Collins

Cambridge IGCSE™

Maths

STUDENT'S BOOK

Also for Cambridge IGCSE™ (9–1)

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Chapter 13

Equations and inequalities

All material labelled Core is also part of the Extended syllabus.

Topics	Level	Key words
1 Solving linear equations	CORE	equation, variable, solution, brackets, solve
2 Setting up equations	CORE	
3 Solving quadratic equations by factorisation	EXTENDED	linear equations, quadratic expressions, quadratic equation, factors, difference of two squares
4 Solving quadratic equations by the quadratic formula	EXTENDED	quadratic formula, coefficients, constant, soluble
5 Solving quadratic equations by completing the square	EXTENDED	completing the square
6 Fractional equations	EXTENDED	
7 Simultaneous equations	CORE	simultaneous linear equations, eliminate, substitute
8 Linear and non-linear simultaneous equations	EXTENDED	linear, non-linear
9 Representing inequalities	CORE	
10 Solving inequalities	EXTENDED	inequality

In this chapter you will learn how to:

CORE	EXTENDED
• Construct simple equations. (C2.5) • Solve linear equations in one unknown. (C2.5 and E2.5) • Solve simultaneous linear equations in two unknowns. (C2.5 and E2.5) • Represent and interpret inequalities, including on a number line. (C2.6 and E2.6)	• Construct expressions, equations and formulas. (E2.5) • Solve simultaneous equations, involving one linear and one non-linear. (E2.5) • Solve quadratic equations by factorisation, completing the square and by use of the quadratic formula. (E2.5) • Solve fractional equations with numerical and linear algebraic denominators. (E2.5) • Construct, solve and interpret linear inequalities, including on a number line. (E2.6)

Why this chapter matters

We use equations to explain some of the most important things in the world.

Three of the most important are shown on this page.

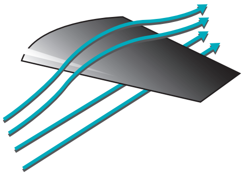
Why does the Moon keep orbiting the Earth and not fly off into space?

This is explained by Newton's law of universal gravitation, which describes the gravitational attraction between two bodies:

$$F = G \times \frac{m_1 \times m_2}{r^2}$$

where F is the force between the bodies, G is the gravitational constant, m_1 and m_2 are the masses of the two bodies and r is the distance between them.

Why don't planes fall out of the sky?



This is explained by Bernoulli's principle, which states that as the speed of a fluid increases, its pressure decreases. This is what causes the difference in air pressure between the top and bottom of an aircraft wing, as shown in the diagram on the left.

In its simplest form, the equation can be written as:

$$p + q = p_0$$

where p = static pressure, q = dynamic pressure and p_0 is the total pressure.

How can a small amount of plutonium release so much energy?

This is explained by Einstein's theory of special relativity, which connects mass and energy in the equation:

$$E = mc^2$$

where E is the energy, m is the mass and c is the speed of light. As the speed of light is nearly 300 000 kilometres per second, the amount of energy in a small mass is huge. If this can be released, it can be used to generate electricity in a nuclear power station.

13.1 Solving linear equations

A teacher gave these instructions to her class.

What algebraic expression represents the teacher's statement?

- Think of a number.
- Double it.
- Add 3.

This is what two of her students said.

Can you work out Kim's answer and the number that Freda started with?

I chose the number 5.

Kim

My final answer was 10.

Freda

Kim's answer will be:

$2 \times 5 + 3 = 13$.

Freda's answer can be set up as an **equation**.

An equation is formed when an expression is put equal to a number or another expression. You should be able to deal with equations that have only one **variable** or letter.

The **solution** to an equation is the value of the variable that makes the equation true.
For example, the equation for Freda's answer is

$$2x + 3 = 10$$

where x represents Freda's number.

The value of x that makes this true is $x = 3\frac{1}{2}$.

To **solve** an equation such as $2x + 3 = 10$, do the same thing to each side of the equation until you have x on its own.

$$2x + 3 = 10$$

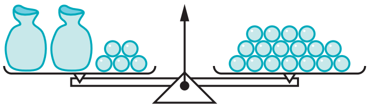
Subtract 3 from both sides: $2x + 3 - 3 = 10 - 3$

$$2x = 7$$

Divide both sides by 2: $\frac{2x}{2} = \frac{7}{2}$

$$x = 3\frac{1}{2}$$

Here is another example.



Mary had two bags, each of which contained the same number of marbles. She also had five spare marbles.

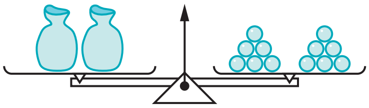
She put the two bags and the five spare marbles on scales and balanced them with 17 single marbles.

How many marbles were there in each bag?

If x is the number of marbles in each bag, then the equation representing Mary's balanced scales is:

$$2x + 5 = 17$$

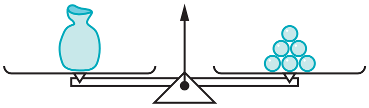
Take five marbles from each pan:



$$2x + 5 - 5 = 17 - 5$$

$$2x = 12$$

Now halve the number of marbles on each pan.



That is, divide both sides by 2:

$$\frac{2x}{2} = \frac{12}{2}$$

$$x = 6$$

Checking the answer gives $2 \times 6 + 5 = 17$, which is correct.

Example 1

Solve each of these equations by 'doing the same to both sides'.

a $3x - 5 = 16$

b $\frac{x}{2} + 2 = 10$

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a Add 5 to both sides

$$3x - 5 + 5 = 16 + 5$$

$$3x = 21$$

Divide both sides by 3.

$$\frac{3x}{3} = \frac{21}{3}$$

$$x = 7$$

Checking the answer gives:

$$3 \times 7 - 5 = 16$$

which is correct.

b Subtract 2 from both sides

$$\frac{x}{2} + 2 - 2 = 10 - 2$$

$$\frac{x}{2} = 8$$

Multiply both sides by 2.

$$\frac{x}{2} \times 2 = 8 \times 2$$

$$x = 16$$

Checking the answer gives:

$$16 \div 2 + 2 = 10$$

which is correct.

EXERCISE 13A

CORE

1 Solve each of these equations by 'doing the same to both sides'. Remember to check that each answer works for its original equation.

> **Advice and Tips**
>
> Be careful with negative numbers.

a $x + 4 = 60$ **b** $3y - 2 = 4$

c $3x - 7 = 11$ **d** $5y + 3 = 18$

e $7 + 3t = 19$ **f** $5 + 4f = 15$

g $3 + 6k = 24$ **h** $4x + 7 = 17$ **i** $5m - 3 = 17$

j $\frac{w}{3} - 5 = 2$ **k** $\frac{x}{8} + 3 = 12$ **l** $\frac{m}{7} - 3 = 5$

m $\frac{x}{5} + 3 = 3$ **n** $\frac{h}{7} + 2 = 1$ **o** $\frac{w}{3} + 10 = 4$

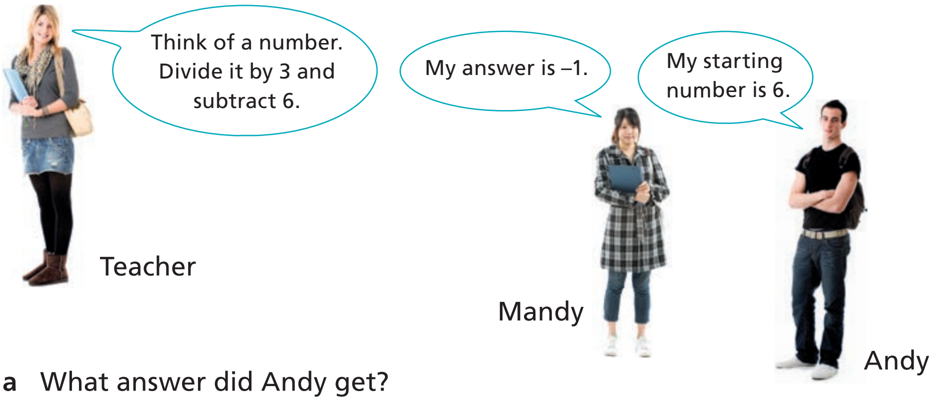
p $\frac{x}{3} - 5 = 7$ **q** $\frac{y}{2} - 13 = 5$ **r** $\frac{f}{6} - 2 = 8$

s $\frac{x + 3}{2} = 5$ **t** $\frac{t - 5}{2} = 3$ **u** $\frac{3x + 10}{2} = 8$

v $\frac{2x + 1}{3} = 5$ **w** $\frac{5y - 2}{4} = 3$ **x** $\frac{6y + 3}{9} = 1$

y $\frac{2x - 3}{5} = 4$ **z** $\frac{5t + 3}{4} = 1$

2



a What answer did Andy get?

b What number did Mandy start with?

3 A teacher asked her class to solve the equation $2x - 1 = 7$.

Mustafa wrote:

$2x - 1 = 7$

$2x - 1 - 1 = 7 - 1$

$2x = 6$

$2x - 2 = 6 - 2$

$x = 4$

Elif wrote:

$2x - 1 = 7$

$2x - 1 + 1 = 7 + 1$

$2x = 8$

$2x \div 2 = 8 \div 2$

$x = 4$

When the teacher read out the correct answer of 4, both students ticked their work as correct.

a Which student used the correct method?

b Explain the mistakes the other student made.

