Two-way range and range-rate observables in a sequential filter

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November 19, 2019

1 Introduction

Imagine that an earth-based satellite dish transmits a signal to a spacecraft over some short interval dt_1 at t_1, that the spacecraft receives that signal over some interval dt_2 at t_2, and immediately transmits it back to the same ground station, which receives it over duration dt_3 at t_3. The two observables in which we are interested are

1. the round-trip time-of-flight, which gives us an approximate range to the spacecraft; and

2. the ratio of signal transmission and reception intervals[1] (the Doppler shift) between transmission and eventual reception, which provides information about the rate at which the range is changing (the range-rate).

While Moyer [1971] has derived models for one-way, two-way, and three-way observables, we are interested only in the two-way solutions, which avoids many of the clock problems which befall our measurements in one-way and three-way methods.

[1] This can also be written as the ratio of frequencies at reception and transmission.

2 Range-rate observable

The most basic form of the range-rate observable is

$$F = \frac{N}{T_c} - f_{\text{bias}} \tag{1}$$

where f_{bias} is $C_4 = 10^6$ (for S-band), N is the number of cycles, and T_c is the time over which those cycles were received. Since

$$N = \int_{T_c} f \, dt \,, \tag{2}$$

where f is the frequency, we can write

$$F = \frac{1}{T_c} \int_{t_3 - T_c/2}^{t_3 + T_c/2} (f - f_{\text{bias}}) dt_3 \,. \tag{3}$$

Moyer's Equation 285 gives an expression for the value in the integral:

$$f - f_{\text{bias}} = C_3 f_q \left(1 - \frac{f_R}{f_T}\right) \tag{4}$$

where f_R is the frequency received, f_T is the transmitted frequency, f_q is the clock frequency (which we treat as being the same at t_1 and t_3), and $C_3 = 96(240/221)$.

According to Moyer, the integral gives a Taylor series:

$$F = C_3 f_q \left(1 - \frac{f_R}{f_T}\right)^* \tag{5}$$

$$\left(1 - \frac{f_R}{f_T}\right)^* = \left(1 - \frac{f_R}{f_T}\right) + \left(\frac{T_c^2}{24}\right) \frac{d^2}{dt_3^2} \left[1 - \frac{f_R}{f_T}\right] \tag{6}$$

The full expansion is quite complicated. Luckily, it can also be expressed more simply as a difference in times-of-flight. The full derivation is not included here, but the result is Equation 480 in Moyer,

$$F = C_3 f_q \frac{\tau_{2_e} - \tau_{2_s}}{T_c} \,, \tag{7}$$

where τ_{2_e} is the round-trip time for the end of the signal, and τ_{2_s} is the same for the start of the signal.[2]

[2] Moyer uses ρ instead of τ, but I find this confusing, since ρ usually indicates a range.

Typically, T_c is the signal duration at receipt (which for our purposes is just over a second). We can write

$$\Delta\tau_2 = \tau_{2_e} - \tau_{2_s} \tag{8}$$

and each time-of-flight is defined in terms of the ranges traversed by the signal,

$$\tau_2 = \frac{r_{12} + r_{23}}{c} \tag{9}$$

where we define

$$r_{12} = \|\mathrm{r}_2 - \mathrm{r}_1\| \tag{10}$$

$$r_{23} = \|\mathrm{r}_3 - \mathrm{r}_2\|\,. \tag{11}$$

If we think of the quantity $\frac{\Delta\tau_2}{T_c}$ as twice the change in position over the receive time interval, divided by c, we can see that it resembles a velocity. We rewrite our observable in terms of the range-rate:

$$F = C_3 f_q \frac{d\tau_2}{dt_3} \tag{12}$$

$$\frac{d\tau_2}{dt_3} = \frac{d}{dt_3}\frac{\|\mathrm{r}_2 - \mathrm{r}_1\| + \|\mathrm{r}_3 - \mathrm{r}_2\|}{c} = \frac{\frac{d}{dt_3}\left[\left((\mathrm{r}_2 - \mathrm{r}_1)^\top (\mathrm{r}_2 - \mathrm{r}_1)\right)^{1/2} + \left((\mathrm{r}_3 - \mathrm{r}_2)^\top (\mathrm{r}_3 - \mathrm{r}_2)\right)^{1/2}\right]}{c}\,. \tag{13}$$

