 2\,\omega t$ over one cycle is zero. Hence the *average power is zero*. Thus the average power dissipation in a pure capacitative circuit is zero, hence the current through it is also *wattless or idel*.

4. Circuit Containing L, C and R

As we have discussed that the emf and current are not in phase. These can be written as,

$$\xi = \xi_0 \sin \omega t$$

and $$I = I_0 \sin (\omega t + \theta)$$

where phase angle θ is given by,

$$\theta = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R}$$

Power at any instant

$$= \xi I$$

$$= \xi_0 I_0 \sin \omega t \sin (\omega t + \theta)$$

$$= \xi_0 I_0 \sin \alpha \sin (\alpha + \theta), \quad \alpha = \omega t$$

$$= \xi_0 I_0 \left[\sin^2 \alpha \cos\theta + \sin \alpha \cos \alpha \sin \theta\right]$$

$$= \xi_0 I_0 \left[\sin^2 \alpha \cos\theta + \frac{1}{2} \sin 2\alpha \sin \theta\right]$$

The average rate of doing work or power over one cycle

$$= \frac{\xi_0 I_0 \cos\theta \int_0^{2\pi} \sin^2 \alpha \, d\alpha \quad \frac{1}{2} \xi_0 I_0 \sin\theta \int_0^{2\pi} \sin 2\alpha \, d\alpha}{\int_0^{2\pi} d\alpha} + \frac{}{\int_0^{2\pi} d\alpha}$$

As we have evaluated above that average of $\sin^2 \alpha = \frac{1}{2}$ and that of $\sin 2\alpha$ is zero. Thus the second part does not contribute to the power is wattless, hence

Average power over one cycle $= \frac{1}{2} \xi_0 I_0 \cos\theta$

$$= \frac{\xi_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} = \cos\theta$$

$$= \xi_{rms} \cdot I_{rms} \cos\theta \qquad \ldots(16.74)$$

The term $\xi_{rms} \cdot I_{rms}$ is called apparent power and $\cos\theta$ is *called power factor*. Thus,

$$\text{Power factor } (\cos\theta) = \frac{\text{True power}}{\text{Apparent power}} \qquad \ldots(16.75)$$

(*i*) Power factor in $L-C-R$ circuit:

$$\cos\theta = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad \ldots(16.76)$$

(*ii*) Power factor in inductive circuit

$$\cos\theta = \frac{R}{\sqrt{R^2+\omega^2 L^2}} \qquad \ldots(16.77)$$

(iii) Power factor in capacitative circuit:

$$\cos\theta = \frac{R}{\sqrt{R^2+\frac{1}{\omega^2 C^2}}} \qquad \ldots(16.78)$$

16.17 CHOKE COIL

As we have discussed the current lags in phase behind $\pi/2$ in a pure inductor and leads in phase by $\pi/2$ in a capacitor when connected in an A.C. circuit. Thus, there is no power loss, as the current through them is wattless. Hence, current in an A.C. circuit is reduced by the inclusion of a capacitor or inductor or both with-out much loss of energy. The *inductor which is used for reducing the current in the A.C. circuit is called a choke or choking coil.* A choke coil consists of thick copper wire wound closely in a large number of turns over a soft iron laminated core. The resistance of the choke coil consists of thick copper wire wound closely in a large number of turns over a soft iron laminated core. The resistance of the choke coil is almost negligible, since loss of power is equal to I^2R. Thus the loss of power is least, though there is some loss of power due to hysteresis which is much loss than I^2R.

Further, the average power dissipated in the choke coil is given by,

$$P = \frac{1}{2}\xi_0 I_0 \cos\theta$$

where the power factor,

$$\cos\theta = \frac{R}{\sqrt{R^2+\omega^2 L^2}}$$

In choke coil L is large while R is very small. Hence $\cos\theta$ is nearly zero. Therefore the power consumed by the coil is extremely small.

Preference of Choke coil over a Resistance

If resistance is used in place of inductance the power loss is I^2R. Hence to avoid this loss, we prefer choke coil.

REVIEW QUESTIONS

16.1 Define complex number. Explain why complex numbers can be used to express sinusoidal quantities.

16.2 Define pure inductance. Show that the current in an inductive circuit lags behind the voltage by $90°$. Plot the currrent, self-induced emf and voltage curve and construct the vector diagram for an inductive circuit.

16.3 Why is a capacitance of practically infinite d.c. resistance? Explain the passage of alternating current through a purely capacitive circuit.

Deduce equation for current in a cpacitive circuit. Plot the current and voltage curves for the capacitive circuit.

16.4 Explain the impedance and reactance in an a.c. circuit. Distinguish between the two.

16.5 Write the equations describing the impedance and reactance for a capacitive, inductive and for the following series circuit:

1. R–L
2. R–C
3. R–L–C

16.6 Derive expressions for current and impedance when an alternating emf is applied to a circuit having (*i*) capacity and resistance in series (*ii*) inductance and resistance in series.

16.7 Derive an expression for the amplitude and phase of the current in a series circuit consisting of a resistance R, a self-inductance L and a capacitance C, when an laternating emf is applied.

16-8 Describe the behavious of a series $R-L-C$ circuit for the case when $X_C > X_L$ and $X_L < X_C$. Discuss the phenomena of resonance in series $R-L-C$ circuit.

16-9 What do you understand by the quality factor (Q) of a circuit? Explain the dependence of the sharpness of the resonance on Q in series $L-C-R$ circuit.

16.10 Derive an expression for Q in terms of bandwidth of the curve.

16.11 Prove that Q is the factor by which the applied voltage is magnified across L and C at resonance. Why LCR circuit is often called an acceptor circuit?

16.12 What is the condition of resonance when an inductor is shunted by a capacitor and explain the paradox of current flowing in the coil and condenser while no current is being supplied by the source at resonance?

16.13 What is the difference between parallel resonance and anti-resonance? Derive an expression for impedance when an inductive coil is shunted by a condenser.

16.14 What do you understand by wattless and power components of an alternating current?

16.15 Find the power in a.c. circuit having (*i*) resistance and capacity (*ii*) resistance and inductance. What is meant by the power factor of a circuit?

PROBLEMS

16.1 A non-inductive resistance of 12 ohms is connected in series with a coil of negligible resistance having an inductance of 0.016 henry. An alternating emf of 130 volts 50 cycles is applied to the circuit. Calculate the current and potential difference across the resistance and the inductance.

16.2 A circuit contains a resistance of 4 ohms and an inductance of 0.68 henry and an alternating effective emf of 500 volts at a frequency of 120 cycles per second is applied to it. Find the value of the effective current in the circuit and power factor.

16.3 An alternating emf of 200 volts and 50 periods per second is applied to a condenser in series with a 20 volt, 5 watt lamp. Find the condenser required to run the lamp.

16.4 A resistance of 10 ohms is joined in series with an inductance of 0.5 henry. What capacitance should be put in series with the combination to obtain the maximum current. What will be the potential difference across the resistance, inductance and capacity? The current is being supplied by 200 volts and 50 cycles mains.

16.5 Alternating emf of 100 virtual volts is applied to a circuit of resistance 0.5 ohm and inductance 0.01 henry, the frequency being 50 cycles/sec. What is the current and lag in the time between emf and current?

16.6 A RLC circuit has R = 2 ohms, L = 0.005 henry and C = 600 microfarad. If potential difference of 100 volts (rms) at frequency 50 applied. Calculate (i) current at resonance (ii) power and (iii) drop in volts along each element of the circuit.

16.7 A coil of resistance 60 ohms and inductance 3 henry is connected in series with a condenser of 4 microfarad and an a.c. supply of 200 volts and 50 c/s. Calculate

1. The impedance in the circuit
2. The phase difference between current and voltage.
3. The potential difference across inductance coil and
4. The potential difference across the capacitor.

16.8 A 60 volt 10 watt lamp is to be run 100 volt 60 cycle mains.
1. Calculate the inductance of the choke required.
2. How much pure resistance would be necessary in the circuit to achieve the same result? What is the disadvantage in this case?

16.9 An inductance of 5 milli henry is connected in parallel with the condenser of capacity 500 microfarad. The two are joined in series with a non-inductive resistance of 4 ohms and a source of alternating emf of 100 volts 50 cycles. Determine the current in the main circuit. At what frequency is the circuit anti-resonant?

16.10 A series circuit containing two pure elements has the following current and the applied voltage,

$\xi = 200 \sin (200t + 50°)$ volts

$I = 4 \cos (2000t + 13.2°)$ amps.

Calculate the values of the elements of the circuit.

ANSWERS

1. 10 amp, 120 volts, 50.24 volts **2.**0.97 amp, cos = θ 1/128.11

3. C = 4.001 microfarad

4. C = 20.28 microfarad, 200 volts, 3140 volts, 3140 volts

5. 31.3 amp, 4.5×10^{-3} sec.

6. 50 amp (rms), 1.16 kilowatt, 47.2 volts, 37.05 volts, 125.08 volts

7. 157.8 ohms, $\tan\theta = -\frac{73}{30}$ lagging, 1190 volts, 1008 volts

8. L = 1.274 henry, R = 240 ohms, Extra loss of energy 6.67 watt

9. 24.82 amp, 100 hertz **10.** R = 30 ohms, C = 12.5 microfarad

SELF-TEST OBJECTIVE QUESTIONS

16.1. An A.C. wave is known by
(*a*) undirectional wave (*b*) bi-directional wave
(*c*) without directional wave (*d*) none of the above

16.2. Vectors are a short hand for the representation of alternating voltages and currents and their use greatly simplifies the problems in A.C. work
(*a*) True (*b*) False

16.3. A choke has
(*a*) A larger number of turns (*b*) Closely spaced turns
(*c*) Necessarily an iron core (*d*) none of these

16.4. Power in A.C. circuit when voltage and current are in phase
(*a*) ξI (*b*) $\xi I \sin\theta$
(*c*) $\xi I \cos\theta$ (*d*) zero

16.5. Impedance is composed of
(*a*) Resistance and capacitance (*b*) Resistance and inductance
(*c*) Resistance and reactance (*d*) Resistance and inductance or capacitance

16.6. At resonance, circuit impedance is
(*a*) Purely capacitive (*b*) Purely inductive
(*c*) Purely resistive (*d*) None of these

16.7. Q factor of a coil is measure of its
(*a*) mutual inductance (*b*) self-inductance
(*c*) retentivity (d) selectivity

16.8. In parallel resonant circuit, at resonance the impedance is
(*a*) Minimum
(*b*) Maximum
(*c*) Equal to difference of inductive and capacitive impedances
(*d*) none of the above

16.9. A series resonant circuit offers
(*a*) zero impedance (*b*) infinite impedance
(*c*) very small finite impedance (*d*) none of these

16.10 At frequencies above resonant frequency, impedance is
(*a*) inductive (*b*) capacitive
(*c*) resistive (*d*) none of these

16.11 At frequency below resonant frequency impedance is
(*a*) inductive (*b*) capacitive
(*c*) resistive (*d*) none of these

16.12 Quality factor Q of a coil is

(*a*) $Q = \frac{\omega L}{R}$ (*b*) $Q = \omega LR$

(*c*) $Q = \frac{R}{\omega L}$ (*d*) $Q = \frac{1}{\omega LR}$

16.13 Bandwidth of a series resonance circuit is

(*a*) $Q_0 f_0$ (*b*) $\frac{R}{2\pi L}$

(*d*) $\frac{R}{4\pi L}$ (*d*) none of these

16.14 Bandwidth in a parallel resonant circuit is

(*a*) $\frac{f_0}{Q_0}$ (*b*) $\frac{Q_0}{f_0}$

(*d*) $\frac{R}{4\pi L}$ (*d*) none of these

16.15 Increasing circuit's resistance R
(*a*) has no effect on bandwidth (*b*) decreases it bandwidth
(*d*) increases its bandwidth (*d*) none of these

ANSWERS

1. (*b*)	**2.** (*a*)	**3.** (*d*)	**4.** (*a*)	**5.** (*c*)	**6.** (*c*)
7. (*d*)	**8.** (*b*)	**9.** (*c*)	**10.** (*a*)	**11.** (*b*)	**12.** (*a*)
13. (*b*)	**14.** (*a*)	**15.** (*c*)			

CHAPTER 17

TRANSFORMERS AND A.C. BRIDGES

17.1 COUPLED CIRCUIT

One of the most important feature of A.C. is that the energy (emfs and currents) can be transferred from one circuit or coil to another circuit or coil easily and in an efficient manner. So we may define the coupled circuits as the circuits so arranged that the energy can be transferred from one circuit to another. Transformer is the simplest example of the coupled circuit.

The circuits are said to be coupled together if they have a common impedance. This common impedance may be a resistance, an inductance or a capacitance or a combination of these and is referred as a coupling element. Coupled circuits are mainly of two types:

1. Simple Coupled Circuits

When coupling element is only resistance, indutance or capacitace.

2. Complex Coupled Circuits

When the coupling element is combination of resistance, inductance or capacitance.

Two circuits may also be coupled together if one of them is in an electric or magnetic field setup by the other. An an example, the oscillating magnetic flux due to a coil in one circuit may induce a voltage in a coil in the second circuit, so that there is a mutual inductance between the two circuits.

Inductively Coupled Circuit

Fig. 17.1 shows two coils indutively coupled. Z_1 is the complex vector impedance connected in series with primary and Z_2 (called load impedance) connected across the secondary. Z_p and Z_s are the complex impedance of the primary and secondary respectively. Z_m is the mutual impedance of two coils, if I_p and I_s are the currents in primary and secondary. By applying Kirchoff's second law to the two meshes, we get,

$$\xi = I_p Z_1 + I_p Z_p + I_s Z_m \qquad \ldots(17.1)$$

and $$0 = I_s Z_2 + I_s Z_s + I_p Z_m \qquad \ldots(17.2)$$

or $$I_s = -\frac{Z_m}{Z_2 + Z_s} I_p \qquad \ldots(17.3)$$

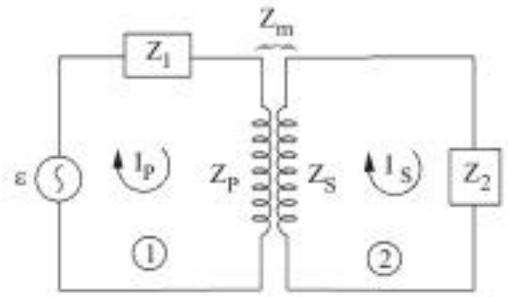


Fig.17.1

Substituting this in eq. (17.1), we have,

$$\xi = I_p (Z_1 + Z_p) - \frac{Z_m^{\,2}}{Z_2 + Z_s} - I_p$$

or $$\xi = I_p \frac{(Z_1 + Z_p)(Z_2 + Z_s) - Z_m^{\,2}}{Z_2 + Z_s}$$

or $$I_p = \frac{(Z_2 + Z_s)\xi}{(Z_1 + Z_p)(Z_2 + Z_s) - Z_m^{\,2}} \qquad \ldots(17.4)$$

Putting this value of Ip in eq. (17.3), we get,

$$I_s = -\frac{Z_m \xi}{(Z_1 + Z_p)(Z_2 + Z_s) - Z_m^{\,2}} \qquad \ldots(17.5)$$

The negative sign on the right hand side of Eq. (17.5) indicates the fact that the secondary current I_s is 180° out of phase with primary current.