Brackets

When you have an equation that contains **brackets**, you first must multiply out the brackets and then solve the resulting equation.

Example 2

Solve the equation $5(x + 3) = 25$.

First multiply out the brackets to get:

$5x + 15 = 25$

Subtract 15: $5x = 25 - 15 = 10$

Divide by 5: $\frac{5x}{5} = \frac{10}{5}$

$x = 2$

An alternative method is to divide by the number outside the brackets.

Example 3

Solve the equation $3(2x - 7) = 15$.

Divide both sides by 3: $2x - 7 = 5$

Add 7: $2x = 12$

Divide by 2: $x = 6$

Make sure you can use both methods.

EXERCISE 13B

CORE

1 Solve each of these equations. Some of the answers may be decimals or negative numbers. Remember to check that each answer works for its original equation. Use your calculator if necessary.

a $2(x + 5) = 16$

b $5(x - 3) = 20$

c $3(t + 1) = 18$

d $4(2x + 5) = 44$

e $2(3y - 5) = 14$

f $5(4x + 3) = 135$

g $4(3t - 2) = 88$

h $6(2t + 5) = 42$

i $2(3x + 1) = 11$

j $4(5y - 2) = 42$

k $6(3k + 5) = 39$

Advice and Tips

Once the brackets have been expanded the equations become straightforward. Remember to multiply *everything* inside the brackets with what is outside.

CORE

l $5(2x + 3) = 27$

m $9(3x - 5) = 9$

n $2(x + 5) = 6$

o $5(x - 4) = -25$

p $3(t + 7) = 15$

q $2(3x + 11) = 10$

r $4(5t + 8) = 12$

2 Fill in values for a, b and c so that the answer to this equation is $x = 4$.

$a(bx + 3) = c$

3 My son is x years old. In five years' time, I will be twice his age and both our ages will be multiples of 10. The sum of our ages will be between 55 and 100. How old am I now?

Advice and Tips

Set up an equation and put it equal to 60, 70, 80, etc. Solve the equation and see if the answer fits the conditions.

Equations with the variable on both sides

When a letter (or variable) appears on both sides of an equation, collect all the terms containing the letter on one side. This is usually the left-hand side of the equation. When there are more of the letters on the right-hand side, it may be easier to turn the equation round. When an equation contains brackets, you must multiply them out first.

Example 4

Solve this equation. $5x + 4 = 3x + 10$

There are more xs on the left-hand side, so leave the equation as it is.

Subtract $3x$ from both sides: $2x + 4 = 10$

Subtract 4 from both sides: $2x = 6$

Divide both sides by 2: $x = 3$

Example 5

Solve this equation. $2x + 3 = 6x - 5$

There are more xs on the right-hand side, so turn the equation round.

$6x - 5 = 2x + 3$

Subtract $2x$ from both sides: $4x - 5 = 3$

Add 5 to both sides: $4x = 8$

Divide both sides by 4: $x = 2$

Example 6

Solve this equation.

$3(2x + 5) + x = 2(2 - x) + 2$

Multiply out both brackets: $6x + 15 + x = 4 - 2x + 2$

Simplify both sides: $7x + 15 = 6 - 2x$

There are more xs on the left-hand side, so leave the equation as it is.

Add $2x$ to both sides: $9x + 15 = 6$

Subtract 15 from both sides: $9x = -9$

Divide both sides by 9: $x = -1$

EXERCISE 13C

CORE

1 Solve each of these equations.

a $2x + 3 = x + 5$ **b** $5y + 4 = 3y + 6$

c $4a - 3 = 3a + 4$ **d** $5t + 3 = 2t + 15$

e $7p - 5 = 3p + 3$ **f** $6k + 5 = 2k + 1$

g $4m + 1 = m + 10$ **h** $8s - 1 = 6s - 5$

Advice and Tips

Remember: 'Change sides, change signs'. Show all your working. Rearrange before you simplify. If you try to do these at the same time you could get it wrong.

2 Hasan says:

Miriam says:

Hasan and Miriam find that they both thought of the same number and both got the same final answer.

What number did they think of?

Advice and Tips

Set up expressions; make them equal and solve.

3 Solve each of these equations.

a $2(d + 3) = d + 12$ **b** $5(x - 2) = 3(x + 4)$

c $3(2y + 3) = 5(2y + 1)$ **d** $3(h - 6) = 2(5 - 2h)$

e $4(3b - 1) + 6 = 5(2b + 4)$ **f** $2(5c + 2) - 2c = 3(2c + 3) + 7$

4 Explain why the equation $3(2x + 1) = 2(3x + 5)$ cannot be solved.

Advice and Tips

Expand the brackets and collect terms on one side as usual. What happens?

5 Explain why these are an infinite number of solutions to this equation.

$2(6x + 9) = 3(4x + 6)$

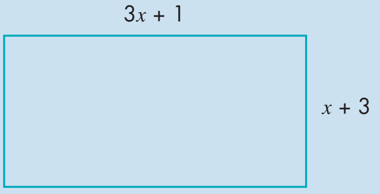
13.2 Setting up equations

You can use equations to represent situations, so that you can solve real-life problems. You can solve many real-life problems by setting them up as linear equations and then solving the equation.

Example 7

The rectangle shown has a perimeter of 40 cm.

Find the value of x.



The perimeter of the rectangle is:

$$3x + 1 + x + 3 + 3x + 1 + x + 3 = 40$$

This simplifies to: $8x + 8 = 40$

Subtract 8 from both sides: $8x = 32$

Divide both sides by 8: $x = 4$

EXERCISE 13D

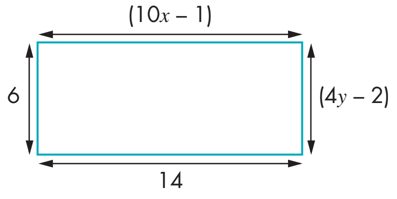
CORE

Set up an equation to represent each situation described below. Then solve the equation. Remember to check each answer.

1 Every day, from Monday to Saturday a man buys a daily paper for d cents. He buys a Sunday paper for 1.80 dollars. His weekly paper bill is 7.20 dollars.

What is the price of his daily paper?

2 The diagram shows a rectangle.



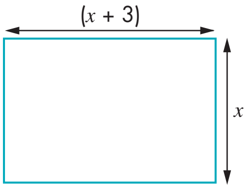
a What is the value of x?

b What is the value of y?

Advice and Tips

Use the letter x for the variable unless you are given a letter to use. Once the equation is set up solve it by the methods shown above.

3 In this rectangle, the length is 3 cm more than the width. The perimeter is 12 cm.



a What is the value of x?

b What is the area of the rectangle?

4 Masha has two bags, each of which contains the same number of sweets. She eats four sweets. She then finds that she has 30 sweets left. How many sweets were there in each bag to start with?

5 Flooring costs \$12.75 per square metre.

The shop charges \$35 for fitting. The final bill was \$137.

How many square metres of flooring were fitted?

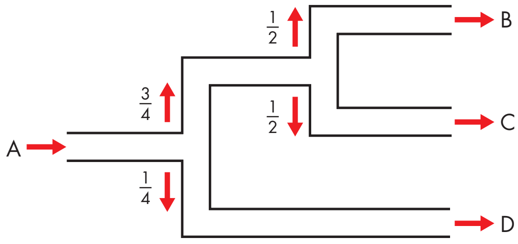
6 Moshin bought eight garden chairs. When he got to the till he used a \$10 voucher as part payment. His final bill was \$56.

a Set this problem up as an equation, using c as the cost of one chair.

b Solve the equation to find the cost of one chair.

7 This diagram shows the traffic flow through a one-way system in a town centre.

Cars enter at A and at each junction the fractions show the proportion of cars that take each route.



a 1200 cars enter at A. How many come out of each of the exits, B, C and D?

b If 300 cars exit at B, how many cars entered at A?

c If 500 cars exit at D, how many exit at B?

8 A rectangular room is 3 m longer than it is wide. The perimeter is 16 m.

Floor tiles cost \$9 for a pack that covers a square metre. How much will it cost to cover the floor?

Advice and Tips

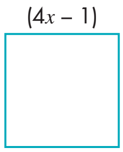
Set up an equation to work out the length and width, then calculate the area.

9 A boy is Y years old. His father is 25 years older than he is. The sum of their ages is 31. How old is the boy?

10 Another boy is X years old. His sister is twice as old as he is. The sum of their ages is 27. How old is the boy?

11 The diagram shows a square.

Find the value of x if the perimeter is 44 cm.

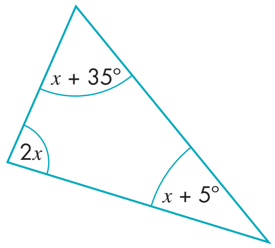


12 Max thought of a number. He then multiplied his number by 3. He added 4 to the answer. He then doubled that answer to get a final value of 38. What number did he start with?

13 The angles of a triangle, in degrees, are $2x$, $x + 5$ and $x + 35$.

a Write down an equation to show that the angles add up to 180 degrees.

b Solve your equation to find the value of x.



CORE

14 Five friends went for a meal in a restaurant. The bill was $\$x$.
They decided to add a \$10 tip and split the bill equally between them.

Each person paid \$9.50.

a Set up this problem as an equation.

b Solve the equation to work out the bill before the tip was added.

