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An introductory treatise on reduced balance laws of Cosserat beams

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This paper serves as an introduction to the variational formulation of Cosserat beams. It provides a detailed derivation and treatment of reduced balance laws of Cosserat beams from the Lagrangian differential equation of motion and Hamilton's principle. Emphasis is given to the details of the derivation, maintaining Bernoulli's assumption of the rigid cross-section. Both the strong form and the weak form of the equilibrium equation for Cosserat beams are derived independently from the infinitesimal stress equilibrium equation. The weak form is then validated by obtaining it from the strong form of the reduced law in a purely mathematical sense. Finally, the strong form is obtained using Hamilton's principle. Once the equations are obtained considering an initially straight reference beam configuration, the balance equation for the beam with initial curved (but unstrained) reference configuration is obtained. The D'Alembert forces are interpreted from the non-inertial director frame of reference and conclusions drawn. The energy conservation law and the conditions associated with it are obtained, establishing the relation between the Lagrangian and Hamiltonian functional for Cosserat beams.

1. Introduction

The mechanics of Cosserat continuum has been a topic of interest since its discovery by Cosserat and Cosserat (1909). Cosserat beam theory is a single manifold problem. The position vector of the midcurve and the directors are the physical parameters that are used to define the state of the beam. This description of the rod falls under the idea of Duhem (1893), where any point in the body is not only described by the position vector, but also by an attached set of vector triad called directors. Cosserat and Cosserat (1909) harnessed this idea to develop the finite strain theory of rods and shells assuming a fixed rectangular Cartesian system. The work by Ericksen and Truesdell (1958) was a mathematical generalization of the work of Cosserat brothers to develop a nonlinear theory of rods and shells. They first considered general differential geometry tools that deal with the transformation from one space to other and then used them to obtain a general description of the undeformed and deformed configuration of the rods. They limited the space to a three-dimensional Euclidean space, thereby developing a complete differential description of the finite strain of the rod. Their work (Ericksen and Truesdell, 1958) also serves as a concise introduction to the history of theory of beams and rods. The work on the finite displacement theory of the rods attributed to Kirchhoff was improvised by Hay (1942), which is in fact a special case of the formulation in Ericksen and Truesdell (1958) obtained by choosing special coordinates. Cohen (1966) developed a comprehensive nonlinear theory of elastic curves for the static case. This work was extended to the dynamic case by Whitman and Desilva (1969). Reissner (1972, 1973, 1981) developed the static finite strain beam theory for the plane case by incorporating the shear deformation using a classical approach. He arrived at non-linear strain displacement relations consistent with equilibrium equations for the static case. Simo (1985) extended the work of Reissner for three-dimensional dynamic case using a director type of approach. Further work by Simo and Vu-Quoc (1991) incorporated the effect of Warping for initially straight beam, maintaining the single manifold nature of the problem. Simo (1985) discussed the balance law considering the uniform straight initial beam configuration, and Iura and Atluri (1989) obtained the governing equations for the initially curved beam configuration using the principle of virtual work. The work by Green and Naghdi is among the first exposition to the theory of elastic rods, developing the mechanics using a classical three-dimensional equations (Green et al., 1974a) and also using Cosserat curves (Green et al., 1974b). Naghdi and Rubin (1982) presented various constraint theories of rods where various classes of deformations were restraints. A relatively recent publication by Brand and Rubin (2007a) dealt with one of such constraint theories of a Cosserat point for numerical solutions of non-linear elastic rods. Significant research on the finite element formulation of the Cosserat beam element is done by Cao et al. (2006),

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Liu et al. (2007), Brand and Rubin (2007b) and Rubin (1985a, 1985b, 2000, 2002). The ability of Cosserat beam theory to capture all kinds of deformations including torsion and shear has been exploited by Todd et al. (2013) and Chadha and Todd (2017) to develop a theory of global shape reconstruction using finite surface strain measurements.

Interested readers are also recommended to refer to the detailed work and references therein by Love (1944), Antman (1972, 1995), Svetlitsky (2000, 2004), Maugin (2017) and Vetyukov (2013). The Cosserat rod is a special case of problems that fall in the domain of micropolar continua, which in turn is a special restraint case of micromorphic continua. An excellent compilation of explanation on micropolar continua (by Altenbach and Eremeyev), micromorphic continua (by Samuel Forest), Electromagnetism and generalized Continua (by Maugin) is found in Altenbach and Eremeyev (2013).