The impldance viewed at the prmary terminals is the ratio of the emf ξ across the primary to the primary current I_p is given by Eq. (17.4)

$$Z = \frac{\xi}{I_p} = \frac{(Z_1 + Z_p)(Z_2 + Z_s) - Z_m^{\,2}}{Z_2 + Z_b}$$

$$= Z_1 + Z_p - \frac{Z_m^{\,2}}{Z_2 + Z_s} \qquad \ldots(17.6)$$

From this expression we see that due to coupling action the primary impedance has been modified. Clearly, eq. (17.6) shows that the presence of secondary *i.e.* the effect of coupling of the secondary circuit is to increase the primary impedance by $\left(-\frac{Z_m^{\,2}}{Z_2 + Z_s}\right)$. If there have been no coupling *i.e.* if secondary coil were not there, Z_m would have been zero and and impednace of primary were $Z_1 + Z_p$.

This additional term $\left(-\frac{Z_m^{\,2}}{Z_2 + Z_s}\right)$ is the *impedance reflected into the primary from the secondary*. In other words the impedance coupled into the primary by the secondary and hence known as *coupled impedance* of the secondary in the primary. Thus,

$$Z_{ref} = -\frac{Z_{oc}^{\ 2}}{Z_2 + Z_s} \qquad \ldots(17.7)$$

17.2 TRANSFORMERS

A transformer is an electrical appliance based on the principle of mutual inductance for converting large current at low voltage to low current at high voltage, and vice versa with very little loss in energy.

It essentially consists of two coils known as primary and secondary wound on a core. The coil to which energy is supplied is called the primary and that from which energy is delivered to the outer circuit is called the secondary. An alternating current in the primary coil set up an alternating magnetic flux in the core. This change in flux by Faraday's law induces an alternating emf in the secondary coil. Thus power is transferred from one coil to the other.

Let N_p and N_s are the number of turns in the primary and secondary coils respectively. Then the magnetic flux linked with the primary coil is $N_p \phi_B A$ and with secondary coil $N_s \phi_B A$ where A is the area and ϕ_B is the magnetic flux linked with each turn of both the coils. The induced e.m.f ξ_p in primary and ξ_s in the secondary is given by Faraday's law,

$$\xi_p = -\frac{d}{dt}(N_p \phi_B A) = -N_p A \frac{d\phi_B}{dt}$$

and $$\xi_s = -\frac{d}{dt}(N_s \phi_B A) = -N_s A \frac{d\phi_B}{dt}$$

Ratio, $$\frac{\xi_s}{\xi_p} = \frac{N_s}{N_p}$$

Now since the resistance of the primary is very low so that there is no energy loss then the induced emf ξ_p in primary will be equal to the applied voltage V_p across the primary. Since secondary may be considered to open then voltage V_s across the secondary will be equal to the induced emf ξ_s. Thus under ideal conditions

$$\frac{V_s}{V_p} = \frac{\xi_s}{\xi_p} = \frac{N_s}{N_p} = K$$

where K is called "transformation ratio".

$$\frac{\text{Secondary voltage}}{\text{Primary voltage}} = \frac{\text{Secondary turns}}{\text{Primary turns}}$$

If I_p and I_s are the currents in the primary and secondary then,

Power in secondary = power in primary

$$V_s I_s = V_p I_p$$

$$\frac{I_p}{I_s} = \frac{V_s}{v_p} = \frac{N_s}{N_p} = K$$

If K > 1 thenthe transformer is called *step up transformer* because $V_s > V_p$.

If K < 1, then the transformer is called *step down transformer* because $V_s < V_p$.

There are two types of transformers:

1. *Core type.* In the core type, the iron core is largely surrounded by the coil as shown in Fig. 17.2 (a). In order to reduce eddy currents, the iron core is built up of laminations.

2. *Shell type.* In the shell type, both the primary and secondary coils are mounted on laminated soft iron core as shown in Fig. 17.2 (b). The coils are very largely surrounded by the iron core. The shell type is better as it entails less leakage of flux.

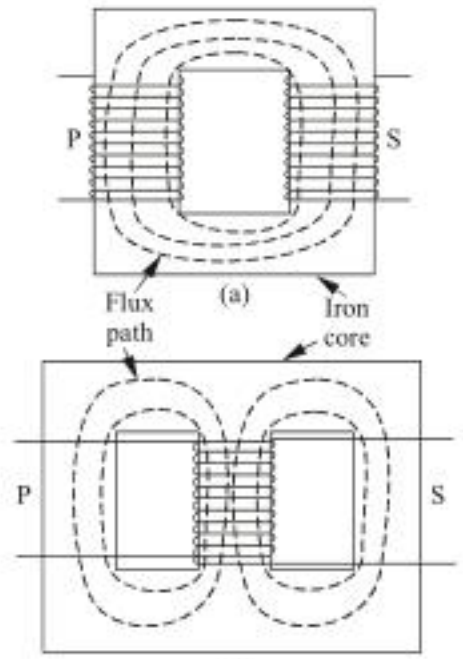


Fig. 17.2

17.3 DETAILED THEORY OF TRANSFORMER

Let us consider the transformer shown in Fig. 17.3. The transformer is an essentially indutively coupled circuit. The comparison of Figs. 17.3 and 17.1 shows that

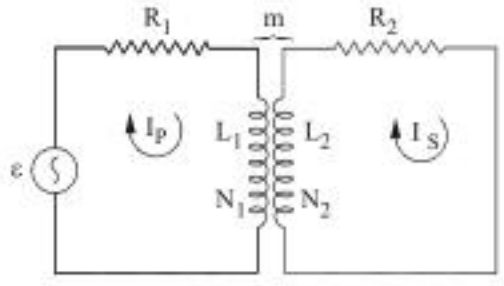


Fig. 17.3

$$\left.\begin{aligned} &Z_1 = R_1, \qquad Z_2 = R_2 \\ &Z_p = j\omega L_1 \text{ and } Z_s = j\omega L_2 \\ \text{and} \quad &Z_m = j\omega M \end{aligned}\right\} \qquad \ldots(17.8)$$

Equation (17.6) for transformer shown in Fig. 17.3 becomes,

$$Z = R_1 + jw L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2}$$

or $$Z = R_1 + j\omega L_1 + \frac{\omega^2 M^2 (R_2 - j\omega L_2)}{R_2^{\,2} + \omega^2 L_2^{\,2}}$$

or $$Z = \left(R_1 + \frac{\omega^2 M^2 R_2}{R_2^{\,2} + \omega^2 L_2^{\,2}} \right) + j \left(\omega L_1 + \frac{\omega^2 M^2 L_2}{R_2^{\,2} + \omega^2 L_2^{\,2}} \right) \quad \ldots(17.9)$$

Clearly, this shown that due to the presence of secondary, the resistance of the primary is increased and the inductive reactance of the primary is reduced. This increase in resistance of the primary is called reflected resistance from secondary into the primary and the corresponding amount by which the reactance of the primary is reduced is called the reflected reactance from the secondary to the primary. Thus,

$$\text{Reflected resistance} = \frac{\omega^2 M^2 R_2}{R_2^{\,2} + \omega^2 L_2^{\,2}}$$

$$\text{Reflected resistance} = \frac{\omega^2 M^2 L_2}{R_2^{\,2} + \omega^2 L_2^{\,2}}$$

In the case of *ideal transformer* the two coils are so wound that the inductive coupling is perfect *i.e.* whole of the flux due to one coil passes through the other. The *coefficient of coupling K is unity*. Thus by eq. (11.37)

$$M = K \sqrt{L_1 L_2}$$

$$M = \sqrt{L_1 L_2}$$

putting this value of M in eq. (17.9), we get,

$$Z = \left(R_1 + \frac{\omega^2 L_1 L_2 R_2}{R_2^{\,2}\, \omega^2 L_2^{\,2}} \right) + j \left(\omega L_1 - \frac{\omega^3 L_1 L_2^{\,2}}{R_2^{\,2} + \omega^2 L_2^{\,2}} \right)$$

If $\omega L_2 \gg R_2$ then, we have by neglecting $R_2^{\,2}$ in $R_2^{\,2} + \omega^2 L_2^{\,2}$.

$$Z = \left(R_1 + \frac{\omega^2 L_1 L_2 R_2}{\omega^2 L_2^{\,2}} \right) + j \left(\omega L_1 - \frac{\omega^3 L_1 L_2^{\,2}}{\omega^2 L_2^{\,2}} \right)$$

or $$Z = R_1 + \frac{L_1 R_2}{L_2} \quad \ldots(17.10)$$

Thus we see that for perfect coupling the reflected reactance from the secondary neutralises the inductive reactance of the primary and the impedance of the primary is purely resistive.

Now, the ratio of the secondary to the primary current from Eq. (17.3) is given by,

$$\frac{I_s}{I_p} = -\frac{Z_m}{Z_2 + Z_s}$$

the negative sign indicates the direction of secondary current is out of 180° in phase with primary current, hence magnitude,

$$\frac{I_s}{I_p} = \frac{Z_m}{Z_2 + Z_s}$$

putting Z_m and Z_s from Eq. (17.8), we get,

$$\frac{I_s}{I_p} = \frac{j\omega M}{R_2 + j\omega L_2}$$

For an ideal transformer, $M = \sqrt{L_1 L_2}$ and under the condition $R_2 \ll \omega L_2$ *i.e.* the reactance of the primary and secondary coils are large in comparison to the generator and load impedance (here resistances). Thus R_2 can be neglected, therefore,

$$\frac{I_s}{I_p} = \frac{j\omega M}{j\omega L_2} = \frac{M}{L_2}$$

$$= \frac{\sqrt{L_1 L_2}}{L_2} = \sqrt{\frac{L_1}{L_2}} \qquad \ldots(17.11)$$

Now by Eq. (11.17) the coefficient of self-inductance L_1 and L_2 are given by

$$L_2 = \frac{\mu_0 N_p^{\,2} A}{l} \text{ and } L_2 = \frac{\mu_0 N_s^{\,2} A}{l}$$

where Np and Ns are the total number of turns of primary and secondary over the length l and A is the area of each turn. Putting these values in Eq. (17.11), we get,

$$\frac{I_s}{I_p} = \sqrt{\frac{L_1}{L_2}} = \frac{N_p}{N_s} \qquad \ldots(17.12)$$

The voltage drop V_p and V_s across the primary and secondary coils by Kirchoff's second law:

$$V_p = \xi - I_p Z_1$$

$$V_s = I_s Z_2$$

where Z_1 and Z_2 are given by Eq. (17.8), the ratio,

$$\frac{V_s}{V_p} = \frac{I_s Z_2}{\xi - I_p Z_1}$$

For an ideal transformer, the Z_1 is much smaller than reactance $(R_1 \ll \omega L_1)$ we can write,

$$\frac{V_s}{V_p} = \frac{I_s Z_2}{\xi}$$

Substituting the magnitude of I_s from Eq. (17.5)

$$\frac{V_s}{V_p} = \frac{Z_m Z_2}{(Z_1 + Z_p)(Z_2 + Z_s) - Z_m^{\,2}}$$

Z_1 can be neglected as above, we get,

$$\frac{V_s}{V_p} = \frac{Z_m Z_2}{Z_p Z_2 + Z_p + Z_s - Z_m^{\,2}}$$

putting Z_p, Z_s and Z_m from Eq. (17.8),

$$\frac{V_s}{V_p} = \frac{j\omega M Z_2}{j\omega L_1 Z_2 - \omega^2 L_1 L_2 + \omega^2 M^2}$$

but $M = \sqrt{L_1 L_2}$, hence,

$$\frac{V_s}{V_p} = \frac{M}{L_1} = \sqrt{\frac{L_2}{L_1}}$$

Therefore by Eq. (17.12)

$$\frac{V_s}{V_p} = \frac{N_s}{N_p} \qquad \ldots(17.13)$$

Transformation Ratio

Eq. (17.13) shows that the ratio of the output voltage to the input voltage is equal to the ratio of turns in the secondary to that in the primary. This ratio is known as transformation ratio.

Step-up Transformer

When $N_s > N_p$ *i.e.* when secondary has greater number of turns than primary, the output voltage V_s is greater than V_p (*i.e.* applied voltage) and the transformer is known as step-up transformer.

Step-down Transformer

When $N_s < N_p$ *i.e.* when the secondary has lesser number of turns than primary, the output voltage V_s is small than V_p *i.e.* applied voltage and the transformer is known as step-down transformer. The comparison of Eqs (17.12) and (17.13) shows that in the *step up transformer voltage increases but current decreases. While in a step-down transformer the voltage decreases but the current increases, by Eqs. (17.12) and (17.13).*

$$\frac{N_s}{N_p} = \frac{V_s}{V_p} = \frac{I_p}{I_s}$$

thus $I_p V_p = I_s V_s$

i.e. the input and output powers are equal to for an ideal transformer. Thus in an ideal transformer there is no loss of energy and has 100 per cent efficiency.

17.4 TRANSFORMER LOSSES

In practice the power delivered to the load resistance in the secondary circuit *i.e.* output power is found to be smaller than the power drawn from the source *i.e.* the input power because of losses due to the imperfect designing. These losses are:

1. Copper Losses

Energy is lost as some *heat ($I^2 R$) is developed* due to flow of current in primary and secondary coils and referred as copper loss. This may be *minimized* by keeping the ohmic *resistance of the winding very small.*

2. Iron Losses

These losses are due to *induced currents* in the core of transformer a part of energy is wasted and core is heated due to these induced currents. This may be minimised if core is made of *laminated sheets of soft iron insulated from each other.*

3. Hysteresis Loss

There is loss of energy in *magnetising* an iron core and taking of through a complete cycle of magnetisation. This loss is known as hysteresis loss. It can be minimised by using soft iron core *having a thin hysteresis loop.*

4. Magnetic Losses

In an actual transformer the whole of the flux produced by one coil does not link with the second so there is some *leakage of magntic flux* resulting in the loss of energy. Losses due to leakage of magnetic flux may be viewed in two ways.

1. The primary leakage due to some part of the flux produced by the current in the primary coil not linking the secondary coil; and
2. The secondary leakage, due to some part of flux produced due to current in the secondary coil not linking the primary coil.

Winding the primary and secondary coil around a closed ring type iron core effecitively reduced these losses.

The efficiency of a given transformer may be defined as,

$$\eta = \frac{\text{power output}}{\text{power delivered by the source}} \times 100$$

$$= \frac{\text{power output}}{\text{power output} + \text{power lost}} \times 100$$

17.5 A.C. BRIDGES

The Wheatstone bridge priciple is also applicable to A.C network with the modiciation that we have to consider impedance instad of resistance. The null point is determined with the help of headphone or vibration galvanometer. The head-phone H indicates minimum sound at balance and points B and D have the same potential. At balance condition for bridge shown in Fig. 17.4, we have,

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

The phase balance condition is adjusted by changing reactances (inductances and capacitances) in the arm. The bridge is most sensitive when all the four impedances are equal and the source and the detector have the same frequency.