15 A teacher asked her class to find three angles of a triangle that were consecutive even numbers.

Tammy wrote:

$$x + x + 2 + x + 4 = 180$$
$$3x + 6 = 180$$
$$3x = 174$$
$$x = 58$$

So the angles are 58°, 60° and 62°.

The teacher then asked the class to find four angles of a quadrilateral that are consecutive even numbers.

Can this be done? Explain your answer.

Advice and Tips

Do the same type of working as Tammy did for a triangle. Work out the value of x. What happens?

16 Maria has a large and a small bottle of cola. The large bottle holds 50 cl more than the small bottle.

From the large bottle she fills four cups and has 18 cl left over.

From the small bottle she fills three cups and has 1 cl left over.

How much cola was there in each bottle?

Advice and Tips

Set up equations for both using x as the amount of cola in a cup. Put them equal but remember to add 50 to the small bottle equation to allow for the difference. Solve for x, then work out how much is in each bottle.

13.3 Solving quadratic equations by factorisation

E

So far, all the equations you have worked with have been **linear equations**. Now you will look at equations that involve **quadratic expressions** such as $x^2 - 2x - 3$, which contain the square of the variable.

Solving the quadratic equation $x^2 + ax + b = 0$

To solve a **quadratic equation** such as $x^2 - 2x - 3 = 0$, you first must be able to factorise it. Work through Examples 8 to 10 below to see how this is done.

Example 8

Solve the equation $x^2 + 6x + 5 = 0$.

This factorises into $(x + 5)(x + 1) = 0$.

The only way this expression can ever equal 0 is if the value of one of the expressions in the brackets is 0.

Hence either $(x + 5) = 0$ or $(x + 1) = 0$

$\Rightarrow x + 5 = 0$ or $x + 1 = 0$

$\Rightarrow x = -5$ or $x = -1$

So the solution is $x = -5$ or $x = -1$.

There are *two* possible values for x.

Example 9

Solve the equation $x^2 + 3x - 10 = 0$.

This factorises into $(x + 5)(x - 2) = 0$.

Hence either $(x + 5) = 0$ or $(x - 2) = 0$

$\Rightarrow x + 5 = 0$ or $x - 2 = 0$

$\Rightarrow x = -5$ or $x = 2$.

So the solution is $x = -5$ or $x = 2$.

Example 10

Solve the equation $x^2 - 6x + 9 = 0$.

This factorises into $(x - 3)(x - 3) = 0$.

The equation has repeated roots.

That is: $(x - 3)^2 = 0$

Hence, there is only one solution, $x = 3$.

EXERCISE 13E

Solve the equations in questions **1–12**.

1 $(x + 2)(x + 5) = 0$

2 $(t + 3)(t + 1) = 0$

3 $(a + 6)(a + 4) = 0$

4 $(x + 3)(x - 2) = 0$

5 $(x + 1)(x - 3) = 0$

6 $(t + 4)(t - 5) = 0$

7 $(x - 1)(x + 2) = 0$

8 $(x - 2)(x + 5) = 0$

9 $(a - 7)(a + 4) = 0$

10 $(x - 3)(x - 2) = 0$

11 $(x - 1)(x - 5) = 0$

12 $(a - 4)(a - 3) = 0$

EXTENDED

EXTENDED

First factorise, then solve the equations in questions **13**–**26**.

13 $x^2 + 5x + 4 = 0$

14 $x^2 + 11x + 18 = 0$

15 $x^2 - 6x + 8 = 0$

16 $x^2 - 8x + 15 = 0$

17 $x^2 - 3x - 10 = 0$

18 $x^2 - 2x - 15 = 0$

19 $t^2 + 4t - 12 = 0$

20 $t^2 + 3t - 18 = 0$

21 $x^2 - x - 2 = 0$

22 $x^2 + 4x + 4 = 0$

23 $m^2 + 10m + 25 = 0$

24 $t^2 - 8t + 16 = 0$

25 $t^2 + 8t + 12 = 0$

26 $a^2 - 14a + 49 = 0$

27 A woman is x years old. Her husband is three years younger.

The product of their ages is 550.

a Set up a quadratic equation to represent this situation.

b How old is the woman?

Advice and Tips

If one solution to a real-life problem is negative, reject it and only give the positive answer.

28 A rectangular field is 40 m longer than it is wide.
The area is 48 000 square metres.

The farmer wants to place a fence all around the field.

How long will the fence be?

Advice and Tips

Let the width be x, set up a quadratic equation and solve it to get x.

First rearrange the equations in questions **29**–**37**, then solve them.

Advice and Tips

You cannot solve a quadratic equation by factorisation unless it is in the form
$x^2 + ax + b = 0$

29 $x^2 + 10x = -24$

30 $x^2 - 18x = -32$

31 $x^2 + 2x = 24$

32 $x^2 + 3x = 54$

33 $t^2 + 7t = 30$

34 $x^2 - 7x = 44$

35 $t^2 - t = 72$

36 $x^2 = 17x - 72$

37 $x^2 + 1 = 2x$

38 A teacher asks her class to solve $x^2 - 3x = 4$.

This is Mario's answer.

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

Hence $x - 4 = 0$ or $x + 1 = 0$

$x = 4$ or -1

This is Sylvan's answer.

$x(x - 3) = 4$

Hence $x = 4$ or $x - 3 = 4 \Rightarrow x = -3 + 4 = -1$

When the teacher reads out the answer of $x = 4$ or -1, both students mark their work as correct.

Who used the correct method and what mistakes did the other student make?

Solving the general quadratic equation by factorisation

The general quadratic equation is of the form $ax^2 + bx + c = 0$ where a, b and c are positive or negative whole numbers. (It is easier to make sure that a is always positive.) Before you can solve any quadratic equation by factorisation, you must rearrange it to this form.

The method is similar to that used to solve equations of the form $x^2 + ax + b = 0$. That is, you have to find two **factors** of $ax^2 + bx + c$ with a product of 0.

Example 11

Solve these quadratic equations. **a** $12x^2 - 28x = -15$ **b** $30x^2 - 5x - 5 = 0$

a First, rearrange the equation to the general form.

$12x^2 - 28x + 15 = 0$

This factorises into $(2x - 3)(6x - 5) = 0$.

The only way this product can equal 0 is if the value of the expression in one of the brackets is 0. Hence:

either $2x - 3 = 0$ or $6x - 5 = 0$

$\Rightarrow 2x = 3$ or $6x = 5$

$\Rightarrow x = \frac{3}{2}$ or $x = \frac{5}{6}$

So the solution is $x = 1\frac{1}{2}$ or $x = \frac{5}{6}$

Note: It is almost always the case that if a solution is a fraction which is then changed into a rounded decimal number, the original equation cannot be evaluated exactly, using that decimal number. So it is preferable to leave the solution in its fraction form. This is called the *rational form*.

b This equation is already in the general form and it will factorise to $(15x + 5)(2x - 1) = 0$ or $(3x + 1)(10x - 5) = 0$.

Look again at the equation. There is a common factor of 5 which can be taken out to give:

$5(6x^2 - x - 1 = 0)$

This is much easier to factorise to $5(3x + 1)(2x - 1) = 0$, which can be solved to give $x = -\frac{1}{3}$ or $x = \frac{1}{2}$.

Special cases

Sometimes the value of b or c may be zero. (Note that if a is zero the equation is no longer a quadratic equation but a linear equation.)

Example 12

Solve these quadratic equations.

a $3x^2 - 4 = 0$

b $4x^2 - 25 = 0$

c $6x^2 - x = 0$

---

a Rearrange to get $3x^2 = 4$.

Divide both sides by 3: $x^2 = \frac{4}{3}$

Take the square root on both sides: $x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$

Note: A square root can be positive or negative. The symbol ± s means that the square root has a positive and a negative value, *both* of which must be used in solving for x.

b You can use the method of part **a** or you can treat this as the **difference of two squares**. This can be factorised to $(2x - 5)(2x + 5) = 0$.

Each set of brackets can be put equal to zero.

$2x - 5 = 0 \Rightarrow x = +\frac{5}{2}$

$2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$ So the solution is $x = \pm\frac{5}{2}$.

c There is a common factor of x, so factorise as $x(6x - 1) = 0$.

There is only one set of brackets this time but each factor can be equal to zero, so $x = 0$ or $6x - 1 = 0$.

Hence, $x = 0$ or $\frac{1}{6}$.

EXERCISE 13F

EXTENDED

Give your answers either in rational form or as mixed numbers.

1 Solve these equations.

a $3x^2 + 8x - 3 = 0$ **b** $6x^2 - 5x - 4 = 0$

c $5x^2 - 9x - 2 = 0$ **d** $4t^2 - 4t - 35 = 0$

e $18t^2 + 9t + 1 = 0$ **f** $3t^2 - 14t + 8 = 0$

g $6x^2 + 15x - 9 = 0$ **h** $12x^2 - 16x - 35 = 0$

i $15t^2 + 4t - 35 = 0$ **j** $28x^2 - 85x + 63 = 0$

k $24x^2 - 19x + 2 = 0$ **l** $16t^2 - 1 = 0$

m $4x^2 + 9x = 0$ **n** $25t^2 - 49 = 0$

o $9m^2 - 24m - 9 = 0$

Advice and Tips

Look out for the special cases where b or c is zero.