To compute the derivative in the expression above, we need to refresh a few identities.[3] Firstly, suppose that vectors a and b are both functions of t. Then

$$\frac{d}{dt}\|\mathrm{a} - \mathrm{b}\| = \frac{d}{dt}\left((\mathrm{a} - \mathrm{b})^\top (\mathrm{a} - \mathrm{b})\right)^{1/2} = \frac{1}{2}\left((\mathrm{a} - \mathrm{b})^\top (\mathrm{a} - \mathrm{b})\right)^{-1/2}\left(2\,(\mathrm{a} - \mathrm{b})^\top (\dot{\mathrm{a}} - \dot{\mathrm{b}})\right) = \frac{(\mathrm{a} - \mathrm{b})^\top (\dot{\mathrm{a}} - \dot{\mathrm{b}})}{\|\mathrm{a} - \mathrm{b}\|}\,, \tag{14}$$

and we call this expression $\mathrm{G}(\mathrm{a}, \mathrm{b})$.

[3] If these don't make sense, a good reference is `https://en.wikipedia.org/wiki/Matrix_calculus#Identities`.

Next, we need to find the differentials of G with respect to each of a, b, $\dot{\mathrm{a}}$, and $\dot{\mathrm{b}}$.

$$\begin{aligned}\frac{\partial G}{\partial \mathrm{a}} &= -\frac{1}{2}\left((\mathrm{a}-\mathrm{b})^\top(\mathrm{a}-\mathrm{b})\right)^{-3/2}(\mathrm{a}-\mathrm{b})^\top(\dot{\mathrm{a}}-\dot{\mathrm{b}})\left(2(\mathrm{a}-\mathrm{b})^\top\right) \\ &\quad + \left((\mathrm{a}-\mathrm{b})^\top(\mathrm{a}-\mathrm{b})\right)^{-1/2}(\dot{\mathrm{a}}-\dot{\mathrm{b}})^\top \\ &= -\frac{(\mathrm{a}-\mathrm{b})^\top(\dot{\mathrm{a}}-\dot{\mathrm{b}})}{\|\mathrm{a}-\mathrm{b}\|^3}(\mathrm{a}-\mathrm{b})^\top + \frac{1}{\|\mathrm{a}-\mathrm{b}\|}(\dot{\mathrm{a}}-\dot{\mathrm{b}})^\top \qquad (15)\end{aligned}$$

and

$$\frac{\partial G}{\partial \mathrm{b}} = \frac{(\mathrm{a}-\mathrm{b})^\top(\dot{\mathrm{a}}-\dot{\mathrm{b}})}{\|\mathrm{a}-\mathrm{b}\|^3}(\mathrm{a}-\mathrm{b})^\top - \frac{1}{\|\mathrm{a}-\mathrm{b}\|}(\dot{\mathrm{a}}-\dot{\mathrm{b}})^\top \qquad (16)$$

$$\frac{\partial G}{\partial \dot{\mathrm{a}}} = \frac{1}{\|\mathrm{a}-\mathrm{b}\|}(\mathrm{a}-\mathrm{b})^\top \qquad (17)$$

$$\frac{\partial G}{\partial \dot{\mathrm{b}}} = -\frac{1}{\|\mathrm{a}-\mathrm{b}\|}(\mathrm{a}-\mathrm{b})^\top \qquad (18)$$

We need only define the Kalman filter state:

$$\mathrm{x} = \begin{bmatrix}\mathrm{r}_2 & \mathrm{v}_2\end{bmatrix}^\top, \qquad (19)$$

where $\mathrm{v} = \dot{\mathrm{r}}_2$.