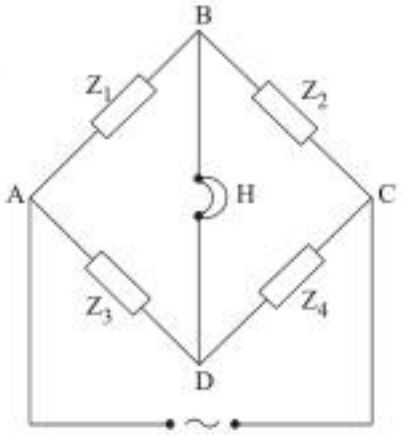


Fig.17.4

17.6 A.C. BRIDGES FOR THE MEASUREMENT OF INDUCTANCES

Few bridges suitable for the measurement of inductances are discussed below:

1. Maxwell's bridge

The briege connections are shown in Fig. 17.5. The balance is obtained by varying capacitance therefore condenser is variable. The inductance is measured in terms of a standard known resistance and capacitance. At the balance point, we have,

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

But here $Z_1 = R_1$

$$\frac{1}{Z_2} = \frac{1}{R_2} + \frac{1}{1/j\omega C}$$

$$\frac{1}{Z_2} = \frac{1}{R_2} + j\omega C$$

$$Z_3 = R_3 + j\omega L$$

$$Z_4 = R_4$$

Therefore,

$$R_1\left(\frac{1}{R_2} + j\omega C\right) = \frac{R_3 + j\omega L}{R_4}$$

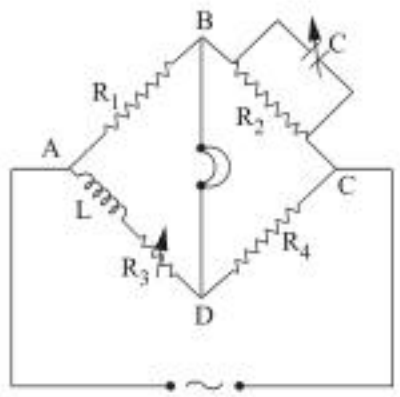


Fig. 17.5

Equating real and imaginary parts on the two sides of this equation, we get,

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \qquad \ldots(1)$$

and $$R_1\omega C = \frac{\omega L}{R_4}$$

or $$L = CR_1R_4 \qquad \ldots(2)$$

Equation (1) and (2) are the two balance conditions. The condition (2) is satisfied by varying capacitance C.

When using the bridge the ratio R_1/R_2 is adjusted to a suitable value and R_3 is varied until the headphone give minimum sound under this condition the potential at the points B and D are equal in magnitude.

Now C is varied until the sound in the headphone reduces to a further minimum. At this the *phase balance* is also obtained. The experiment is repeated alternately varhing R_3 and C. However, the perfect balance will never be obtained due to some self-capacity (or distributed capacity) of the inducatance.

In order to avoid the mutual coupling between the different branches of the bridge which will cause the residual sounds in the head phone, long wires should not be used for circuit connection and sufficient space should be allowed between them.

2. Owen bridge

This is a sensitive bridge for the measurement of inductance in terms of resistance and a fixed standard capacitance. The bridge connections are shown in Fig. 17.6. The inductor of which self-inductance to be measured is connected in series with a non-inductive variable resistance R_3. C_1 and C_2 are fixed capacitor R_4 is a fixed resistance.

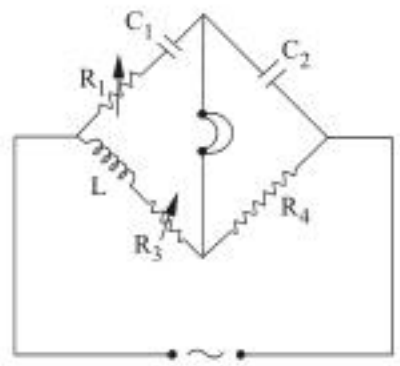


Fig. 17.6

At the balance point,

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

But $$Z_1 = R_1 + \frac{1}{j\omega C_1}$$

$$Z_2 = \frac{1}{j\omega C_2}$$

$Z_3 = R_3 + j\omega L$, where R_3 also includes the resistance of the inductance

$Z_4 = R_4$

Hence, $$\frac{R_1 + \dfrac{1}{j\omega C_1}}{\dfrac{1}{j\omega C_2}} = \frac{R_3 + j\omega L}{R_4}$$

or $$\left(R_1 + \frac{1}{j\omega C_1}\right) j\omega C_2 = \frac{R_3 + j\omega L}{R_4}$$

Equating the real and imaginary parts, we have,

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} \qquad \ldots(1)$$

and $$C_2 R_1 = \frac{L}{R_4} \qquad \ldots(2)$$

or $$L = C_2 R_1 R_4$$

The condition (1) is satisfied by varying R_3 and (2) by varying R_1 till the sound is minimum in both the cases. The small inductances of the connecting leads and terminals referred as residuals inductances can be eliminated, using this bridge as follows. Obtain two balances, 1st with L in the circuit and then with L shortcircuited, if R_1 and R'_1 are repective values of R_1, for the two balances, then we have,

$$L = C_2 R_4 (R_1 - R'_1) \qquad \ldots(3)$$

3. Anderson's Bridge

This is one of the *most accurate bridge which is commonly used* for the measurement of self inductance. The connections are shown in Fig. 17.7.

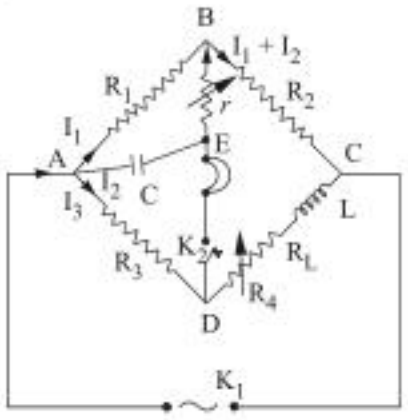


Fig. 17.7

At first a steady balance is obtained with the cell and galvanometer connected with the help of keys K_1 and K_2 respectively and adjusting the resistance in the four arms of the bridge. At balance,

$$\frac{R_1}{R_2} = \frac{R_3}{R_4 + R_L} \qquad \ldots(1)$$

Next an A.C. source and a headphone are connected and an inductive balance is obtained by adjusting the resistance *r*.

When the current in the headphone is zero *i.e.* the bridge is balance the potential drop between A and E will be the same as that between A and D. Thus,

$$\frac{I_2}{j\omega C} = I_3 R_3 \qquad \ldots(2)$$

At balance the potential drop along ABC is the same as along ADC thus by Kirchoff's seocnd law, we get,

$$I_1 R_1 + (I_1 + I_2) R_2 = I_3 (R_3 + R_4 + R_L + j\omega L)$$

or $$I_1 (R_1 + R_2) + I_2 R_2 = I_3 (R_3 + R_4 + R_L + j\omega L) \qquad \ldots(3)$$

where R_L is the resistance of the inductance. Since there is no source of emf in mesh AEBA we have by Kirchoff's second law,

$$I_2 \left(\frac{1}{j\omega C} + r \right) = I_1 R_1 \qquad \ldots(4)$$

Putting the value of I_3 from (2) and (3) we get,

$$I_1 (R_1 + R_2) + I_2 R_2 = \frac{I_2}{j\omega C R_3} (R_3 + R_4 + R_L + j\omega L)$$

or $$I_1 (R_1 + R_2) + I_2 = \left(\frac{R_3 + R_4 + R_L + j\omega L}{j\omega C R_3} - R_2 \right) \qquad \ldots(5)$$

Now putting the value of I_1 from (4) in (5)

$$\frac{I_2}{R_1} \left(\frac{1}{j\omega C} + r \right) (R_1 + R_2) = I_2 \left[\frac{R_3 + R_3 + R_L + j\omega L}{j\omega C R_3} - R_2 \right]$$

or $$\frac{1}{R_1} (R_1 + R_2) \left(\frac{1}{j\omega C} + r \right) = \frac{R_3 + R_3 + R_L + j\omega L}{j\omega C R_3} - R_2 \qquad \ldots(6)$$

Equating the real and imaginary parts, we get,

$$\left(1 + \frac{R_2}{R_1} \right) r = \frac{L}{C R_3} - R_2$$

or $$L = C R_3 \left[R_2 + \left(1 + \frac{R_2}{R_1} \right) r \right] \qquad \ldots(7)$$

and $$\left(1 + \frac{R_2}{R_1} \right) \frac{1}{\omega C} = \frac{R_3 + R_4 + R_L}{\omega C R_3}$$

or $$\frac{R_2}{R_1} = \frac{R_4 + R_L}{R_3} \qquad \ldots(8)$$

comparison of Eq. (8) and Eq. (1) shows that it is the condition for D.C. balance, and Eq. (7) is for A.C. balance. The inductive balance is obtained by adjusting R_4 and r. In practice we set $R_1 = R_2$, then eq. (7) and (8) becomes,

$$L = C R_3 \left[R_2 + 2r\right] \qquad \ldots(9)$$

and $$R_L = R_3 - R_4 \qquad \ldots(10)$$

Thus L and R_L are determined in terms of known quantities.

17.7 A.C. BRIDGES FOR THE MEASUREMENT OF CAPACITANCE

Few bridge suitable for the measurement of capacitance are discussed below:

1. De Sauty's A.C. Bridge

This is used to determine the capacity of an unknown capacitor in terms of the capacitance of a standard dnown condenser. The capacitors and two non-inductive variable resistances are connected in a Wheastone bridge as shown in Fig. 17.8 usually one resistance is given a fixed value and other is varied till the balance is obtained. At balance,

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

$$\frac{\dfrac{1}{j\omega C_1}}{R_2} = \frac{\dfrac{1}{i\omega C_2}}{R_4}$$

or $$\frac{C_2}{C_1} = \frac{R_2}{R_4}$$

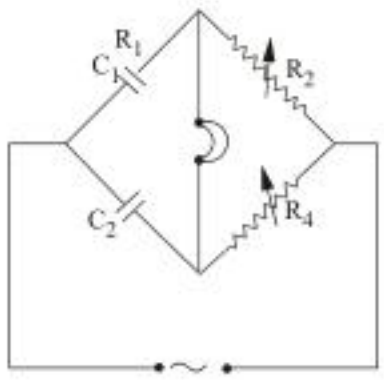


Fig. 17.8

or $$C_2 = C_1 \frac{R_2}{R_1}$$

This bridge is also used for *cmparing two capacitances.*

2. Wein's bridge

This is basically the De Sauty's bridge but with slight modification. In this bridge an additional non-inductive variable resistance R_3 is included in series with the standard capacitor as shown in Fig. 17.9. With this modification the bridge can be used to determine the power factor of the capacitor C_1. At balance.

$$\frac{R_1 + \dfrac{1}{j\omega C_1}}{R_2} = \frac{R_3 + \dfrac{1}{j\omega C_3}}{R_4}$$

or $$R_1 R_4 + \frac{R_4}{j\omega C_1} = R_2 R_3 + \frac{R_2}{j\omega C_3}$$

On equating the real and imaginary parts, we get,

$$R_1 R_4 = R_2 R_3$$

or $$\frac{R_1}{R_2} = \frac{R_3}{R_4} \qquad \ldots(1)$$

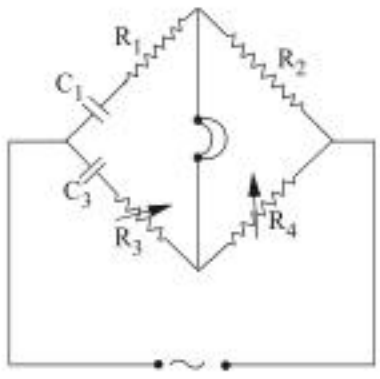


Fig. 17.9

and $$\frac{R_4}{C_1} = \frac{R_2}{C_3}$$

or $$C_1 = C_3 \frac{R_4}{R_2} \qquad \ldots(2)$$

Thus C_1 can be calculated. The power factor of the capacitance C_1 is,

$$\cos\theta = \frac{R_1}{\sqrt{R_1 + (1/\omega^2 C_1^2)}} \qquad \ldots(3)$$

R_1 is known by Eq. (1) and C_1 by Eq. (2), hence the power factor may be calculated by Eq. (3). If

$\frac{1}{\omega} C_1 \gg R$, then

$$\cos\theta = \omega C_1 R_1$$

3. Schering Bridge

Schering bridge is particularly used for the *accurate mesurement of small capacitance* and also for capacitance determination in general. The connections are shown in Fig. 17.10. The condenser under test is C_1. The bridge is balanced by varying R_4 and C_4. At balance, we have,

$$\frac{R_1 + \frac{1}{j\omega C_1}}{R_2} = \frac{\frac{1}{j\omega C_3}}{Z_4}$$

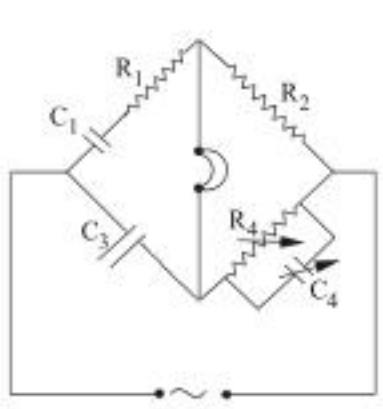


Fig. 17.10

where $$\frac{1}{Z_4} = \frac{1}{R_4} + \frac{1}{\frac{1}{j\omega C_4}} = \frac{1}{R_4 \, j\omega C_4}$$

Therefore,

$$\frac{R_1 + \frac{1}{j\omega C_1}}{R_2} = \frac{1}{j\omega C_3}\left(\frac{1}{R_4} + j\omega C_4\right)$$

or $$R_1 + \frac{1}{j\omega C_1} = \frac{R_2}{j\omega C_3 R_4} + \frac{R_2 C_4}{C_3}$$

Equating the real and imaginary components, we get,

$$R_1 = R_2 \frac{C_4}{C_1} \qquad \ldots(1)$$

and $$\frac{1}{C_1} = \frac{R_2}{C_3 R_4}$$

or $$C_1 = \frac{R_4}{R_2} C_3 \qquad \ldots(2)$$

By eq. (2), C_1 can be calculated.

By eq. (1), the effective resistance R_1 can be calculated.

17.8 ROBINSON'S BRIDGE FOR DETERMINING THE FREQUENCY OF AN A.C. SOURCE

Robinson had modified the Wien's capacity bridge so that the balance condition depends upon the frequency of the A.C. source. The modified bridge is shown in Fig. 17.11.

At balance, we have,

$$\frac{Z_1}{R_2} = \frac{R_3 + \dfrac{1}{j\omega C_3}}{R_4}$$

or $$\frac{R_4}{R_2} = \frac{1}{Z_1}\left(R_3 + \frac{1}{j\omega C_3}\right)$$

But $$\frac{1}{Z_1} = \frac{1}{R_1} + j\omega C_1$$

Therefore,

$$\frac{R_4}{R_2} = \left(R_3 + \frac{1}{j\omega C_3}\right)\left(\frac{1}{R_1} + j\omega C_1\right)$$

$$= \frac{R_3}{R_1} + \frac{C_1}{C_3} + \frac{1}{j\omega C_3 R_1} + j\,\omega\, C_1 R_3$$

Equating real parts we get,

$$\frac{R_4}{R_2} = \frac{R_3}{R_1} + \frac{C_1}{C_3} \qquad \ldots(1)$$

Equating the imaginary parts, we get,

$$\frac{1}{\omega C_3 R_1}\,\omega C_1 R_3 = 0$$

or $$\omega^2 = \frac{1}{C_1 C_3 R_1 R_3} \qquad \ldots(2)$$

Generally, for convenience, we get,

$$C_1 = C_3 \text{ and } R_1 = R_3$$

then, $$\omega^2 = \frac{1}{C_1^{\,2} R_1^{\,2}}$$

or $$f^2 = \frac{1}{4\pi^2 C_1^{\,2} R_1^{\,2}}$$

or $$f^2 = \frac{1}{2\pi C_1 R_1}$$

Hence f is known in termsof C_1 and R_1 and is easily calculated.