2 Rearrange these equations into the general form and then solve them.

- **a** $x^2 - x = 42$
- **b** $8x(x + 1) = 30$
- **c** $(x + 1)(x - 2) = 40$
- **d** $13x^2 = 11 - 2x$
- **e** $(x + 1)(x - 2) = 4$
- **f** $10x^2 - x = 2$
- **g** $8x^2 + 6x + 3 = 2x^2 + x + 2$
- **h** $25x^2 = 10 - 45x$
- **i** $8x - 16 - x^2 = 0$
- **j** $(2x + 1)(5x + 2) = (2x - 2)(x - 2)$
- **k** $5x + 5 = 30x^2 + 15x + 5$
- **l** $2m^2 = 50$
- **m** $6x^2 + 30 = 5 - 3x^2 - 30x$
- **n** $4x^2 + 4x - 49 = 4x$
- **o** $2t^2 - t = 15$

3 Here are three equations.

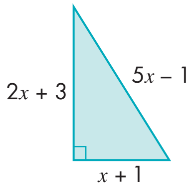
A: $(x - 1)^2 = 0$ B: $3x + 2 = 5$ C: $x^2 - 4x = 5$

- **a** Give some mathematical fact that equations A and B have in common.
- **b** Give a mathematical reason why equation B is different from equations A and C.

4 Pythagoras' theorem states that the sum of the squares of the two short sides of a right-angled triangle equals the square of the long side (hypotenuse).

A right-angled triangle has sides $5x - 1$, $2x + 3$ and $x + 1$ cm.

- **a** Show that $20x^2 - 24x - 9 = 0$.
- **b** Find the area of the triangle.



13.4 Solving quadratic equations by the quadratic formula

E

Many quadratic equations cannot be solved by factorisation because they do not have simple factors. Try to factorise, for example, $x^2 - 4x - 3 = 0$ or $3x^2 - 6x + 2 = 0$. You will find it is impossible.

One way to solve this type of equation is to use the **quadratic formula**. This formula can be used to solve *any* quadratic equation that can be solved (is **soluble**).

The solution of the equation $ax^2 + bx + c = 0$ is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a and b are the **coefficients** of x^2 and x respectively and c is the **constant** term.

This is the quadratic formula, which you should memorise.

Example 13

Solve $5x^2 - 11x - 4 = 0$, giving solutions correct to 2 decimal places.

Take the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and put $a = 5$, $b = -11$ and $c = -4$, which gives:

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(5)(-4)}}{2(5)}$$

Note that the values for a, b and c have been put into the formula in brackets. This is to avoid mistakes in calculation. It is a very common mistake to get the sign of b wrong or to think that -11^2 is -121. Using brackets will help you do the calculation correctly.

$$x = \frac{11 \pm \sqrt{121 + 80}}{10} = \frac{11 \pm \sqrt{201}}{10}$$

$$\Rightarrow x = 2.52 \text{ or } -0.32$$

Note: The calculation has been done in stages. With a calculator it is possible just to work out the answer, but make sure you can use your calculator properly. Otherwise, break the calculation down. Remember the rule: 'if you try to do two things at once, you will probably get one of them wrong'.

Note: If you are asked to solve a quadratic equation to one or two decimal places, you can be sure that it can be solved *only* by the quadratic formula.

EXERCISE 13G

EXTENDED

Use the quadratic formula to solve the equations in questions **1–15**. Give your answers to 2 decimal places.

1 $2x^2 + x - 8 = 0$

2 $3x^2 + 5x + 1 = 0$

3 $x^2 - x - 10 = 0$

4 $5x^2 + 2x - 1 = 0$

5 $7x^2 + 12x + 2 = 0$

6 $3x^2 + 11x + 9 = 0$

7 $4x^2 + 9x + 3 = 0$

8 $6x^2 + 22x + 19 = 0$

9 $x^2 + 3x - 6 = 0$

10 $3x^2 - 7x + 1 = 0$

11 $2x^2 + 11x + 4 = 0$

12 $4x^2 + 5x - 3 = 0$

Advice and Tips

Use brackets when substituting and do not try to work two things out at the same time.

13 $4x^2 - 9x + 4 = 0$

14 $7x^2 + 3x - 2 = 0$

15 $5x^2 - 10x + 1 = 0$

16 A rectangular lawn is 2 m longer than it is wide.

The area of the lawn is 21 m^2. The gardener wants to edge the lawn with edging strips, which are sold in lengths of $1\frac{1}{2}$ m. How many will she need to buy?

17 Shaun is solving a quadratic equation, using the formula.

He correctly substitutes values for a, b and c to get:

$$x = \frac{3 \pm \sqrt{37}}{2}$$

What is the equation Shaun is trying to solve?

18 Hasan uses the quadratic formula to solve $4x^2 - 4x + 1 = 0$.

Miriam uses factorisation to solve $4x^2 - 4x + 1 = 0$.

They both find something unusual in their solutions.

Explain what this is, and why.

13.5 Solving quadratic equations by completing the square

E

Another method for solving quadratic equations is **completing the square**. You learned about this in section 12.7. You can use this method as an alternative to the quadratic formula.

There are three basic steps in rewriting $x^2 + px + q$ in the form $(x + a)^2 + b$.

Step 1: Ignore q and just look at the first two terms, $x^2 + px$.

Step 2: Rewrite $x^2 + px$ as $\left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$.

Step 3: Bring q back to get $x^2 + px + q = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q$.

Example 14

Rewrite each expression in the form $(x \pm a)^2 \pm b$.

a $x^2 + 6x - 7$

b $x^2 - 8x + 3$

---

a Ignore –7 for the moment.

Rewrite $x^2 + 6x$ as $(x + 3)^2 - 9$.

(Expand $(x + 3)^2 - 9 = x^2 + 6x + 9 - 9 = x^2 + 6x$. The 9 is subtracted to get rid of the constant term when the brackets are expanded.)

Now bring the –7 back, so $x^2 + 6x - 7 = (x + 3)^2 - 9 - 7$

Combine the constant terms to get the final answer:

$$x^2 + 6x - 7 = (x + 3)^2 - 16$$

b Ignore +3 for the moment.

Rewrite $x^2 - 8x$ as $(x - 4)^2 - 16$.

(Note that you still subtract $(-4)^2$, as $(-4)^2 = +16$.)

Now bring the +3 back, so $x^2 - 8x + 3 = (x - 4)^2 - 16 + 3$.

Combine the constant terms to get the final answer:

$$x^2 - 8x + 3 = (x - 4)^2 - 13$$

Example 15

Rewrite $x^2 + 4x - 7$ in the form $(x + a)^2 - b$. Hence solve the equation $x^2 + 4x - 7 = 0$, giving your answers to 2 decimal places.

Note that:

$$x^2 + 4x = (x + 2)^2 - 4$$

So:

$$x^2 + 4x - 7 = (x + 2)^2 - 4 - 7 = (x + 2)^2 - 11$$

When $x^2 + 4x - 7 = 0$, you can rewrite the equations completing the square as:

$$(x + 2)^2 - 11 = 0$$

Rearranging gives $(x + 2)^2 = 11$.

Taking the **square root** of both sides gives:

$$x + 2 = \pm\sqrt{11}$$

$$\Rightarrow x = -2 \pm \sqrt{11}$$

This answer could be left like this, but you are asked to calculate it to 2 decimal places.

$$\Rightarrow x = 1.32 \text{ or } -5.32 \text{ (to 2 decimal places)}$$

To solve $ax^2 + bx + c = 0$ when a is not 1, start by dividing through by a.

Example 16

Solve by completing the square.

$2x^2 - 6x - 7 = 0$

Divide by 2: $x^2 - 3x - 3.5 = 0$

$x^2 - 3x = (x - 1.5)^2 - 2.25$

So: $x^2 - 3x - 3.5 = (x - 1.5)^2 - 5.75$

When: $x^2 - 3x - 3.5 = 0$

then: $(x - 1.5)^2 = 5.75$

$x - 1.5 = \pm\sqrt{5.75}$

$x = 1.5 \pm\sqrt{5.75}$

$x = 3.90$ or $x = -0.90$

EXERCISE 13H

EXTENDED

1 Write an equivalent expression in the form $(x \pm a)^2 - b$.

a $x^2 + 4x$ **b** $x^2 + 14x$ **c** $x^2 - 6x$ **d** $x^2 + 6x$
e $x^2 - 3x$ **f** $x^2 - 9x$ **g** $x^2 + 13x$ **h** $x^2 + 10x$
i $x^2 + 8x$ **j** $x^2 - 2x$ **k** $x^2 + 2x$

2 Write an equivalent expression in the form $(x \pm a)^2 - b$.
Question **1** will help with **a** to **h**.

a $x^2 + 4x - 1$ **b** $x^2 + 14x - 5$ **c** $x^2 - 6x + 3$ **d** $x^2 + 6x + 7$
e $x^2 - 3x - 1$ **f** $x^2 + 6x + 3$ **g** $x^2 - 9x + 10$ **h** $x^2 + 13x + 35$
i $x^2 + 8x - 6$ **j** $x^2 + 2x - 1$ **k** $x^2 - 2x - 7$ **l** $x^2 + 2x - 9$

3 Solve each equation by completing the square. Leave your answer as a surd where necessary.
The answers to question **2** will help.

a $x^2 + 4x - 1 = 0$ **b** $x^2 + 14x - 5 = 0$ **c** $x^2 - 6x + 3 = 0$
d $x^2 + 6x + 7 = 0$ **e** $x^2 - 3x - 1 = 0$ **f** $x^2 + 6x + 3 = 0$
g $x^2 - 9x + 10 = 0$ **h** $x^2 + 13x + 35 = 0$ **i** $x^2 + 8x - 6 = 0$
j $x^2 + 2x - 1 = 0$ **k** $x^2 - 2x - 7 = 0$ **l** $x^2 + 2x - 9 = 0$

4 Solve by completing the square. Give your answers to 2 decimal places.

a $x^2 + 2x - 5 = 0$ **b** $x^2 - 4x - 7 = 0$ **c** $x^2 + 2x - 9 = 0$

5 Solve these equations by completing the square. Leave your answer as a surd.

a $2x^2 - 6x - 3 = 0$ **b** $4x^2 - 8x + 1 = 0$ **c** $2x^2 + 5x - 10 = 0$
d $0.5x^2 - 7.5x + 8 = 0$

EXTENDED

6 Ahmed rewrites the expression $x^2 + px + q$ by completing the square. He does this correctly and gets $(x - 7)^2 - 52$.