2.1 Range-rate measurement partial

We now have all the tools we need to compute the measurement partial

$$\begin{aligned}H &= \frac{\partial F}{\partial \mathrm{x}} \\ &= \begin{bmatrix}\frac{\partial F}{\partial \mathrm{r}_2} & \frac{\partial F}{\partial \mathrm{v}_2}\end{bmatrix}\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial F}{\partial \mathrm{r}_2} &= \frac{C_3 f_q}{c}\left(\frac{\partial G(\mathrm{r}_2,\mathrm{r}_1)}{\partial \mathrm{r}_2} + \frac{\partial G(\mathrm{r}_3,\mathrm{r}_2)}{\partial \mathrm{r}_2}\right) \\
&= \frac{C_3 f_q}{c}\Bigg(-\frac{(\mathrm{r}_2-\mathrm{r}_1)^\top(\mathrm{v}_2-\mathrm{v}_1)}{r_{12}^3}(\mathrm{r}_2-\mathrm{r}_1)^\top + \frac{1}{r_{12}}(\mathrm{v}_2-\mathrm{v}_1)^\top \\
&\qquad + \frac{(\mathrm{r}_3-\mathrm{r}_2)^\top(\mathrm{v}_3-\mathrm{v}_2)}{r_{23}^3}(\mathrm{r}_3-\mathrm{r}_2)^\top - \frac{1}{r_{23}}(\mathrm{v}_3-\mathrm{v}_2)^\top\Bigg)
\end{aligned} \tag{20}$$

$$\begin{aligned}
\frac{\partial F}{\partial \mathrm{v}_2} &= \frac{C_3 f_q}{c}\left(\frac{\partial G(\mathrm{r}_2,\mathrm{r}_1)}{\partial \mathrm{v}_2} + \frac{\partial G(\mathrm{r}_3,\mathrm{r}_2)}{\partial \mathrm{v}_2}\right) \\
&= \frac{C_3 f_q}{c}\left(\frac{1}{r_{12}}(\mathrm{r}_2-\mathrm{r}_1)^\top - \frac{1}{r_{23}}(\mathrm{r}_3-\mathrm{r}_2)^\top\right)
\end{aligned} \tag{21}$$

2.2 Measurement covariance

The measurement covariance for the Doppler observable ought to be constant regardless of range. A single measurement ought to have a $\sigma_{\dot{\rho}} = 1$ mm/s. However, the measurements are expressed as a frequency, so we need to convert:

$$R_{\mathrm{doppler}} = \left(\frac{C_e f_q}{c}\sigma_{\dot{\rho}}\right)^2 . \tag{22}$$

3 Range observable

The observable for two-way range is approximately Eq. 9.[4] Preliminary analysis (not shown) suggests two-way range information does not improve the state covariance in the context of Doppler measurements, so we don't perform a detailed derivation. The measurement partial is

$$H = \frac{1}{c}\left(\frac{(\mathrm{r}_2-\mathrm{r}_1)^\top}{r_{12}} - \frac{(\mathrm{r}_3-\mathrm{r}_2)^\top}{r_{23}}\right) , \tag{23}$$

which is basically identical to the range-rate observable with respect to the changing velocity.

[4]The full expression is Equation 379 by Moyer.

Since the observable is a round-trip time, we must divide our expected $\sigma_\rho = 2$ m by the speed of light to get our measurement covariance:

$$R_{\text{range}} = \left(\frac{\sigma_\rho}{c}\right)^2 . \tag{24}$$

References

Theodore D. Moyer. Mathematical formulation of the Double-Precision Orbit Determination Program (DPODP). Technical report, Jet Propulsion Laboratory, Pasadena, California, U.S., 1971.