For this setting, Eq. (1) gives,

$$R_4 = 2R_2$$

Therefore, R_1 must be kept equal to two times R_2 while adjusting the bridge for balance.

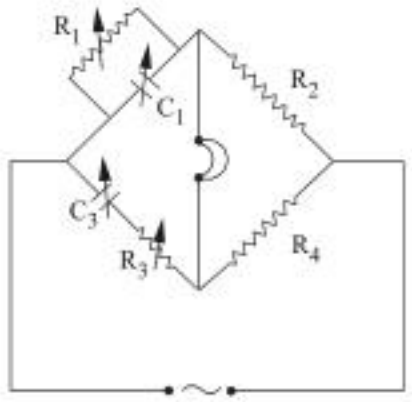


Fig. 17.11

17.9 ROTATING MAGNETIC FIELD OR PRINCIPLE OF AN A.C. INDUCTION MOTOR

If there are two coils placed at right angles to each other carrying A.C. current of the same maximum value and frequency but differing in pahse by $\pi/2$ they give rise to a rotating magnetic field whose magnitude remains constant but the direction changes continuously.

Consider two coils AB and CD placed at right angles to each other and carrying alternating current as shown in Fig. 17.12 (*a*) each coil will produce its own magnetic field. Let I_1 and I_2 be the current and I_0 be its peak value. Suppose I_1 leads I_2 by a phase angle $\pi/2$. Then,

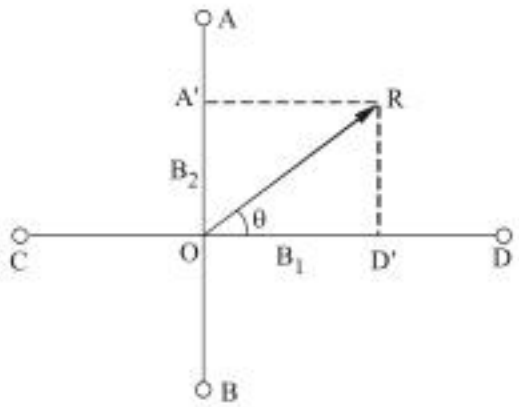


Fig. 17.12 (*a*)

$$I_1 = I_0 \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \ldots(1)$$

$$I_2 = I_0 \sin \omega t \qquad \ldots(2)$$

The magnetic fields produced by these crossed currents are also at right angles to each other because the magnetic field due to each coil will be in phase with respect to respective currents. The strength of each magnetic field being proportional to the current. Let B_1 and B_2 be the magnetic fields produced by the current I_1 and I_2 respectively and B_0 is the maximum value of the field.

Then,

$$B_1 = B_0 \sin\left(\omega t + \frac{\pi}{2}\right) = B_0 \cos \omega t \qquad \ldots(3)$$

$$B_2 = B_0 \sin \omega t$$

If the field B_1 is due to the coil AB and B_2 the field due to the coil CD they may be represented by the vectors OD′ and OA′, the resultant B is vector OR. The magnitude,

$$B = \sqrt{B_1^{\,2} + B_2^{\,2}}$$

$$B = \sqrt{B_0^{\,2} \cos^2 \omega t + B_0 \sin^2 \omega t}$$

or $$B = B_0 \qquad \ldots(5)$$

And its direction is given by,

$$\tan \theta = \frac{B_2}{B_1}$$

where θ is an angle between OR and B_1

or $$\tan \theta = \frac{B_0 \sin \omega t}{B_0 \cos \omega t} = \tan \omega t$$

or $$\theta = \omega t \qquad \ldots(6)$$

Thus, the resultant magnetic field is constant in magnitude and rotates with constant angular velocity as shown in Fig. 17.12 (*b*).

This shows that,

1. when $\omega t = 0$, $\theta = 0$, B is along OA
2. when $\omega t = \frac{\pi}{4}$, $\theta = \frac{\pi}{4}$ B is along OR
3. when $\omega t = \frac{\pi}{2}$, $\theta = \frac{\pi}{2}$ B is along OC
4. when $\omega t = \pi$, $\theta = \pi$ B is along OD
5. when $\omega t = 2\pi$, $\theta = 2\pi$ B is along OA

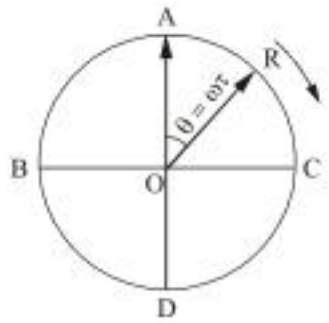


Fig. 17.12 (b)

Thus the direction of resultant magnetic field B represented by OR changes with time. *Such a magntic field is called rotating magnetic field.* In this case when I_1 leads I_2 by $\pi/2$ the direction is clockwise.

In case the current I_2 leads I_1 by $\pi/2$, the magnetic field rotates the anti-clockwise direction.

If now a conductor or coil is placed at 0 in a rotating magnetic field, induced currents will be produced in the conductor. Hence the conductor will begin to move in the direction of the rotating field so that the relative velocity between the conductor and rotating magnetic field may become least in accordance with Lenz's law.

A couple acts on a conductor placed in a rotating magnetic field. Let the rectangular coil CC be placed in such a rotating magnetic field B, as shown in Fig. 17.12 (c). The total magnetic flux passing through the coil at any instance *t* is $\phi_m = \int_v \vec{B} \cdot d\vec{S}$.

$$\phi_B = BA \sin \omega t$$

where A is the effective area of the coil and ωt is the angle between $\vec{B}$ and the plane of the coil at any instant *t*. The induced emf

$$\xi_{ind} = -\frac{d\phi_B}{dt} = -BA\,\omega \cos \omega t$$

$$= BA\,\omega \sin\left(\omega t - \frac{\pi}{2}\right)$$

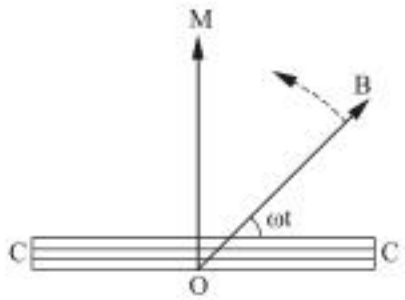


Fig. 17.12 (c)

The current will lag behind the emf by an angle

$= \tan^{-1} \frac{\omega L}{R}$ and given by,

$$I = \frac{BA\,\omega}{\sqrt{\omega^2 L^2 + R^2}} \sin\left(\omega t - \frac{\pi}{2} - \alpha\right) \qquad \ldots(7)$$

By *definition the magnetic moment of the coil is* $m = AI$ and is perpendicular to the plane of the coil being directed along the normal OM.

The moment of the couple acting on the coil tending to turn it in the direction of B is,

$$\tau = \vec{m} \times \vec{B}$$

$$= m\,B \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= -BAI \cos \omega t \qquad (\text{as } m = IA)$$

Substituting the value of I from eq. (7) we get,

$$\tau = \frac{B^2 A^2 \omega}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\omega t - \frac{\pi}{2} - \alpha\right) \cos \omega t$$

$$= \frac{B^2 A^2 \omega}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \alpha) \cos \omega t$$

$$= \frac{B^2 A^2 \omega}{\sqrt{R^2 + \omega^2 L^2}} (\cos^2 \omega t \cos \alpha + \sin \omega t \cos \omega t \sin \alpha)$$

The aveage value of $\sin \omega t \cos \omega t$ or $\frac{1}{2}\sin \omega t$ is zero for one cycle and of $\cos^2 \omega t = \frac{1}{2}$. Hence the mean couple acting on the coil is,

$$\tau_{mean} = \frac{1}{2} \frac{B^2 A^2 \omega}{\sqrt{R^2 + \omega^2 L^2}} \cos \alpha$$

$$= \frac{1}{2} \frac{B^2 A^2 \omega}{\sqrt{R^2 + \omega^2 L^2}} \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$$

$$= \frac{1}{2} \frac{B^2 A^2 \omega R}{(R^2 + \omega^2 L^2)} \quad \ldots(8)$$

This mean couple acts in the direction of rotation of the magnetic field. Eq. (8) shows that the magnitude of the couple depends on the magnitude of ω Eq. (8) can be written as,

$$\tau_{mean} = \frac{B^2 A^2 \omega R}{2(R^2 + \omega^2 L^2)}$$

$$= \frac{B^2 A^2 \omega R}{2(\omega L - R)^2 + 4\omega L R}$$

The couple is maximum when denominator is least that is, when $\omega L - R = 0$

or $\omega L = R$

or $\tan \alpha = \frac{\omega L}{R} = 1$

or $\alpha = \frac{\pi}{4} = 45^\circ$

Therefore, the value of α for maximum couple is 45° and the value of maximum couple is obtained by putting $R = \omega L$.

$$\tau_{max} = \frac{B^2 A^2 \omega \cdot \omega L}{2 \times 2\omega^2 L^2}$$

$$= \frac{1}{4} \frac{B^2 A^2}{L} \quad \ldots(9)$$

Fig. 17.12 (*d*) shows the variation of couple with speed. Along Y-axis we take,

$$\frac{\omega}{R^2 + \omega^2 L^2} = \frac{\omega}{Z^2}$$

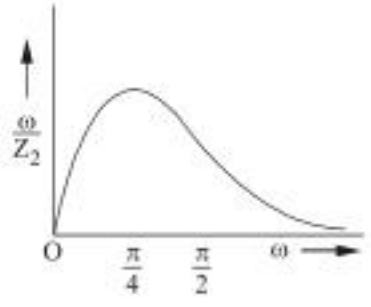


Fig. 17.12 (*d*)

against the ω or frequency $f = \frac{\omega}{2\pi}$ along x-axis. The couple is maximum when

$$\omega = \frac{R}{L} \; i.e. \; \alpha = \frac{\pi}{4}$$

17.10 SINGLE PHASE INDUCTION MOTOR

The motion of a conductor in rotating magnetic field has many practical consequences, important among then is A.C. motor.

Fig. 17.13 (*a*) shows the diagram of a single phase induction motor. Two pairs of electromagnetic AB and CD carrying alternating current of same periodicity but differing in phase by $\pi/2$ are called starter. A conductor or coil (R) is mounted between the two pairs undergoes continuous rotation under the influence of resulting rotating magnetic field. This is called the rotator R.

In practice the rotator of an induction motor consists of a number of heavy copper bars parallel to the axis of rotation. These bars are connected to heavy copper rings at the ends. This is known as squirrel cage rotator as shown in Fig. 17.13 (*b*).

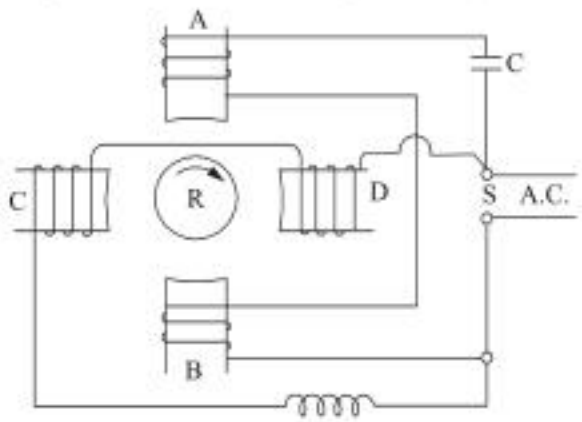


Fig. 17.13 (*a*)

Squirrel cage rotor

Fig. 17.13 (*b*)

Thus the current in the two electromagnetic are

$$I_1 = I_0 \sin(\omega t + \theta_1)$$

and $$I_2 = I_0 \sin(\omega t + \theta_2)$$

As the currents are differeing by $\frac{\pi}{2}$ then

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

The variation of current I_1 and I_2 are shown in Fig. 17.13 (*c*). At the origin O, the current I_1 is zero and I_2 has maximum negative value of and the flux is directed towards the right but at the instant 1, I_1 is maximum, I_2 is zero and flux is directed upwards. Thus the flux on the rototor is continuously rotating anticlockwise and inducing a current in the rotator which sets the rotator into rotation in the direction of rotating field and motor starts.

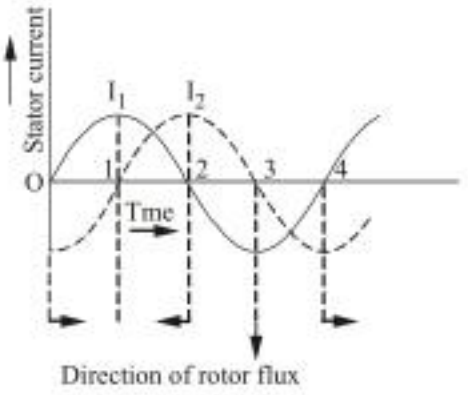


Fig. 17.13 (*c*)

17.11 SKIN EFFECT

A steady current flowing in a wire is distributed uniformly throughout the cross-section of the wire. On the other hand an

alternating current of high frequency is not distributed uniformly but there is a concentration of current on the outer surface of the conductor. If the frequency of the current is very high, the current is almost wholly confined to the surface layer. This concentration on the outer surface of conductor to the surface layer. This concentration on the outer surface of the conductor is called "skin effect".

In an ideal resistor, the current is in phase with the applied emf, consequently phase angle is zero. But actual resistors may deviate from the ideal of a zero phase angle. Furthermore, the a.c. resistance of a conductor is not in general the same as its d.c. resistance but is larger and increases with frequency. This effect is largely the result of *self-induced emfs* which are different for different paths in the conductor. These emfs increase with frequency, since by Faraday's law they depend on the rate of change of the magnetic flux; and they are least for conducting paths linked by the least flux. The result is that the current density distribution is latered from its d.c. pattern; and this is turn implies an increase of resistance, however, for the d.c., current density distribution is such as to minimise resistance. In isolated wires, the a.c. current density is greatest at the surface of the wire, giving rise to the phenomena of the "skin effect", while in coils the effect may be even larger and is more complex because of the magnetic field of the entire coil.

Skin effect may also be related to the fact that the high frequency electromagnetic waves are strongly absorbed* in a good conductor.

REVIEW QUESTIONS

17.1 Define coupled circuits and coefficient of coupling. Show that the effective impednace of the primary circuit is modified when it is coupled inductively with a secondary circuit.

17.2 Give the construction and working of a transformer. What are the energy losses? How are they reduced?

17.3 Deduce the condition of balance for A.C. Bridge. Give the theory and working of Maxwell's bridge. How would you use it to determine the inductance of a coil.

17.4 Describe an accurate bridge method for the determination (i) of the self-inductance of a coil and (ii) of the capacitance of the capacitor.

17.5 Describe with relevant theory, Anderson's bridge for finding the self-inductance of the coil.

17.6 How Robinson's bridge is used for measuring frequency of the A.C. mains?

17.7 Explain how a rotating magnetic field is produced and show that how it can be used in the construction of a motor.

17.8 Describe the principle involved in the construction of an induction motor.

17.9 Discuss the phenomena of skin effect.

SELF-TEST OBJECTIVE QUESTIONS

17.1 At higher frequencies the flow of current through bars is

(*a*) uniform throughout the cross-section (*b*) concentrated along the centre
(*c*) concentrated on the outer layers (*d*) none of the above

17.2 Skin effect is more relevant at

(*a*) low frequences (*b*) high frequencies
(*c*) very low frequencies (*d*) low voltages

17.3 Skin effect depends upon factors like permeability μ and resistivity ρ alongwith frequency f. It is proportional to

(*a*) $\mu \frac{f}{\rho}$ (*b*) $\frac{\rho}{\mu f}$

(*c*) $f \frac{\rho}{\mu}$ (*d*) $\frac{\mu}{\rho}$

*For further details see Electromagnetic by I.S. Grant and W.R. Phillps.