What are the values of p and q?

7 Jorge writes the steps to solve $x^2 + 6x + 7 = 0$ by completing the square. He writes them on sticky notes. Unfortunately he drops the sticky notes and they get out of order. Try to put the notes in the correct order.

Add 2 to both sides	*Subtract 3 from both sides*	*Write $x^2 + 6x + 7 = 0$ as $(x + 3)^2 - 2 = 0$*	*Take the square root of both sides*

13.6 Fractional equations

E

Look at these equations.

$$\frac{x+4}{x-4} = 5 \qquad \frac{10}{x+1} = \frac{x}{x-2} \qquad \frac{3}{x-1} + \frac{12}{x+1} = 2$$

In each equation, there is an algebraic expression in the denominator of a fraction.

To solve an equation like this, multiply each side by the denominator.

To solve $\frac{x+4}{x-4} = 5$ multiply each side by $x - 4$

$$x + 4 = 5(x - 4)$$

Multiply out the bracket and rearrange.

$$x + 4 = 5x - 20$$

$$4 + 20 = 5x - x$$

$$24 = 4x, \text{ so } x = 6$$

You can check that this is a solution.

If $x = 6$, then $\frac{x+4}{x-4} = \frac{6+4}{6-4} = \frac{10}{2} = 5$

The equation $\frac{10}{x+1} = \frac{x}{x-2}$ has two algebraic expressions in the denominator.
Multiply by both on each side.

$$10(x - 2) = x(x + 1)$$

Multiply out the brackets and rearrange.

$$10x - 20 = x^2 + x$$

$$0 = x^2 - 9x + 20$$

Advice and Tips

$x + 1$ cancels on the left and $x - 2$ cancels on the right.

You can factorise this quadratic expression.

$(x-4)(x-5) = 0$ so $x = 4$ or 5. There are two solutions to this equation.

Check: if $x = 4$ then $\frac{10}{x+1} = \frac{10}{5} = 2$ and $\frac{x}{x-2} = \frac{4}{2} = 2$

if $x = 5$ then $\frac{10}{x+1} = \frac{10}{6} = \frac{5}{3}$ and $\frac{x}{x-2} = \frac{5}{3}$

Example 17

Solve the equation $\frac{3}{x-1} + \frac{12}{x+1} = 2$

Multiply both sides by $x - 1$ and $x + 1$.

$$3(x+1) + 12(x-1) = 2(x-1)(x+1)$$

Multiply out the brackets and rearrange.

$$3x + 3 + 12x - 12 = 2(x^2 - 1)$$

$$15x - 9 = 2x^2 - 2$$

$$0 = 2x^2 - 15x + 7$$

You can factorise this quadratic expression.

$$(2x-1)(x-7) = 0$$

Either $2x - 1 = 0$ so $x = \frac{1}{2}$ or $x - 7 = 0$ so $x = 7$

Exercise 13I

EXTENDED

1 Solve these equations.

a $\frac{x}{2} + \frac{x}{3} = 4$ b $\frac{x}{3} - \frac{x}{5} = 8$ c $\frac{x}{2} + \frac{2x}{3} = 4$

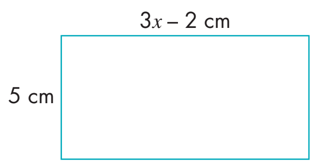
2 Solve these equations.

a $\frac{12}{x} = 20$ b $\frac{x+10}{x-2} = 4$ c $\frac{3x-5}{x-5} = 8$

d $\frac{x}{x+2} = 5$ e $\frac{6x-1}{2x+3} = 2$ f $\frac{4-x}{7+x} = 10$

3 The area of this rectangle is $12x + 2$ cm^2.

Find the value of x.



EXTENDED

4 Solve these equations.

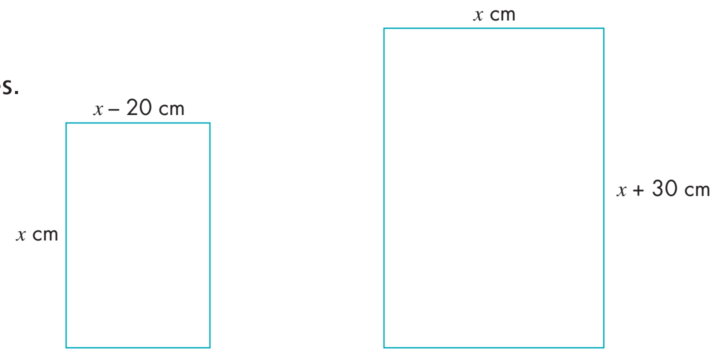
a $\frac{x+6}{x} = \frac{x}{x-3}$ **b** $\frac{x+5}{x} = \frac{x}{x-4}$ **c** $\frac{x+3}{x} = \frac{x}{4}$

d $\frac{x+1}{x} = \frac{2x+5}{x+4}$ **e** $\frac{x+4}{x+1} = \frac{x}{6}$ **f** $\frac{x}{x+3} = \frac{5}{x+5}$

5 These rectangles are similar.

$\frac{\text{length}}{\text{width}}$ is the same for both rectangles.

Work out the value of x.



6 Solve these equations.

a $\frac{8}{x+1} + \frac{10}{x+2} = 4$ **b** $\frac{9}{2x+1} - \frac{4}{x+1} = 1$

c $\frac{2}{x-1} + \frac{1}{x-3} = 1$ **d** $\frac{5}{2x-1} + \frac{10}{3x+1} = 2$

7 **a** $\frac{2}{x} + \frac{x}{2} = 2$

Show that this equation has only one solution.

b $\frac{2}{x} + \frac{x}{2} = 2.5$

Solve this equation.

c $\frac{2}{x} + \frac{x}{2} = 4$

Show that the solutions of this equation are $x = 4 \pm 2\sqrt{3}$.

13.7 Simultaneous equations

All the equations we have looked at so far have just one unknown.

Sometimes there is more than one unknown variable in a problem. In these cases, we will have several **simultaneous equations** to solve.

Example 18

Tariq is twice as old as Meera. Their total age is 39 years. How old are they?

Suppose Tariq is x years old and Meera is y years old.

Tariq is twice as old as Meera: $x = 2y$ (equation 1)

Their total age is 39 years: $x + y = 39$ (equation 2)

We have two unknowns and two equations to use to find them.

Substitute $2y$ for x in equation 2:

$2y + y = 39$

$\Rightarrow 3y = 39$

$\Rightarrow y = 13$

Now use equation (1) to find x: $x = 2 \times 13 = 26$

Tariq is 26 and Meera is 13.

EXERCISE 13J

1 Solve each of these pairs of simultaneous equations.

a $x + y = 15$, $y = 2x$ **b** $x = 3y$, $x + y = 24$ **c** $x + y = 60$, $y = 4x$

2 Solve each of these pairs of simultaneous equations.

a $y = x + 12$, $y = 3x$ **b** $y = x - 10$, $x = 5y$ **c** $x + 4 = y$, $y = 9x$

3 Solve each of these pairs of simultaneous equations.

a $x + y = 20$, $x - y = 6$ **b** $y + x = 23$, $y - x = 5$ **c** $x + y = 6$, $x - y = 14$

4 Solve each of these pairs of simultaneous equations.

a $y = 2x + 3$, $y = 8x$ **b** $x + y = 20$, $y = 3x - 2$ **c** $y = 2x + 4$, $y = 10 - x$

5 Carmen and Anish are carrying some books. There are 40 books altogether.

Carmen has 4 times as many as Anish.

How many do they each have?

6 Ari writes down two numbers. The total is 37. The difference between them is 14.

What are the numbers?

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7 Luis records the temperature at midday and again at midnight.

He notices that the temperatures add up to 5 and the difference between them is 11.

What are the temperatures?

8 Carlos has $\$x$ and Sarah has $\$y$.

a Together they have \$75. Write an equation to show this.

b Sarah has twice as much as Carlos. Write an equation to show this.

c Solve the equations to find x and y.

9 The mass of a plate is x g and the mass of a cup is y g.

a The total mass is 300 g. Write an equation to show this.

b The plate is 60 g heavier than the cup. Write an equation to show this.

c Find the values of x and y.