17.4 For step down transformer the number of turns in the primary is
(*a*) less than that in the secondary (*b*) much less than that in the secondary
(*c*) more than that in the secondary (*d*) none of the above

17.5 For an ideal transformer on load
(*a*) $\frac{E_p}{E_s} = \frac{N_p}{N_s}$ (*b*) $\frac{E_p}{E_s} = \frac{N_s}{N_p}$
(*c*) both (a) and (b) (*d*) none of (a) and (b)

17.6 The laminations of the transformer are there to
(*a*) increase impedance (*b*) reduce transients
(*c*) reduce eddy currents (*d*) increase the transformer ratio effectively

17.7 Transformers are least affected by
(*a*) copper losses (*b*) iron losses
(*c*) hysteresis (d) mechanic losses

17.8 An ideal transformer does not change
(*a*) power (*b*) current
(*c*) voltage (*d*) none of the above

17.9 Most commonly A.C. bridge circuit used for measurement of capacitance is
(*a*) Maxwell-Wein bridge (*b*) Hay's bridge
(*c*) De-saughty bridge (*d*) Schering bridge

17.10 A Maxwell bridge is used for the measurement of
(*a*) inductance only
(*b*) capacitance only
(*c*) both inductance and capacitance
(*d*) none of these

17.11 Wein bridge find applincation in
(*a*) Harmonic distortion analyser (*b*) Resistance measurement
(*c*) Frequency measurement (*d*) None of these.

ANSWERS

1. (*c*)	**2.** (*b*)	**3.** (*b*)	**4.** (*c*)	**5.** (*a*)	**6.** (*c*)
7. (*d*)	**8.** (*a*)	**9.** (*d*)	**10.** (*a*)	**11.** (*c*)	

CHAPTER 18

CIRCUIT ANALYSIS

18.1 CLASSIFICATION OF CIRCUITS

Electric circuits may be classified as follows :

(a) Direct current and alternating current circuits

According to the kind of current flowing in the circuit.

(b) Single or multi-mesh circuits or networks

According to the number of closed paths in which current is flowing.

(c) Circuits with a single or several energy sources

According to the number of sources in the circuit.

(d) Linear and non-linear circuits

According to the volt ampere characteristics of the circuit elements.

18.2 IMPORTANT TERMS OF CIRCUIT ANALYSIS

We often use the following terms in the circuit analysis :

1. **Node.** A node is a point or junction where two or more branches meet and are connected electrically.
2. **Branch.** A branch of network is defined as a section along which the current remains constant. Branches are of two types :

   (a) Branches with no source of emf are said to be passive,

   (b) Branches containing sources are called active branches.
3. **Mesh (or Loop).** A closed conducting path (or circuit) made up of several branches is called a mesh of an electrical network.
4. **Network.** An electrical network may be defined as any electrical circuit containing impedances and generators.
5. **An ideal voltage generator.** It is a power source that contains no internal series impedance and maintains a constant potential difference across its terminals, no matter how much current is drawn. Fig 18.1(*a*) shows an ideal voltage generator.

(*a*) (*b*)

Fig. 18.1

6. **Real or Practical Voltage Generator.** In practice any physical source of power necessarily has an internal impedance, as shown in Fig. 18.1(*b*), in series with ideal voltage generator.
7. **An ideal current generator.** It is a power source that contains an infinite internal parallel impedance. It is ideal in the sense that it maintains a constant current through a circuit no matter what the load impedance is. An ideal current generator is shown in Fig. 18.2(*a*).

Fig. 18.2

8. **A practical current generator.** It has finite impedance in parallel with the ideal current generator, as shown in Fig. 18.2(*b*)

18.3 LAWS AND THEOREMS FOR CIRCUIT ANALYSIS

The linear circuits or networks can be analysed by Ohm's law and complex network by Kirchoff's laws. But in the case of complex networks, the process becomes lengthy and tedious. To make the calculations easy a number of network theorems have been formulated. Some of them are listed below:

1. The reciprocity theorem
2. Superposition theorem
3. Thevenin's theorem
4. Norton's theorem
5. Maximum power transfer theorem

We shall discuss them in detail separately.

18.4 THE RECIPROCITY THEOREM

The Kirchoff's laws which apply to D.C. networks are also applicable to A.C. networks. In addition to the two Kirchoff's laws, there is a third law, known as the reciprocity theorem which is very useful in analysing networks. The statement of the theorem is :

In any passive two terminal network of conductors, consider any two branches P and Q. Then the current in one branch e.g., P, due to the emf E in another branch Q, is equal to the current in the latter branch Q when the same emf E is placed in the former branch P. In other words, the interchange of an ideal current source and an ideal voltage source in passive linear two terminal circuit will not change the voltmeter and ammeter readings.

18. 5 SUPERPOSITION THEOREM

In any linear network, if there are more than one generators (or energy sources) and containing impedances, then the current flowing in any element is the vector sum of the currents that are separately caused to flow in that element by each energy source, with all other generators replaced.

Consider the two voltage sources E_1 and E_2 and three impedances Z_1, Z_2 and Z_3 as shown in Fig.18.3 (a). Fig. 18.3 (b) and Fig. 18.3 (c) represent the equivalent circuits illustrating superposition theorem. By superposition theorem,

$$I_1 = I_1' + I_1''$$

and $$I_2 = I_2' + I_2''$$

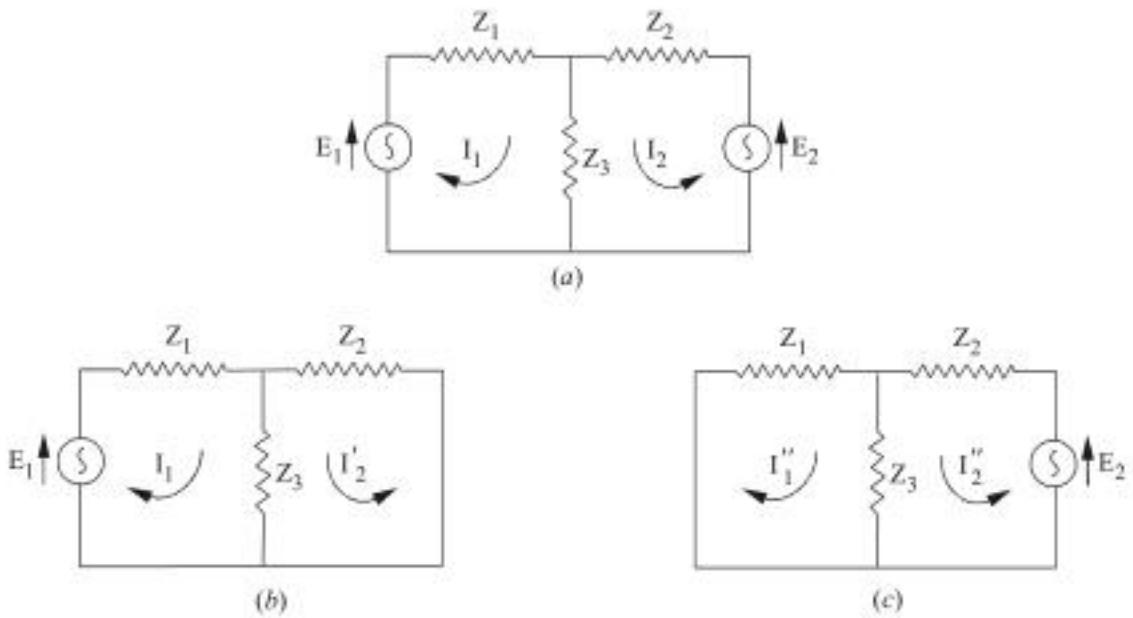


Fig. 18.3

18.6 THEVENIN'S THEOREM

To avoid the lengthy calculations in network analysis we have a valuable tool known as Thevenin's Theorem. The essence of this theorem is to simplify that portion of the circuit which is of lesser interest and replace the network by an *equivalent circuit which, for certain purposes, has the same characteristics as the original*. Thus by Thevenin's theorem we get a most useful equivalent circuit.

According to this theorem, any two terminal *linear network containing linear impedances and one or more generators can be replaced with an equivalent circuit, consisting of an impedance* (Z′) *in series with an emf (i.e. a voltage source)* E'. The value of emf (E′) is that which is presented to the two terminals when the external load is removed (open circuit voltage), *i.e.*, when the output current is zero. And the value of the impedance (Z′) is that which is presented to the output terminals when all the sources of the emf are short-circuited or replaced by impedances equal to their internal impedances.

To illustrate the application of Thevenin's theorem consider the network shown in Fig.18.4 (*a*). As shown in Fig. 18.4 (*b*) all the circuit except the load impedance Z_L has been simplified to a single generator E' and series impedance Z' equal to the impedance of the two terminal network. This combination of a voltage source in series with a source impedance makes up an equivalent voltage generator known also as the Thevenin's equivalent. No matter how complicated the original circuit, it can be simplified in this manner so that a single generator and series impedance gives an equivalent circuit.

Fig. 18.4

As an example of the applications of the theorem, consider a two terminal network, containing one active mesh and other passive mesh as shown in Fig. 18.5 (*a*) where Z_L is the load impedance. Fig. 18.5 (b) represents the Thevenin's equivalent network.

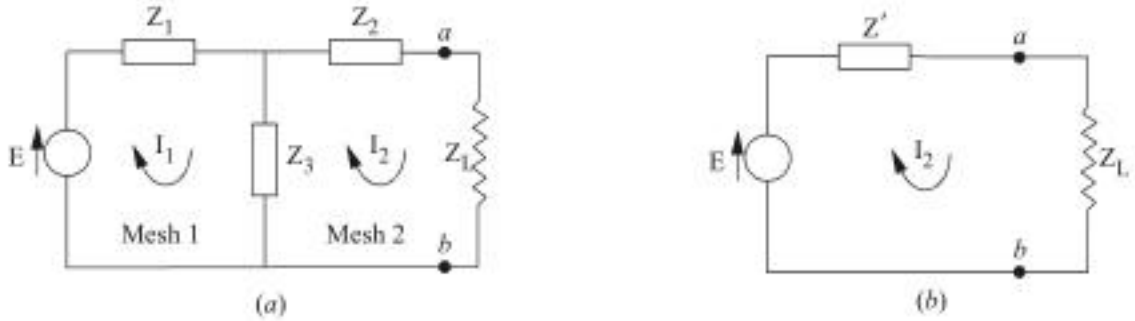


Fig. 18.5

I_1 and I_2 are the loop currents flowing in active and passive network. Applying the Kirchoff's voltage law in two meshes, we get

$$Z_1 I_1 + Z_3 I_1 - Z_3 I_2 = E \qquad \ldots(18.1)$$

and $$(Z_2 + Z_3 + Z_1) I_2 - Z_3 I_1 = 0 \qquad \ldots(18.2)$$

By equation (18.2)

$$I_1 = \frac{(Z_2 + Z_3 + Z_L)\, I_2}{Z_3}$$

Putting this in eqn. (18.1)

$$(Z_1 + Z_3)\frac{(Z_2 + Z_3 + Z_L)}{Z_3} I_2 - Z_3 I_2 = E$$

or $$(Z_1 + Z_3)(Z_2 + Z_3 + Z_1) I_2 - Z_3^{\,2} I_2 = EL_3$$

or $$I_2 = \frac{EZ_3}{(Z_1 + Z_3)(Z_2 + Z_3 + Z_L) - Z_3^{\,2}}$$

$$I_2 = \frac{\dfrac{E\,Z_3}{Z_1 + Z_3}}{Z_2 + Z_3 + Z_L - \dfrac{Z_3^{\,2}}{Z_1 + Z_3}}$$

or $$I_2 = \frac{\dfrac{EZ_3}{Z_1 + Z_3}}{Z_2 + Z_L + \dfrac{Z_1 Z_3}{Z_1 + Z_3}} \qquad \ldots(18.3)$$

The open circuit voltage E' is found by determining the voltage across the point *ab* when Z_L is disconnected as shown in Fig. 18.5 (c). When Z_L is disconnected, there is no voltage drop across Z_2 and current by Ohm's law will be given by,

$$I = \frac{E}{Z_1 + Z_3}$$

and the voltage will be open circuit voltage,

$$E' = I\,Z_3 = \frac{EZ_3}{Z_1 + Z_3} \qquad \ldots(18.4)$$

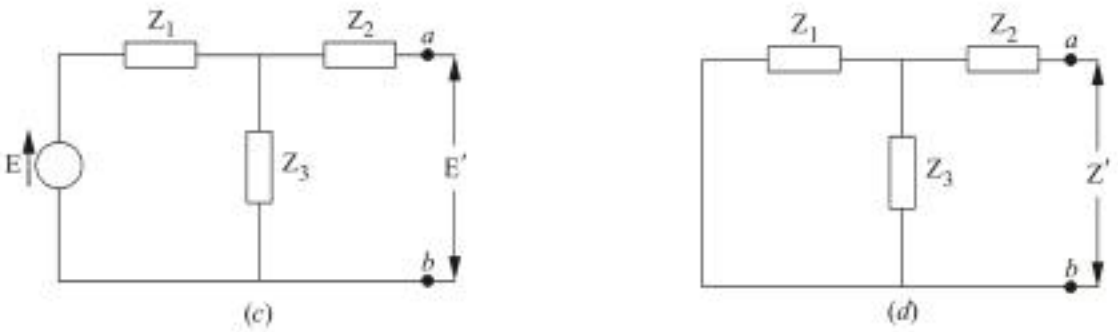


Fig. 18.5

The equivalent impedance is found by looking back from the points *a* and *b* with all emf generators short circuited as shown in Fig. 18.5 (*d*). From this figure. we see that Z_1 and Z_3 are in parallel and Z_2 is in series with this combination. Thus

$$Z' = Z_2 + Z,$$

where $$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_3}$$

or $$Z = \frac{Z_1 Z_3}{Z_1 + Z_3}$$

Therefore,

$$Z' = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3} \qquad \ldots(18.5)$$

Hence by eqns. (18.4) and (18.5), Eqn. (18.3) can be written as,

$$I_2 = \frac{E'}{Z' + Z_L} \qquad \ldots(18.6)$$

Now from equivalent circuit Fig.(18.5 *b*),

$$I_2 = \frac{E'}{Z' + Z_L} \qquad \ldots(18.7)$$

Eqns. (18.6) and (18.7) are the same ; hence theorem is proved.

Thus by knowing E' and Z'. I_2 can be calculated. In a like manner the Thevenin's equivalent circuit can be found for any network.

18.7 NORTON'S THEOREM

This theorem gives an equivalent circuit in cases where current sources are of major interest than emfs; for example, the transistor circuits. Thus the resultant equivalent circuit contains *a current generator instead of voltage generator.*

This theorem states that *the current in a load impedance* Z_L *connected to two output terminals of a network consisting of one or more generators and impedances is the same as if the load impedance were connected to a constant current source (or generator)* I' *in shunt with an impedance* Z'. The value of generated current I' is that which flows through the two terminals connected to the load

when they are short-circuited together and the shunt impedance Z' is that which appears at these two terminals when all sources of emf inside the network are replaced by their internal impedances.
To illustrate the application of Norton's theorem, consider the network as shown in Fig. 18.6 (*a*) same as Fig. 18.5 (*a*). Here, the equivalent circuit is obtained by Norton's theorem as shown in Fig. 18.6 (*b*).