10 Ahmed is x years old. His mother is y years old.

a Ahmed is 26 years younger than his mother. Write an equation to show this.

b The total of Ahmed's age and his mother's age is 50 years. Write an equation to show his.

c Find the age of Ahmed and his mother.

11 The length of Chen's car is x m. The length of Ari's car is y m.

a Chen's car is 0.4 m shorter than Paola's. Write an equation to show this.

b The total length of the two cars is 8.6 m. Write an equation to show this.

c Find the length of Paola's car.

A pair of **simultaneous linear equations** is exactly that — two linear equations for which you want the same solution, and which you therefore *solve together*. For example,

$x + y = 10$ has many solutions:

$x = 2, y = 8$

$x = 4, y = 6$

$x = 5, y = 5$ …

and $2x + y = 14$ has many solutions:

$x = 2, y = 10$

$x = 3, y = 8$

$x = 4, y = 6$ …

But only *one* solution, $x = 4$ and $y = 6$, satisfies both equations at the same time.

In the last section you looked at some simple examples. You can now look at ways of solving more complicated examples of simultaneous equations.

Elimination method

One way to solve simultaneous equations is by the *elimination method*. There are six steps in this method.

Step 1 is to balance the coefficients of one of the variables.

Step 2 is to **eliminate** this variable by adding or subtracting the equations.

Step 3 is to solve the resulting linear equation in the other variable.

Step 4 is to **substitute** the value found back into one of the previous equations.

Step 5 is to solve the resulting equation.

Step 6 is to check that the two values found satisfy the original equations.

Example 19

Solve the equations: $6x + y = 15$ and $4x + y = 11$.

Label the equations so that the method can be clearly explained.

$6x + y = 15$ (1)

$4x + y = 11$ (2)

Step 1: Since the y-term in both equations has the same coefficient there is no need to balance them.

Step 2: Subtract one equation from the other. (Equation (1) minus equation (2) will give positive values.)

(1) – (2) $2x = 4$

Step 3: Solve this equation: $x = 2$

Step 4: Substitute $x = 2$ into one of the original equations. (Usually it is best to the one with smallest numbers involved.)

So substitute into: $4x + y = 11$

which gives: $8 + y = 11$

Step 5: Solve this equation: $y = 3$

Step 6: Test the solution in the original equations. So substitute $x = 2$ and $y = 3$ into $6x + y$, which gives $12 + 3 = 15$ and into $4x + y$, which gives $8 + 3 = 11$. These are correct, so you can confidently say the solution is $x = 2$ and $y = 3$.

Example 20

Solve these equations. $5x + y = 22$ (1)

$2x - y = 6$ (2)

Step 1: Both equations have the same y-coefficient but with *different* signs so there is no need to balance them.

Step 2: As the signs are different, *add* the two equations, to eliminate the y-terms.

(1) + (2) $7x = 28$

Step 3: Solve this equation: $x = 4$

Step 4: Substitute $x = 4$ into one of the original equations, $5x + y = 22$,

which gives: $20 + y = 22$

Step 5: Solve this equation: $y = 2$

Step 6: Test the solution by putting $x = 4$ and $y = 2$ into the original equations, $2x - y$, which gives $8 - 2 = 6$ and $5x + y$ which gives $20 + 2 = 22$. These are correct, so the solution is $x = 4$ and $y = 2$.

Substitution method

This is an alternative method. The method you use depends very much on the coefficients of the variables and the way that the equations are written in the first place. There are five steps in the substitution method.

Step 1 is to rearrange one of the equations into the form $y = \ldots$ or $x = \ldots$.

Step 2 is to substitute the right-hand side of this equation into the other equation in place of the variable on the left-hand side.

Step 3 is to expand and solve this equation.

Step 4 is to substitute the value into the $y = \ldots$ or $x = \ldots$ equation.

Step 5 is to check that the values work in both original equations.

Example 21

Solve the simultaneous equations: $y = 2x + 3$, $3x + 4y = 1$.

Because the first equation is in the form $y = \ldots$ you can use the substitution method.

Again label the equations to help with explaining the method.

$y = 2x + 3$ (1)

$3x + 4y = 1$ (2)

Step 1: As equation (1) is in the form $y = \ldots$ there is no need to rearrange an equation.

Step 2: Substitute the right-hand side of equation (1) into equation (2) for the variable y.

$3x + 4(2x + 3) = 1$

Step 3: Expand and solve the equation. $3x + 8x + 12 = 1$, $11x = -11$, $x = -1$

Step 4: Substitute $x = -1$ into $y = 2x + 3$: $y = -2 + 3 = 1$

Step 5: Test the values in $y = 2x + 3$, which gives $1 = -2 + 3$ and $3x + 4y = 1$, which gives $-3 + 4 = 1$. These are correct so the solution is $x = -1$ and $y = 1$.

EXERCISE 13K

1 Solve these simultaneous equations.

In question **1** parts **a** to **i** the coefficients of one of the variables are the same so there is no need to balance them. Subtract the equations when the identical terms have the same sign. Add the equations when the identical terms have opposite signs. In parts **j** to **l** use the substitution method.

a $4x + y = 17$
$2x + y = 9$

b $5x + 2y = 13$
$x + 2y = 9$

c $2x + y = 7$
$5x - y = 14$

d $3x + 2y = 11$
$2x - 2y = 14$

e $3x - 4y = 17$
$x - 4y = 3$

f $3x + 2y = 16$
$x - 2y = 4$

g $x + 3y = 9$
$x + y = 6$

h $2x + 5y = 16$
$2x + 3y = 8$

i $3x - y = 9$
$5x + y = 11$

j $2x + 5y = 37$
$y = 11 - 2x$

k $4x - 3y = 7$
$x = 13 - 3y$

l $4x - y = 17$
$x = 2 + y$

2 In this sequence, the next term is found by multiplying the previous term by a and then adding b. a and b are positive whole numbers.

3 14 47 … …

a Explain why $3a + b = 14$.

b Set up another equation in a and b.

c Solve the equations to solve for a and b.

d Work out the next two terms in the sequence.

Balancing coefficients in one equation only

You could solve all the examples in Exercise 13I, question **1** by adding or subtracting the equations in each pair, or by substituting without rearranging. This does not always happen. The next examples show what to do when there are no identical terms, or when you need to rearrange.

Example 22

Solve these equations. $3x + 2y = 18$ (1)

$2x - y = 5$ (2)

Step 1: Multiply equation (2) by 2. There are other ways to balance the coefficients but this is the easiest and leads to less work later. With practice, you will get used to which will be the best way to balance the coefficients.

2 × (2) $4x - 2y = 10$ (3)

Label this equation as number (3).

Be careful to multiply every term and not just the y-term. You could write:

$2 \times (2x - y = 5) \Rightarrow 4x - 2y = 10$ (3)

Step 2: As the signs of the y-terms are opposite, add the equations.

(1) + (3) $7x = 28$

Be careful to add the correct equations. This is why labelling them is useful.

Step 3: Solve this equation: $x = 4$

Step 4: Substitute $x = 4$ into any equation, say $2x - y = 5 \Rightarrow 8 - y = 5$

Step 5: Solve this equation: $y = 3$

Step 6: Check: (1), $3 \times 4 + 2 \times 3 = 18$ and (2), $2 \times 4 - 3 = 5$, which are correct so the solution is $x = 4$ and $y = 3$.

Example 23

Solve the simultaneous equations: $3x + y = 5$ (1)

$5x - 2y = 12$ (2)

Step 1: Multiply the first equation by 2: $6x + 2y = 10$ (3)

Step 2: Add (2) + (3): $11x = 22$

Step 3: Solve: $x = 2$

Step 4: Substitute back: $3 \times 2 + y = 5$

Step 5: Solve: $y = -1$

Step 6: Check: (1) $3 \times 2 - 1 = 5$ and (2) $5 \times 2 - 2 \times -1 = 10 + 2 = 12$, which are correct.

EXERCISE 13L

CORE

1 Solve parts **a** to **c** by the substitution method and the rest by first changing one of the equations in each pair to obtain identical terms for one unknown, and then adding or subtracting the equations to eliminate those terms.

a $5x + 2y = 4$, $4x - y = 11$

b $4x + 3y = 37$, $2x + y = 17$

c $x + 3y = 7$, $2x - y = 7$

d $2x + 3y = 19$, $6x + 2y = 22$

e $5x - 2y = 26$, $3x - y = 15$

f $10x - y = 3$, $3x + 2y = 17$

g $3x + 5y = 15$, $x + 3y = 7$

h $3x + 4y = 7$, $4x + 2y = 1$

i $5x - 2y = 24$, $3x + y = 21$

j $5x - 2y = 4$, $3x - 6y = 6$

k $2x + 3y = 13$, $4x + 7y = 31$

l $3x - 2y = 3$, $5x + 6y = 12$

2 **a** Francesca is solving the simultaneous equations $4x - 2y = 8$ and $2x - y = 4$.

She finds a solution of $x = 5$, $y = 6$ which works for both equations.

Explain why this is not a unique solution.

b Dimitri is solving the simultaneous equations $6x + 2y = 9$ and $3x + y = 7$.

Why is it impossible to find a solution that works for both equations?

Balancing coefficients in both equations

In some cases, you will need to change *both* equations to obtain identical terms. The next example shows you how to do this.

Note: The substitution method is not suitable for these types of equation as you end up with fractional terms.

Example 24

Solve these equations. $4x + 3y = 27$ (1)

$5x - 2y = 5$ (2)

You need to change both equations to obtain identical terms in either x or y. However, you can see that if you make the y-coefficients the same, you will add the equations. Addition is always safer than subtraction, so this is obviously the better choice. Do this by multiplying the first equation by 2 (the y-coefficient of the second equation) and the second equation by 3 (the y-coefficient of the first equation).

Step 1: (1) × 2 or 2 × ($4x + 3y = 27$) $\Rightarrow$ $8x + 6y = 54$ (3)

(2) × 3 or 3 × ($5x - 2y = 5$) $\Rightarrow$ $15x - 6y = 15$ (4)

Label the new equations (3) and (4).

Step 2: Eliminate one of the variables: (3) + (4): $23x = 69$

Step 3: Solve the equation: $x = 3$

Step 4: Substitute into equation (1): $12 + 3y = 27$

Step 5: Solve the equation: $y = 5$

Step 6: Check: (1), $4 \times 3 + 3 \times 5 = 12 + 15 = 27$ and (2), $5 \times 3 - 2 \times 5 = 15 - 10 = 5$, which are correct so the solution is $x = 3$ and $y = 5$.

EXERCISE 13M

CORE

1 Solve these simultaneous equations.

a $2x + 5y = 15$
$3x - 2y = 13$

b $2x + 3y = 30$
$5x + 7y = 71$

c $2x - 3y = 15$
$5x + 7y = 52$

d $3x - 2y = 15$
$2x - 3y = 5$

e $5x - 3y = 14$
$4x - 5y = 6$

f $3x + 2y = 28$
$2x + 7y = 47$

g $2x + y = 4$
$x - y = 5$

h $5x + 2y = 11$
$3x + 4y = 8$

i $x - 2y = 4$
$3x - y = -3$

j $3x + 2y = 2$
$2x + 6y = 13$

k $6x + 2y = 14$
$3x - 5y = 10$

l $2x + 4y = 15$
$x + 5y = 21$

m $3x - y = 5$
$x + 3y = -20$

n $3x - 4y = 4.5$
$2x + 2y = 10$

o $x - 5y = 15$
$3x - 7y = 17$

CORE

2 Here are four equations.

A: $5x + 2y = 1$

B: $4x + y = 9$

C: $3x - y = 5$

D: $3x + 2y = 3$

Here are four sets of (x, y) values.

(1, –2), (–1, 3), (2, 1), (3, –3)

Match each pair of (x, y) values to a pair of equations.

Advice and Tips

You could solve each possible set of pairs but there are six to work out. Alternatively you can substitute values into the equations to see which work.

3 Find the area of the triangle enclosed by these three equations.

$y - x = 2$ $x + y = 6$ $3x + y = 6$

4 Find the area of the triangle enclosed by these three equations.

$x - 2y = 6$ $x + 2y = 6$ $x + y = 3$

Advice and Tips

Find the point of intersection of each pair of equations, plot the points on a grid and use any method to work out the area of the resulting triangle.

13.8 Linear and non-linear simultaneous equations

E

You have already seen the method of substitution for solving **linear** simultaneous equations.

You can use a similar method when you need to solve a pair of equations, one of which is linear and the other of which is **non-linear**. But you must always substitute from the linear into the non-linear.

Example 25

Solve these simultaneous equations.

$x^2 + y^2 = 5$

$x + y = 3$

Call the equations (1) and (2):

$x^2 + y^2 = 5$ (1)

$x + y = 3$ (2)

Rearrange equation (2) to obtain:

$x + = 3 - y$

Substitute this into equation (1), which gives:

$(3 - y)^2 + y^2 = 5$

Expand and rearrange into the general form of the quadratic equation:

$9 - 6y + y^2 + y^2 = 5$

$2y^2 - 6y + 4 = 0$

Divide by 2:

$y^2 - 3y + 2 = 0$

Factorise:

$(y - 1)(y - 2) = 0$

$\Rightarrow y = 1$ or 2

Substitute for y in equation (2):

When $y = 1$, $x = 2$ and when $y = 2$, $x = 1$

Note that you should always give answers as a pair of values in x and y.

Example 26

Find the solutions of the pair of simultaneous equations: $y = x^2 + x - 2$ and $y = 2x - 4$

This example is slightly different, as both equations are given in terms of y,

$2x + 4 = x^2 + x - 2$

Rearranging into the general quadratic:

$x^2 - x - 6 = 0$

Factorising and solving gives:

$(x + 2)(x - 3) = 0$

$x = -2$ or 3

Substituting back to find y:

When $x = -2$, $y = 0$

When $x = 3$, $y = 10$

So the solutions are (–2, 0) and (3, 10).

EXERCISE 13N

EXTENDED

1 Solve these pairs of linear simultaneous equations using the substitution method.

a $2x + y = 9$
$x - 2y = 7$

b $3x - 2y = 10$
$4x + y = 17$

c $x - 2y = 10$
$2x + 3y = 13$

2 Solve these pairs of simultaneous equations.

a $xy = 2$
$y = x + 1$

b $xy = -4$
$2y = x + 6$

3 Solve these pairs of simultaneous equations.

a $x^2 + y^2 = 25$
$x + y = 7$

b $x^2 + y^2 = 9$
$y = x + 3$

c $x^2 + y^2 = 13$
$5y + x = 13$

4 Solve these pairs of simultaneous equations.

a $y = x^2 + 2x - 3$
$y = 2x + 1$

b $y = x^2 - 2x - 5$
$y = x - 1$

c $y = x^2 - 2x$
$y = 2x - 3$

5 Solve these pairs of simultaneous equations.

a $y = x^2 + 3x - 3$ and $y = x$

b $x^2 + y^2 = 13$ and $x + y = 1$

c $x^2 + y^2 = 5$ and $y = x + 1$

d $y = x^2 - 3x + 1$ and $y = 2x - 5$

e $y = x^2 - 3$ and $y = x + 3$

f $y = x^2 - 3x - 2$ and $y = 2x - 6$

g $x^2 + y^2 = 41$ and $y = x + 1$

6 Ravi's phone number has 6 digits: XY1290

He notices that the sum of the first 2 digits is 12 and the sum of the squares of the first two digits is 90.

a Write down two equations using X and Y.

b What is Ravi's phone number?

7 Samara says:

I am 4 years older than my brother. The difference between the squares of our ages is 80.

How old is Samara?

8 Salman is thinking of two numbers.

The sum of the squares of the numbers is 85. The square of the sum of the numbers is 121

a If the numbers are x and y, write down two equations connecting x and y.

b Work out the two numbers.

13.9 Representing inequalities

There are four inequality signs:

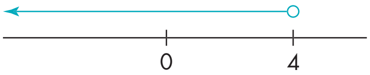
$<$ means ‘less than’

$>$ means ‘greater than’

$\leqslant$ means ‘less than or equal to’

$\geqslant$ means ‘greater than or equal to’

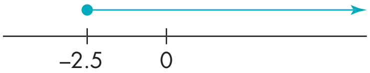
$x < 4$ means ‘x is less than 4’. You can show the possible values of x on a number line.



The open circle shows that 4 is not included.

x could be 1.5 or 0 or –3.1 or –75 but it could not be 4.

$x \geqslant -2.5$ means that x is greater than or equal to –2.5.



The closed circle shows that –2.5 is included.

x could be –2.5 or 0 or 4 but it could not be –3 or –10

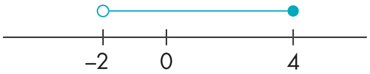
You can write inequalities the other way around, but you must change the sign.

$-3 < 1$ is the same as $1 > -3$

$x < 4$ is the same as $4 > x$

$x \geqslant -2.5$ is the same as $-2.5 \leqslant x$

Look at this diagram.



It shows numbers that are greater than –2 and less than or equal to 4

You can write this inequality: $-2 < x \leqslant 4$

It shows that x is between –2 and 4; –2 is not included; 4 is included.

Exercise 13O

CORE

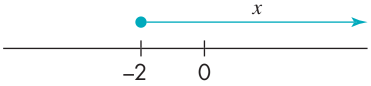
1 Show each inequality on a number line.

a $x \geqslant 3$ **b** $x < 5.2$ **c** $x \leqslant -10$

d $x > -4.5$ **e** $x \leqslant -4.5$ **f** $x < -30$

CORE

2 Write inequalities for these diagrams.

a


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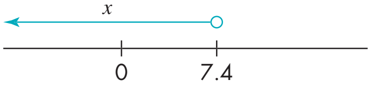
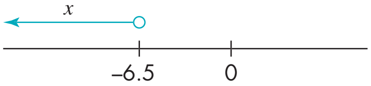
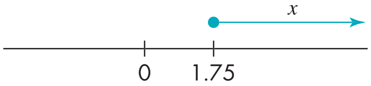


c


d


3 x is a positive integer and $x < 6$

Write all the possible values of x.

4 x is an integer and $x^2 \leqslant 10$

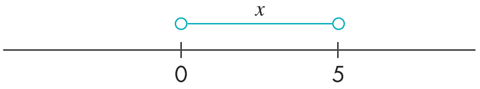
Write all the possible values of x.

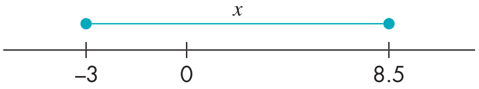
5 Show each inequality on a number line.