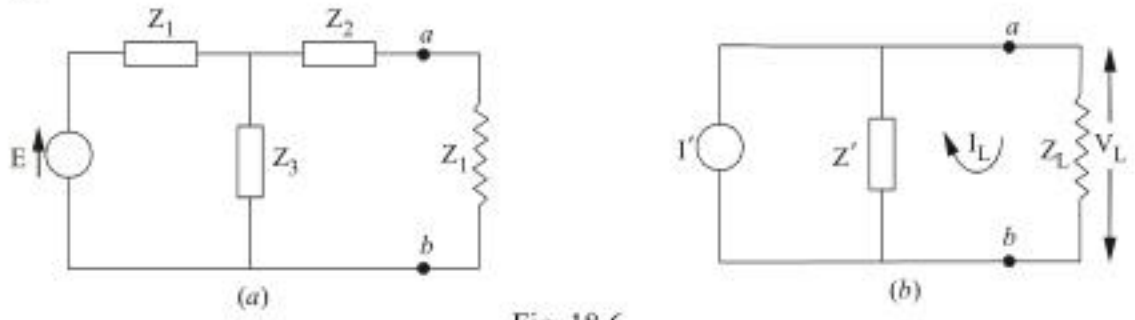


Fig. 18.6

The current in the load impedance Z_L is given by.

$$I_L = \frac{V_L}{Z_L}$$

From Fig. 18.6 (b), the voltage across the load impedance Z_L is by Ohm's law,

$$V_L = I'Z$$

where, $$\frac{1}{Z} = \frac{1}{Z'} + \frac{1}{Z_L}$$

or $$Z = \frac{Z' Z_L}{Z' + Z_L}$$

Hence,

$$V_L = \frac{I' Z' Z_L}{Z' + Z_L}$$

Therefore,

$$I_L = \frac{V_L}{Z_L} = \frac{I' Z'}{Z' + Z_L}$$

$$= \frac{I'}{1 + \frac{Z_L}{Z'}} \qquad \ldots(18.8)$$

As stated above, the magnitude of the current I' is found by short-circuiting the output terminals, as shown in Fig. 18.6 (*c*). Applying Kirchoff's law to the two meshes shown in Fig.18.6 (*c*),

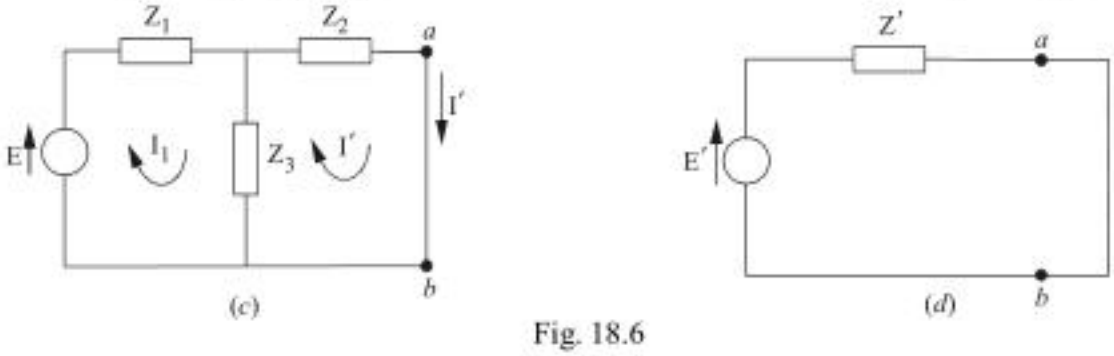


Fig. 18.6

$$I_1(Z_1+Z_3)-I'Z_3=E$$

and $$(Z_2+Z_3)I'-I_1Z_3=0$$

$$I_1=\frac{(Z_2+Z_3)I'}{Z_3}$$

Hence,

$$(Z_1+Z_3)\frac{(Z_2+Z_3)I'}{Z_3}-I'Z_3=E$$

or $$(Z_1+Z_3)(Z_2+Z_3)I'-Z_3^2I'=EZ_3$$

or $$I'=\frac{EZ_3}{(Z_1+Z_3)(Z_2+Z_3)-Z_3^2}$$

or $$I'=\frac{EZ_3}{Z_1Z_2+Z_1Z_3+Z_2Z_3} \quad \ldots(18.9)$$

The parellel impedance Z' is found by looking back from the output terminals when all sources are short-circuited and replaced by their internal impedances as shown in Fig 18.5 (*d*). Therefore, for equivalent circuit shown in Fig. 18.6 (*b*),

$$Z'=Z_2+\frac{Z_1Z_3}{Z_1+Z_3} \quad \ldots(18.10)$$

Thus knowing I' and Z', we can find the current through the load impedance Z_L by Eqn. (18.8).

Relationship between Thevenin's Equivalent Circuit and Norton's Equivalent Circuit

From the above discussion we can conclude that it is possible to represent any network either by Thevenin's or Norton's equivalent circuit. The current I_2 for Thevenin equivalent circuit shown in Fig 18.5 (*b*) is given by Eqn. (18.6) *i.e.*

$$I_2=\frac{E'}{Z'+Z_L} \quad \ldots(18.11)$$

And for Norton's equivalent by Eqn. (18.8), *i.e.*

$$I_L=\frac{I'Z'}{Z'+Z_L} \quad \ldots(18.12)$$

I' is the short-circuit current of the Thevenin's equivalent network of Fig. 18.5(*b*) as shown in Fig. 18.6 (*d*).

$$I'=\frac{E'}{Z'} \quad \ldots(18.13)$$

With this value of I' *we see that Eqns.(18.11) and (18.12) are the same. Hence Eqn.(18.13) clearly indicates that circuits 18.6 (b) and 18.5 (b) are the same*. Thus, we conclude that circuit shown in Fig. 18.6 (*b*) is equivalent to circuit shown in Fig. 18.5 (*b*) which is equivalent to original circuit shown in Fig. 18.5 (*a*) or 18.6 (*a*).

Limitations on Thevenin's and Norton's Theorems

Be careful to note the wording of the two theorems. The equivalence obtained by Thevenin's or Norton's equivalent circuit *holds for the load current and not for the conditions within generator.* Thus the limit is imposed on these theorems.

To be more specific, as an illustration to this point, the power loss within the generator of Fig. 18.5 (*a*) is not the same as the power loss within the equivalent generator of Fig. 18.5 (*b*).

For simplifying the calculation, suppose all the impedances be pure resistances. Then, for circuit [Fig. 18.5(*a*)] under no load conditions, *i.e.* $Z_L \to 0$, the power loss within the generator is $P = \frac{E^2}{R_1 + R_3}$, whereas the power loss within the Thevenin equivalent [Fig. 18.5(*b*)] under the same no load condition is zero.

Therefore the Thevenin's or Norton's equivalent circuits cannot be used to calculate circuit efficiencies.

18.8 MAXIMUM POWER THEOREM

In many electronic circuits it is required to transfer efficiently the maximum electric power from the circuit to the load. Let us examine the condition when the power transfer is maximum. The condition is given by maximum power theorem. This theorem states *that the maximum power will be delivered to the load when the load resistance is equal to the internal resistance of the network delivering the power.*

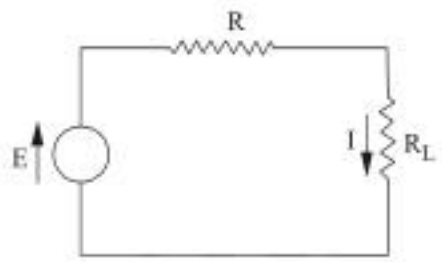


Fig. 18.7 (*a*)

Let us consider two terminal network being represented by its Thevenin's equivalent circuit shown in Fig. 18.7 (*a*). The load is resistance R_L. The prime on the equivalent source and equivalent impedance (here resistance) have been eliminated for convenience.

According to Joule's law, the power delivered to the load resistance is,

$$P = I^2 R_L$$

$$= \left(\frac{E}{R + R_L}\right)^2 R_L \qquad \ldots(18.14)$$

$$= \frac{E^2/R_L}{(1 + R/R_L)^2} \qquad \ldots(18.15)$$

The power generated in the load resistance will be maximum when the internal resistance R will be minimum *i.e.*, zero. Thus, when $R = 0$, $P = P_M + E^2/R_L$.

Therefore, Eq.(18.15) becomes,

$$P = \frac{P_M}{\left(1 + \frac{R}{R_L}\right)^2} \qquad \ldots(18.16)$$

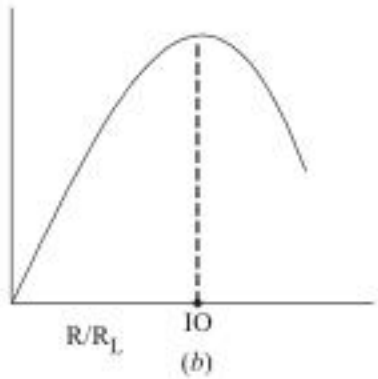
Equation (18.16) shows that P is smaller than P_M. *From Eqn.(18.14), it is clear that the power in the load is zero if the load resistance* (R_L) *is very small and is also zero when the load resistance is very large.* Thus, there must be some particular value (an optimum value) of load resistance for which the power in R_L is the maximum.

For finding this optimum value for maximum power transfer, differentiate Eqn. (18.15) with respect to R_L and equate the result to zero,

$$\frac{dP}{dR_L} = E^2 \frac{d}{dR_L}\left[\frac{1}{R_L} \cdot \frac{1}{\left(1+\frac{R}{R_L}\right)^2}\right] = 0$$

$$= E^2\left[\left[-\frac{2\left(1+\frac{R}{R_L}\right)}{\left(1+\frac{R}{R_L}\right)^4}\left(-\frac{R}{R_L{}^2}\right)\frac{1}{R_L} + \frac{1}{\left(1+\frac{R}{R_L}\right)^2}\left(-\frac{1}{R_L{}^2}\right)\right] = 0\right.$$

$$= \frac{E^2}{R_L{}^2\left(1+\frac{R}{R_L}\right)^2}\left[\frac{\frac{2R}{R_L}}{1+\frac{R}{R}} - 1\right] = 0$$



(*b*)

Fig. 18.7

or $$\frac{\frac{2R}{R_L}}{1+\frac{R}{R_L}} - 1 = 0$$

or $$\frac{2R}{R_L} = 1 + \frac{R}{R_L}$$

or $$R_L = R. \qquad \ldots(18.17)$$

Thus the theorem is proved when the power transferred is maximum ; the load is said to be matched to the circuit. If a graph is plotted between the load power and the R/R_L, the curve so obtained is shown in Fig. 18.7(*b*).

Further the power taken from the source P_S is given by,

$$P_S = I^2 R + I^2 R_L$$

For maximum power $R = R_L$ Therefore, the ratio of the maximum power transferred to the power taken from the source is,

$$\frac{P_{max}}{P_S} = \frac{I^2 R_L}{I^2 R_L + I^2 R_L} = \frac{1}{2} \qquad \ldots.(18.18)$$

Thus, the maximum efficiency is 50 per cent, which means that half of the power is lost with internal resistance of the network.

18.9 MAXIMUM POWER TRANSFER THEOREM IF THE INTERNAL IMPEDANCE OF THE SOURCE IS NOT A PURE RESISTANCE

Under this condition the different matching conditions are required. *The theorem states that the maximum power will be received by a network from another circuit connected to its two terminals when the impedance of the receiving network is varied to make the impedance looking into the network at its two terminals as conjugate to each other.*

Suppose a two terminal active linear network connected to a two terminal passive linear network as shown in Fig. 18.8. The network to the left of *a* .*b* terminals represents the Thevenin's equivalent network and Z_L represents the equivalent impedance of a passive linear network.

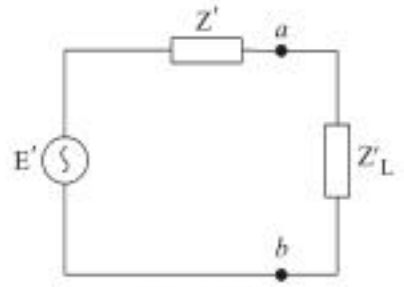


Fig. 18.8

For a source impedance containing reactive element and resistance, we have,

$$Z' = R' + jX' \qquad \ldots(18.19)$$

and similarly $Z_L = R_1 + jX_L$ …(18.20)

According to maximum power transfer theorem Z' and Z_L are conjugate to each other, *i.e*.,

$$Z_L = R' - jX_L' \qquad \ldots(18.21)$$

Let I be the current in the network, then

$$I = \frac{E'}{Z' + Z_L} \qquad \ldots(18.22)$$

or $$I = \frac{E'}{(R' + R_L) + j(X' + X_L)} \qquad \ldots(18.23)$$

The power delivered to the load is,

$$P_L = I^2 R_L \text{ (substituting I)}$$

$$P_L = \frac{(E')^2 R_L}{(R' + R_L)^2 + (X' + X_L)^2} \qquad \ldots(18.24)$$

If X_L is varying, then maximum power will be calculated by putting.

$$\frac{\delta P_L}{\delta X_L} = 0 \qquad \ldots(18.25)$$

For eq. (18.24),

$$\frac{\delta P_L}{\delta X_L} = -\frac{2(E')^2 R_L (X_L + X')}{[(R_L + R')^2 + (X_L + X')^2]^2} = 0$$

i.e, when numerator is zero.

$$X_L + X' = 0$$

or $$X_L = -X' \qquad \ldots(18.26)$$

After substituting $X_L + X' = 0$ in Eqn. (18.24), we get,

$$(P_L)_{max} = \frac{(E')^2 R_L}{(R_L + R')^2} \qquad \ldots(18.27)$$

If R_L is varying, then maximum power will be calculated by putting,

$$\frac{\delta (P_L)_{max}}{\delta R_L} = 0$$

Hence,

$$\frac{\delta (P_L)_{max}}{\delta R_L} = \frac{E'^2 (R_L + R')^2 - 2 (E')^2 R_L (R_L + R')}{(R_L + R')^2} = 0$$

i.e. when numerator is zero. Thus,

$$E'^2 = (R_L + R') \left[R_L + R' - 2R_L \right] = 0$$

or $$\left[R' - R_L \right] = 0$$

or $$R_L = R' \qquad \ldots(18.28)$$

Substituting X_L and R_L from Eqns. (18.26) and (18.28) in Eqn. (18.20), we get,

$$Z_L = R_L + j\, X_L = R' = j\, X' \qquad \ldots(18.29)$$

i.e, conjugate of Z. *Hence the theorem is proved.*

Example 18.1 *Find the open circuit voltage and Thevenin's resistance of two terminal network shown in Fig. 18.9.*

Solution. The open circuit voltage by Eqn. (18.4) is

$$E' = \frac{E Z_3}{Z_1 + Z_3} = \frac{100 \times 30}{20 + 30} = 60 \text{ V}$$

and the Thevenin's resistance by Eqn.(18.5),

$$Z' = Z_2 + \frac{Z_1 Z_2}{Z_1 + Z_3}$$

As given $Z_2 = 0$

$$Z' = 0 + \frac{20 \times 30}{20 + 30} = 12\Omega$$

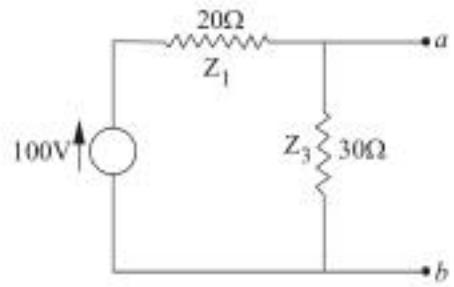


Fig. 18.9

Example 18.2 *For the network shown in Fig.18.10, calculate the current through load impedance using Thevenin's theorem.*