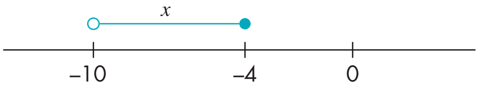
a $2 \leqslant x \leqslant 5$ **b** $-6 < x < -3$ **c** $-4 \leqslant x \leqslant 4$

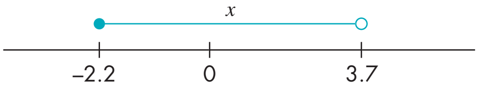
d $0 < x \leqslant 7$ **e** $-1.5 \leqslant x < 4.4$ **f** $-100 < x \leqslant -64$

6 Write inequalities for these diagrams.

a


b


c


d


7 x is an integer and $-3.5 \leqslant x \leqslant 2.5$

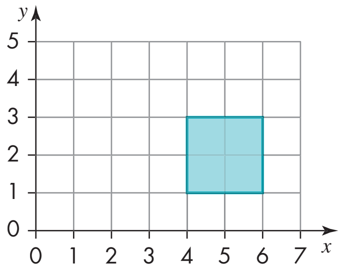
Write all the possible values of x.

8 **a** For points inside the square, $4 < x < 6$

Write an inequality for the y coordinates.

b For the points inside a rectangle, $1 < x < 4$ and $2 < y < 6$

Draw the rectangle on a coordinate grid.



9 A ticket costs \$$x$.

The cost is more than \$10 but not more than \$15.

a Write an inequality for x.

b Show your inequality on a number line.

10 A child is at least 3 years old but less than 8 years old.

a If the age of the child is x years, write an inequality for x.

b Show your inequality on a number line.

13.10 Solving inequalities

E

Inequalities behave similarly to equations. You use the same rules to solve linear inequalities as you use for linear equations.

There are four inequality signs:

$<$ means 'less than'

$>$ means 'greater than'

$\leqslant$ means 'less than or equal to'

$\geqslant$ means 'greater than or equal to'.

Be careful. Never replace the inequality sign with an equals sign.

Example 27

Solve $2x + 3 < 14$.

Rewrite this as: $2x < 14 - 3$

$2x < 11$

Divide both sides by 2: $\frac{2x}{2} < \frac{11}{2}$

$\Rightarrow x < 5.5$

This means that x can take any value below 5.5 but *not* the value 5.5.

You can use a number line to show the solution to the last example.

The open circle shows that 5.5 is not included in the solution.

If you multiply or divide by a negative number when you are solving an inequality you must change the sign.

'less than' becomes 'more than'

'more than' becomes 'less than'

Example 28

a Solve the inequality $10 - 2x \leqslant 3$

b Show the solution on a number line.

a Subtract 10 from both sides: $-2x \leqslant -7$

Divide both sides by -2: $x \geqslant 3.5$

Note that you must reverse inequality signs when multiplying or dividing both sides by a negative number. So the inequality has changed in the last line.

You could solve example 25 in a different way:

$$10 - 2x \leqslant 3$$

Add $2x$ to both sides: $10 \leqslant 2x + 3$

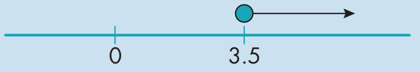
Subtract 3 from both sides: $7 \leqslant 2x$

Divide both sides by 2: $3.5 \leqslant x$

The sign does not change this time.

$3.5 \leqslant x$ is equivalent to $x \geqslant 3.5$

b



Here is a double inequality: $-5 \leqslant 2x + 4 \leqslant 20$

It means that the value of $2x + 4$ is between -5 and 20 inclusive.

Solve the inequality by doing the same thing to all three terms.

Subtract 4: $-9 \leqslant 2x \leqslant 16$

Divide by 2: $-4.5 \leqslant x \leqslant 8$

Here is the solution on a number line:

Check:

If $x = -4.5$ then $2x + 4 = -9 + 4 = -5$, the smallest value.

If $x = 8$ then $2x + 4 = 16 + 4 = 20$, the largest value.

EXERCISE 13P

EXTENDED

1 Solve these linear inequalities. Show each solution on a number line.

a $x + 4 < 7$ **b** $t + 5 > 3$ **c** $p + 12 \geqslant 2$

d $2x - 3 < 7$ **e** $4y + 5 \leqslant 17$ **f** $3t + 4 > 13$

g $\frac{x}{2} + 4 < 7$ **h** $\frac{y}{5} + 6 \leqslant 3$ **i** $\frac{t}{3} - 2 \geqslant 4$

j $3(x - 2) < 15$ **k** $5(2x + 1) \leqslant 35$ **l** $2(4t - 3) \geqslant 36$

2 Write down the largest integer value of x that satisfies each inequality.

a $x - 3 \leqslant 5$, where x is positive

b $x + 2 < 9$, where x is positive and even

c $3x - 11 < 40$, where x is a square number

d $5x - 8 \leqslant 15$, where x is positive and odd

e $2x + 1 < 19$, where x is positive and prime

3 Write down the smallest integer value of x that satisfies each inequality.

a $x - 2 \geqslant 9$, where x is positive

b $x - 2 > 13$, where x is positive and even

c $2x - 11 \geqslant 19$, where x is a square number

4 Ahmed went to town with \$20 to buy two CDs. His bus fare was \$3. The CDs were both the same price. When he reached home he still had some money in his pocket. What was the most each CD could cost?

Advice and Tips

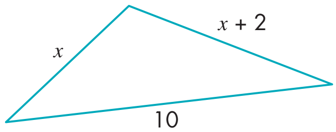
Set up an inequality and solve it.

5 **a** Explain why you cannot make a triangle with three sticks of length 3 cm, 4 cm and 8 cm.

b Three sides of a triangle are x, $x + 2$ and 10 cm.

x is a whole number.

What is the smallest value x can take?



6 Five cards have inequalities and equations marked on them.

| $x > 0$ | $x < 3$ | $x \geqslant 4$ | $x = 2$ | $x = 6$ |

The cards are shuffled, laid face down and then turned over, one at a time.

If the possible values on two consecutive cards have any numbers in common, then a point is scored.

If they do not have any numbers in common, then a point is deducted.

a The first two cards below score –1 because $x = 6$ and $x < 3$ have no numbers in common.

Explain why the total for this combination scores 0.

| $x = 6$ | $x < 3$ | $x > 0$ | $x = 2$ | $x \geqslant 4$ |

b What does this combination score?

| $x > 0$ | $x = 6$ | $x \geqslant 4$ | $x = 2$ | $x < 3$ |

c Arrange the cards to give a maximum score of 4.

7 Solve these linear inequalities.

a $4x + 1 \geqslant 3x - 5$ **b** $5t - 3 \leqslant 2t + 5$ **c** $3y - 12 \leqslant y - 4$

d $2x + 3 \geqslant x + 1$ **e** $5w - 7 \leqslant 3w + 4$ **f** $2(4x - 1) \leqslant 3(x + 4)$

EXTENDED

8 Solve these linear inequalities.

a $\frac{x+4}{2} \leqslant 3$ b $\frac{x-3}{5} > 7$ c $\frac{2x+5}{3} < 6$

d $\frac{4x-3}{5} \geqslant 5$ e $\frac{2t-2}{7} > 4$ f $\frac{5y+3}{5} \leqslant 2$

9 Solve these inequalities.

a $2 < 3x - 7 < 20$ b $-8 \leqslant 4x + 2 \leqslant 8$ c $-6 < 2(x - 5) \leqslant 0$

d $-4 \leq \frac{x+10}{5} < 2$ e $5 < 4(x + 2) < 10$ f $-6 \leqslant \frac{x}{3} - 1 \leqslant 5$

10 $0 \leqslant x \leqslant 10$

a Work out an inequality for the value of $5x - 3$

b Work out an inequality for the value of $5(x - 3)$

11 $10 < 2x + 3 < 15$

a Show the possible values of x on a number line.

b If x is an integer, write the possible values of x.

12 In this question n is always an integer.

a Find the largest possible value of n if $2n + 3 < 12$.

b Find the largest possible value of n if $\frac{n}{5} < 20$.

c Find the smallest possible value of n if $3(n - 7) \geqslant 10$.

d Find the smallest possible value of n if $\frac{6n-2}{7} \geqslant 9$.

e Find the smallest possible value of n if $3n + 14 \leqslant 8n - 13$.

13 a If $20 - x > 4$, which of these numbers are possible values of x?

–10 0 10 20 30

b Solve the inequality $20 - x > 4$.

14 Solve these inequalities.

a $15 - x > 6$

b $18 - x \leqslant 7$

c $6 \geqslant 9 - x$

15 Solve these inequalities.

a $20 - 2x \leqslant 5$ b $3 - 4x \geqslant 11$

c $25 - 3x > 7$ d $2(6 - x) < 9$

e $\frac{10-2x}{5} \leqslant 4$ f $\frac{8-4x}{3} > 2$

Check your progress

Core

- I can construct simple equations
- I can solve linear equations in one unknown
- I can solve simultaneous linear equations in two unknowns

Extended

- I can solve quadratic equations by factorisation, by completing the square or by use of the formula
- I can solve simultaneous equations where one equation is linear and the other is quadratic
- I can construct and solve linear inequalities
- I can solve fractional equations