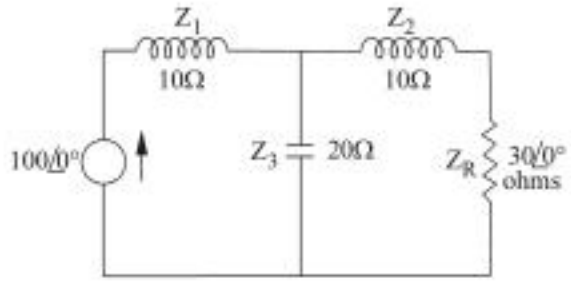


Fig. 18.10

Solution. By eqns.(18.4) and (18.5) we have

$$E' = \frac{E Z_3}{Z_1 + Z_3} = \frac{(100 + j0)(-20j)}{j10 - j20}$$

$$= \frac{(100 + j0)(-j20)}{-j10} = (200 + j0) \text{ volts}$$

and $$Z' = Z_2 + \frac{Z_1 Z_2}{Z_1 + Z_3}$$

$$= j10 + \frac{(j10)(-j20)}{j10 - j20} = j10 + j20 = j30\ \Omega$$

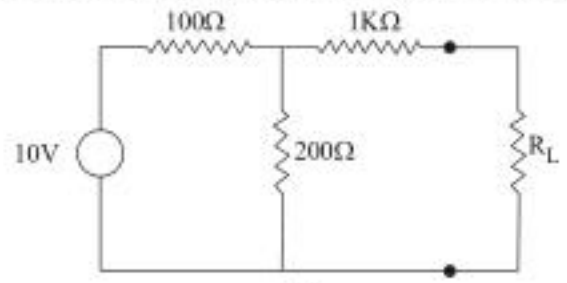
Therefore, $$I_R = \frac{E'}{Z' + Z_R} = \frac{200 + j0}{j30 + 30 + j0} = \frac{200}{30 + j30}$$

or $$I_R = (3.33 - j3.33)\ \text{amp.}$$

or $$I_R = \sqrt{\left[(3.33)^2 + (3.33)^2\right]}\ \tan^{-1} - \frac{3.33}{3.33}$$

$$= 4.72 \angle -45^\circ\ \text{app.}$$

Example 18.3 *What is the Norton's equivalent of the network shown in Fig. 18.11(a).*



(*a*)

Fig. 18.11

Solution. The Norton's equivalent circuit's current is given by eq.(18.9) *i.e.*

$$I' = \frac{EZ_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

or $$I' = \frac{10 \times 200}{100 \times 1000 + 100 \times 200 + 200 \times 1000}$$

or $$I' = \frac{1}{160}\ \text{amp.}$$

and equivalent circuit's resistance by eqn.(18.10) *i.e.*

$$Z' = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3}$$

$$= 1000 + \frac{100 \times 200}{100 + 200}$$

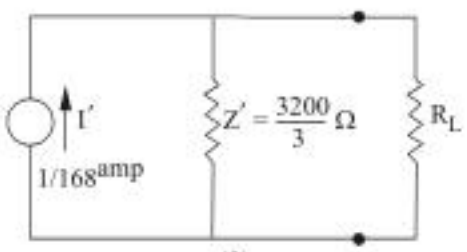
$$= 1000 + \frac{200}{3} = \frac{3200}{3}$$

Therefore, by eqn.(18.8), the current is

$$I_L = \frac{I'}{1 + \frac{Z_L}{Z'}} = \frac{1/160}{1 + \frac{R_L}{\frac{3200}{3}}}$$

$$= \frac{20}{3200 + 3R_L}$$

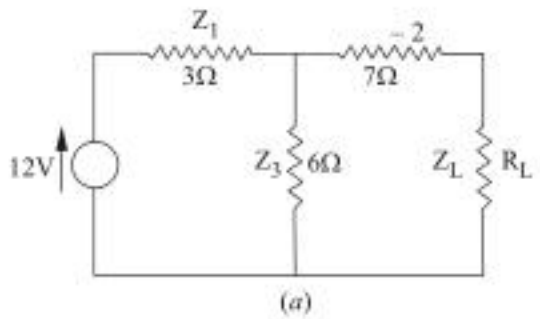
The Norton's equivalent circuit is shown in Fig.18.11(*b*).



(*b*)

Fig. 18.11

Example 18.4 *Convert the linear network shown in Fig.(18.12b) into Thevenin's equivalent network and then into Norton's equivalent network and show that the power delievered to the load in both the cases is same.*



(*a*)

Fig. 18.12

Solution. For converting the network of Fig.18.12(*a*) into Thevenin's equivalent circuit, the open circuit voltage E′ by eqn.(18.4)

$$E' = \frac{E Z_3}{Z_1 + Z_3} = \frac{12 \times 6}{3+6} = 8 \text{ volts}$$

and the equivalent impedance by eq.(18.5)

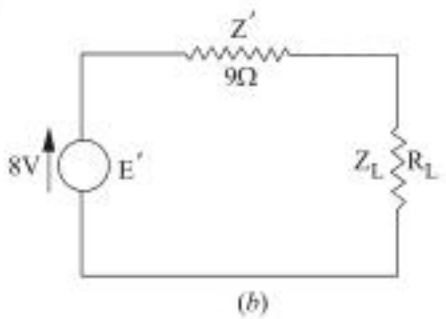
$$Z' = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3}$$

$$= 7 + \frac{3 \times 6}{3+6} = 9\,\Omega$$

The Thevenin's equivalent circuit is shown in Fig.(18.12*b*).

Power delivered to the load is

$$P_L = I^2 R_L = \left(\frac{8}{9+R_L}\right)^2 R_L \text{ watt} \qquad \ldots(1)$$



(*b*)

Fig. 18.12

For converting the network into Norton's equivalent circuit by Eqn.(18.8), we have current through load

$$I_L = \frac{I'}{1 + \frac{Z_L}{Z'}}$$

where I' by Eqn.(18.9) is

$$I' = \frac{E Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

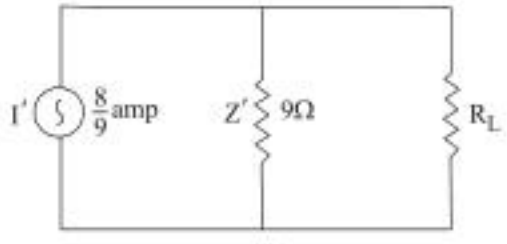
$$= \frac{12 \times 6}{3 \times 7 + 3 \times 6 + 7 \times 6} = \frac{8}{9} \text{ amp.}$$

and Z' by Eqn.(18.19),

$$Z' = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3}$$

$$Z' = 7 + \frac{3 \times 6}{3 + 6} = 9\,\Omega$$

The Norton's equivalent circuit is shown in Fig. 18.12 (*c*).



(*c*)

Fig. 18.12

Therefore,

$$I_L = \frac{\frac{8}{9}}{1 + \frac{R_L}{9}} = \frac{8}{9 + R_L}$$

$$\text{Power} = \left(\frac{8}{9 + R_L} \right)^2 R_L \text{ watt} \qquad \ldots(2)$$

From(1) and (2), we see that in both the cases power delivered to the load is same.

Example 18.5 *For the network shown in Fig. 18.13, find the maximum power.*

Solution. The current I is given by

$$I = \frac{100}{4 + R}$$

Power absorbed by the resistance R is

$$P = I^2 R$$

$$= \frac{1000}{(4 + R)^2} R$$

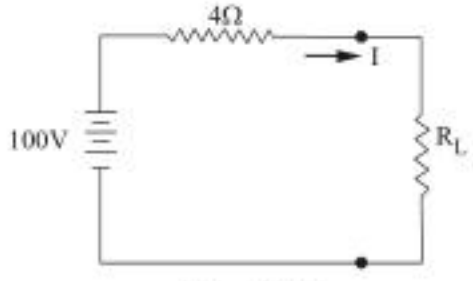


Fig. 18.13

To find the resistance for maximum power,

$$\frac{dP}{dR} = \frac{1000\,(4 + R)^2 - 2000\,(4 + R)\,R}{(4 + R)^4} = 0$$

which gives R = 4 ohms

The load current is

$$I = \frac{100}{4+4} = 12.5 \text{ amp.}$$

and the maximum power is

$$P_{max} = (12.5)^2 4$$
$$= 625 \text{ watt.}$$

REVIEW QUESTIONS

18.1. Find the short circuit current and Thevenin resistance of the two terminal network shown in Fig. 18.14.

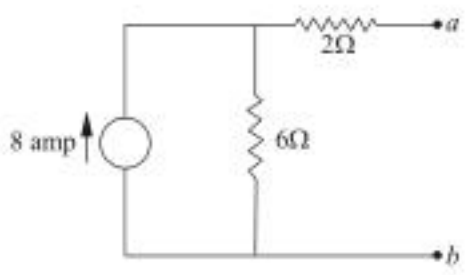


Fig. 18.14

18.2. Calculate the current through the galvanometer branch in an unbalanced Wheatstone Bridge shown in Fig,18.15 using Kirchhoff's laws and Thevenin theorem.

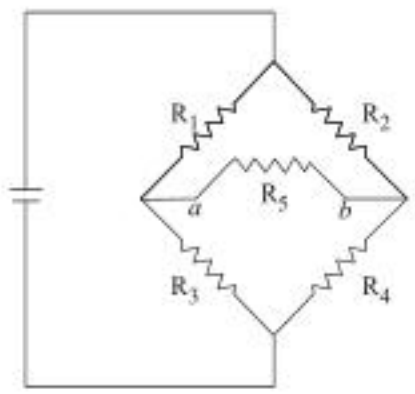


Fig. 18.15

18.3. Find the value of R_L which will absorb maximum power and determine this maximum power in the network shown in Fig. 18.16.

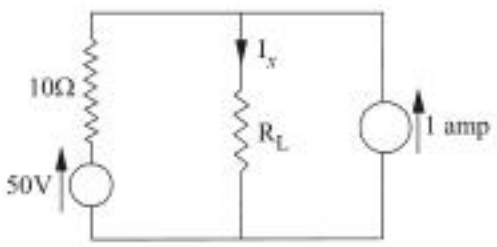


Fig. 18.16

18.4. Explain reciprocity theorem and superposition theorem.

18.5. State and prove Thevenin's and Norton's theorems. What are the limitations of these theorems?

18.6. State and prove the maximum power theorem for two terminal network.

18.7. Show that a two terminal network connected across the output of generator will absorb maximum power from the generator when the load impedance is the conjugate of the internal impedance of the generator.

PROBLEMS

18.1 Find I_1, using Thevenin's theorem, in Fig. 18.17 (a) and (b)

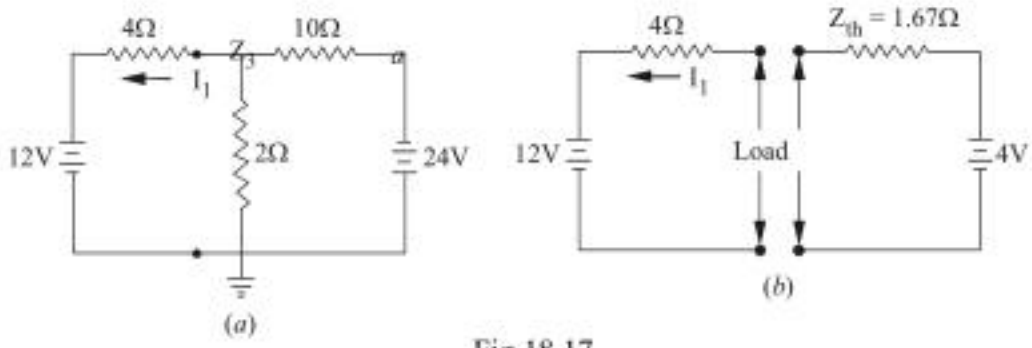


Fig.18.17

18.2. Find the current I_{AB} in Fig 18.18.

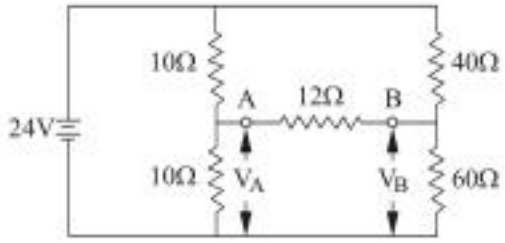


Fig.18.18

18.3. Find I_L in Fig 18.19.

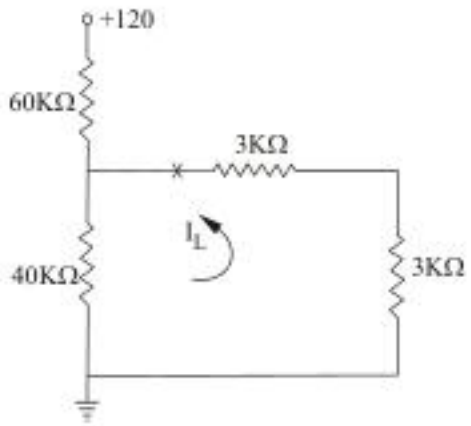


Fig.18.19

18.4. Find I_L in Fig 18.20.

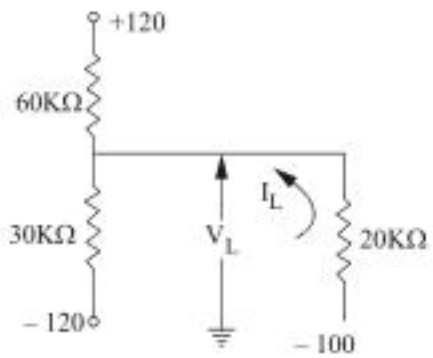


Fig.18.20

18.5. Find the current and voltage between terminals A and B in Fig. 18.21.

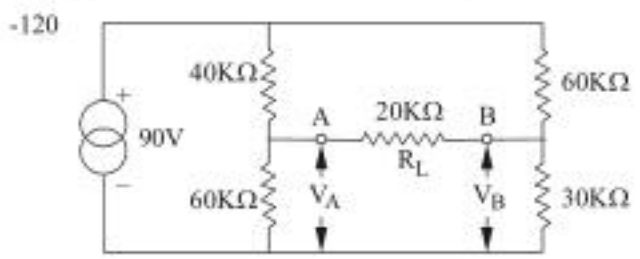


Fig.18.21

18.6. Convert the Thevenin's equivalent circuit of Fig. 18.22 to Norton's in Fig. 18.23, and show that they are the same by solving for the load current in each.

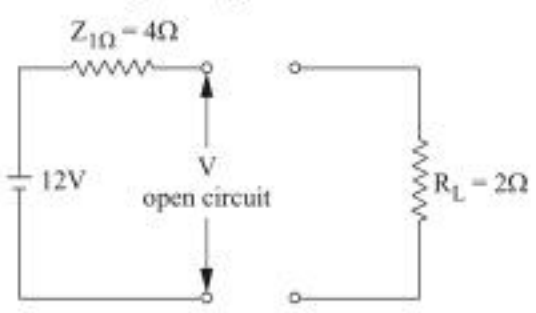


Fig.18.22

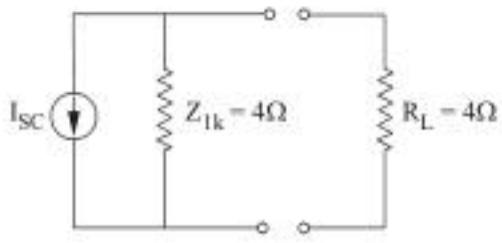
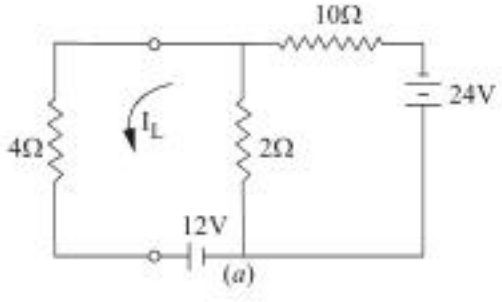


Fig.18.23

18.7. Find the current in the 4.ohm resistor by Norton's theorem (See Figs. 18.24(a) and 18.24(b).



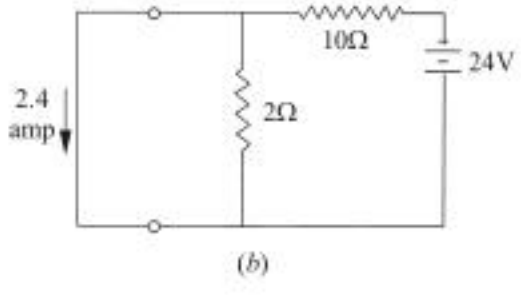


Fig.18.24

18.8. Convert the circuit of Fig. 18.25 from the voltage source to form an equivalent current source.

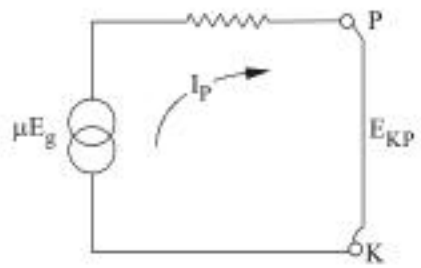


Fig.18.25

18.9. Using Thevenin's theorem, find the current through 100 ohm resistance connected across terminals AB in the circuit shown in Fig. 18.26.

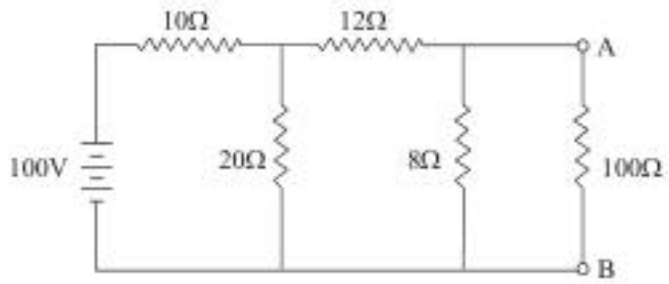


Fig.18.26

18.10. (a) What value should R_L have for maximum power transfer in Fig. 18.27?

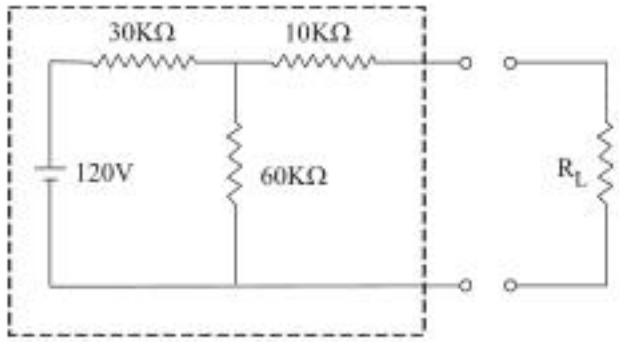


Fig.18.27

(b) What is the power dissipated in R_L when matched.

18.11. A generator develops 200V and has an internal resistance of 100 ohm. Find the power delivered to a load of (1) 100 ohm (2) 300 ohm. Comment on the result.

ANSWERS

1. 2.82 amp **2.** – 0.0375 amp **3.** $I_L = 1.6$ mA

4. $I_L = 1.33$ mA **5.** $V_{AB} = -7.5$ volts; $I_{AB} = 0.38$ mA

6. Load current in 2 amp in both circuits **7.** 2.82 amp.

9. 0.19 A **10.** 30 K ohm, 53.3 mw

11. 100 watts and efficiency is 50 per cent only; 75 watts and efficiency is 75 per cent

SELF TEST OBJECTIVE QUESTIONS

18.1 There is no relation between Thevenin's theorem and Norton's theorem
(*a*) true (*b*) false

18.2 Thevenin's and Norton's circuits are equivalent at
(*a*) all frequencies
(*b*) only at frequencies at which these are computed
(*c*) at resonant frequency only
(*d*) at no frequency

18.3 Open circuit voltage is the potential difference between two points having
(*a*) zero impedance (*b*) infinite impedance
(*c*) reactive impedance (*d*) none of these

18.4 Short circuit current flows through a load
(*a*) of zero impedance (*b*) which is reactive
(*c*) which is not necessarily reactive (*d*) of infinite impedance

18.5 Thevenin's theorem reduces a two terminal network to a
(*a*) current generator in parallel with an impedance
(*b*) voltage generator in series with an impedance
(*c*) one terminal network
(*d*) none of the above

18.6 In a Thevenin equivalent circuit, load can be found out by applying
(*a*) Kirchhoff's current law (*b*) Kirchhoff's voltage law
(*c*) Superposition theorem (*d*) None of these

18.7 Thevenin's and Norton's theorems can be applied to networks with
(*a*) D.C sources only
(*b*) A.C. sources only
(*c*) Both D.C. and A.C. or a combination of the two
(*d*) None of these

18.8 Load current can be found out by applying to the Nortonised circuit
(*a*) Kirchhoff's voltage law (*b*) Node analysis techniques
(*c*) Superposition theorem (*d*) None of these

18.9 Norton's theorem reduces a big electrical network to a one
(*a*) with only two terminals (*b*) with only one part
(*c*) with only one generator and one impedance (*d*) none of these

18.10 According to maximum power transfer theorem, for maximum power transfer
(*a*) Source impedance should be very large as compared to load impedance.
(*b*) Source impedance should be very small as compared to load impedance.
(*c*) Source impedance should be complex conjugate of load impedance
(*d*) none of these

18.11 In application of superposition theorem, we are required to solve as many circuits as there are
(a) meshes (b) nodes
(c) sources (d) none of these

ANSWERS

1. (*b*) **2.** (*b*) **3.** (*b*) **4.** (*a*) **5.** (*b*) **6.** (*z*)
7. (*c*) **8.** (*b*) **9.** (*c*) **10.** (*c*) **11.** (*c*)

APPENDIX I

SOME FUNDAMENTAL AND DERIVED CONSTANTS

Name	*Symbol*	*Value*
Speed of light	C	3.00×10^{8} mets/sec.
Permeability constant	μ_0	1.26×10^{-6} henry[a]/metre
Permittivity constant	ϵ_0	8.85×10^{-12} farad[b]/metre
Elementary charge	e	1.60×10^{-19} coul
Avagadro constant	N	6.02×10^{23} molo
Electron rest mass	m	9.11×10^{-31} kg
Proton rest mass	m_p	1.67×10^{-27} kg
Neutron rest mass	m_n	1.67×10^{-27} kg
Planck constant	h	6.63×10^{-23} joul sec.
e/m of electron	e/m	1.76×10^{11} coul/kg
Quantum/charge ratio	h/e	4.14×10^{-15} joul sec/coul
Bohr radius	a_0	5.29×10^{-11} metre
Bohr magneton	μ_e	9.27×10^{-24} joul/tesla
Nuclear magneton	μ_N	5.05×10^{-27} joule/tesla[c]
Proton magnetic moment	μ_p	1.41×10^{-26} joule/tesla
Universal gas constant	R	8.31 joules/mole °K
Boltzmann constant	K	1.38×10^{-23} joule/ °K
Gravitational constant	G	6.67×10^{-11} nt m^2/kg^2

a henry = Weber/ampere = Newton.metre/ampere^2
b Farad = Coulomb/volt
c Tesla = Weber/metre^2

APPENDIX II

SYMBOLS, DIMENSIONS AND UNITS OF ELECTRICAL AND MAGNETIC QUANTITIES

All dimensions are in MKSC i.e. LMTQ which is Rationalised MKS unit system. The term rationalised means to remove the factor 4π (or 2π) that would otherwise appear in many formulas in electricity and magnetism. But the internationally accepted System International Unit i.e. S.I. units are those which have ampere as the unit for all electrical and magnetic quantities. Thus in S.I. system we have MKSA i.e. LMTI where I is ampere. Thus the S.I. dimensions are obtained by replacing Q in rationalised MKS system by TI (because I = Q/T). The units given in Table are the derived practical units.

Quantity	*Symbol*	*Dimensions in rationalised MKS i.e. LMTQ*	*Dimensions in S.I. System i.e. LMTI*	*Derived practical units*
Capacitance	C	$L^2 M^{-1} T^2 Q^2$	$L^2 M^{-1} T^4 I^2$	Farad
Charge	q	Q	TI	Coulomb/m^3
Charge density (Volume)	ρ	$L^{-3}Q$	L^{-3} TI	Coulomb/m^3
Charge density (surface)	σ	$L^{-2}Q$	L^{-2} IT	Coulomb/m
Charge density (linear)	λ	$L^{-1}Q$	L^{-1} TI	Coulomb/m
Conductivity	σ	$L^{-3} M^{-1} T Q^2$	$L^{-3} M^{-1} T^3 I^2$	$(Ohm.metre)^{-1}$
Conductance	g	$L^{-2} M^{-1} T Q^2$	$L^{-2} M^{-1} T^3 I^2$	Ohm^{-1} = mho
Current	I	$T^{-1} Q$	I	ampere
Current density (surface)	J	$I^{-2} T^{-1} Q$	$L^{-2}I$	amp.$metre^2$
Current density (linear)	J_L	$L^{-1}T^{-1}Q$	$L^{-1}I$	amp/metre
Electric displacement	D	$L^{-2} Q$	L^{-2} TI	Coul/$metre^2$
Electric field	E	$L M T^{-2} Q^{-1}$	$LMT^{-3} I^{-1}$	Volts/metre
Electric dipole moment	p	LQ	LTI	Coul-metre
Electric polarisation	P	$L^{-2} Q$	L^{-2} TI	Coul-$metre^2$

Quantity	*Symbol*	*Dimensions in rationalised MKS i.e. LMTQ*	*Dimensions in S.I. System i.e. LMTI*	*Derived practical units*
Electric flux	ϕE	$L^3 M T^{-2} Q^{-1}$	$L^3 M T^{-3} I^{-1}$	Volt metre
Electric potential	V	$L^2 M T^{-2} Q^{-1}$	$L^2 M T^{-3} I^{-1}$	Volt
Electric-motive force	ξ	$L^2 M T^1 Q^{-1}$	$L^2 M T^{-3} I^{-1}$	Volt
Inductance	L	$L^2 M Q^{-2}$	$L^2 M T^{-2} I^{-2}$	Henry
Magnetic dipole moment	*m*	$L^2 T^{-1} Q$	$L^2 I$	Amp-metre^2
Magnetic field strength	H	$L^{-1} T^{-1} Q$	$L^{-1} I$	Amp/metre
Magnetic flux	ϕB	$L^2 M T^{-1} Q^{-1}$	$L^2 M T^{-2} I^{-1}$	weber = volt.sec
Magnetic inductance	B	$M T^{-1} Q^{-1}$	$M T^{-2} I^{-2}$	$\text{Tesla} = \frac{\text{weber}}{\text{metre}^2}$
Magnetic scalar potential	V_m	$T^{-1} Q$	I	ampere
Magnetic vector potential	A	$L M T^{-1} Q^{-1}$	$L^2 M T^{-2} I^{-1}$	$\frac{\text{henry.amperes}}{\text{metre}}$
Magnetisation[a]	M	$L^{-1} T^{-1} Q$	$L^{-1} I$	ampere/metre
Magneto-motive force	ξ_m	$T^{-1} Q$	I	(amp-turn)
Permeability	μ	$L M Q^{-2}$	$L M T^{-2} I^{-2}$	henry/metre
Permittivity	ϵ	$L^{-3} M^{-1} T^2 Q^2$	$L^{-3} M T^4 I^2$	Farads/metre
Polarisability[b]	α	$M^{-1} T^2 Q^2$	$M^{-1} T^4 I^2$	Farad.metre^2
Resistance	R	$L^2 M T^{-1} Q^{-2}$	$L^2 M T^{-3} I^{-2}$	Ohm
Resistivity	σ	$L^3 M T^{-1} Q^{-2}$	$L^2 M T^{-3} I^{-2}$	ohm.metre
Reluctance	R_L	$M^{-1} L^{-2} Q^2$	$M^{-1} L^{-2} T^2 I^2$	Henry^{-1}
Voltage	V	$L^2 M T^{-2} Q^{-1}$	$L^2 M T^{-3} I^{-1}$	Volt

(a) Defined by $\overline{M} = \frac{1}{\mu_0}(\overline{B} - \mu_0 \overline{H})$

(b) $\overline{P} = \alpha \overline{E}$

SOURCE BOOKS

		Author
1.	An Introduction to Vector Analysis	B. Hague John Wiley & Sons Inc. New York, London
2.	Physics Part II	David Halliday & Robert Resnick Wiley Eastern.
3.	Foundation of Electromagnetic Theory	John R. Reitz and Frederick J. Milford Indian Book Company
4.	Electricity and Magnetism-Barkley Physics Course. Vol. II	Edward M. Purcell
5.	Lectures on Physics, Vol. II	Feynman
6.	Dielectirc materials	B. Tareev Mir. Publisher Moscow
7.	Electromagnetics	John D. Karua McGraw Hill Book Co. Inc.
8.	Classical Electrodynamics	John David Jackson John Wiley & Sons Inc. New York / London
9.	Classical Electricity and Magnetism	Wolfgang K.H. Panofsky and Melba. Philips Indian Book Co.
10.	Electromagnetic Fields	Samuel Seely McGraw Hill Book Co. Inc
11.	Electricity & Magnetism	Edson Ruther Peck McGraw Hill Book Co. Inc.
12.	Electromagnetism	John C. Slater and Nathaniel H. Frank McGraw Hill Book Co. Inc.
13.	Electromagnetism	I.S. Grant, W.R. Phillips John Wiley & Sons Ltd., Chicheater
14.	Concepts in Electricity and Magnetism	Reuben Benumop, Holt, Rinehart and Winston-New York
15.	The Physics of Electricity and Magnetism	William Taussig Scott John Wiley & Sons Inc.
16.	Electricity and Magnetism	Glag D. Jefimenko Appelton Century-Crofts New York
17.	Magnetism and Electricity	H.E. Hardey Macmillan & co. Ltd., London
18.	Vacuum Tubes	Karl R. Spangenberg McGraw Hill Book Co. Inc.
19.	Electronics	Jacob Millimen & Samuel Seely McGraw Hill Book Co. Inc.
20.	Atomic Physics	J.B. Rajan S. Chand & Co.
21.	Electronic Circuit Analysis, Vol I	Philip Cutler & Hardy Hoover McGraw Hill Book Co. Inc.
22.	Modern Transistor Circuits	John M Carroll McGraw Hill Book Co. Inc.
23.	Transistors	Dennis Le Croissotte Prentice Hall of India New Delhi

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