It is evident that the problem of Cosserat rods has been well treated in the past. While the aforementioned references are seminal contributions to this field of mechanics, they do not tend to elaborate the details of the inherent physical relationships or the connections particularly appreciative to the engineers. We focus on illustrating various ideas by means of rigorous mathematical derivations and illustrative schematic diagrams wherever possible, attempting to deliver the matter in a simplified yet complete manner. The detail with which the derivations are performed and the results explained in the defined domain of the discussion sets this work apart from the references mentioned above. We believe that the primary novelty of our work is that the mathematical details and interpretations it encompasses would help the reader get acquainted with a method of rational reasoning of the description of finite strains and the governing differential equation of three dimensional, geometrically exact Cosserat beams.

Unlike the general work of Ericksen and Truesdell (1958), we limit our discussion to the classical Cosserat beam formulation with orthonormal director triad and fixed Cartesian reference system. We outline the tensor algebra and variational principles required to derive the strong and weak form. We discuss about a method to uniquely define the shear angles and obtain the curvature terms as a function of pitch and yaw angles. A careful interpretation of the finite strain vector obtained as a result of superimposition of strain due to curvature, elongation, shear, and torsion is presented. We also present a detailed discussion on the variation of the director triad and parametrization of the orthogonal rotation tensor using Rodrigues method with an explanatory example. The pioneering work of Ibrahimbegovic et al. (1995) on vector like parametrization of three-dimensional finite rotations details the kind of parametrization described in this paper. Another approach on parametrization of rotation tensor is the quaternion method, which is explained in the work of Argyris (1982). We carefully develop the deformation gradient tensor of the beam assuming the undeformed state of the rod to be naturally curved. We culminate the section on the deformation gradient tensor by presenting a clear exposition of the finite strain vector of the rod referenced to the curved reference configuration (Section 3.1.3).

Since the balance laws in both weak and strong forms are at the heart of finite element analysis, we firmly believe that it is beneficial to obtain these equations in more elucidated and detailed fashion, using both an infinitesimal equilibrium equation and the Hamilton–Lagrange principle. The results obtained here will be directly used to generalize the theory of shape reconstruction developed by the authors and to investigate the conservation laws of Cosserat beams. In this paper, we do not specifically assume that the midcurve passes either through the geometric centroid or the mass centroid of the beam but rather leave its location general. We obtain the equations for the initially straight configuration and finally achieve the same for an initially curved (but strain-free) reference configuration. To clearly demonstrate the importance of the terms involved in the equation of motion, we interpret the motion as viewed from the director frame of reference. We also obtain the energy conservation law from Hamilton's principle, thereby establishing a transformational link between the total energy and Lagrangian functional for Cosserat beams. This sets a foundation for our further work on conservation laws of Cosserat rods as a problem of symmetries in the Noether Theorem sense.

The remainder of the paper is arranged as follows: Section 2 details the geometric formulation, defines the deformation parameters (Section 2.1), and finally outlines the required mathematical tools (Section 2.2). Section 3 derives the deformation gradient tensor (Section 3.1) and the variation of deformation parameters (Section 3.3), defines the stresses, and presents the reduced force and moment in the classical sense (Section 3.4). Section 4 presents the derivation of the Strong form of the reduced balance law. Section 5 deals with the derivation and interpretation of the weak form of equation from the infinitesimal Lagrangian equation (Section 5.1) and validating the weak form by obtaining it from the strong form (Section 5.2). Section 6 comprehends the derivation of strong form from the Hamilton's equation of motion. Section 7 presents a linear constitutive law relating the reduced forces with the reduced strain parameters. Section 8 deals with the energy conservation for the Cosserat beam. Finally, Section 9 draws some conclusions and describes the scope of future research in the field.

2. Kinematic model and mathematical tools

2.1. Geometry and deformation parameters

The beam configuration is defined by a midcurve and the family of cross-sections. The beam can possess different cross-sections varying smoothly. The cross-sections are assumed to be rigid, and as such, the *Poisson* and *warping effects* are ignored (refer Appendix A.5 for more details on this assumption). The initial shape of the structure may be curved or straight and is assumed to be unstrained. We begin by assuming that the initially curved reference beam Ω^c deforms to some current configuration Ω. Consider a fixed orthogonal Cartesian triad $\{\boldsymbol{E_i}\}$. Any configuration of the structure is described by the locus of the geometric centroids of the family of cross-sections called the mid-curve, defined by the position vector $\boldsymbol{\varphi}(\xi_1)$ parametrized by the undeformed arc-length $\xi_1 \in [0, L_0]$, where L_0 is the total length of the mid-curve in the undeformed configuration, or

$$\boldsymbol{\varphi}(\xi_1) = \varphi_i \boldsymbol{E}_i. \quad (1)$$

The parameter $\blacksquare(\xi_1)$ represents the cross-section of the beam at an arc length ξ_1 and is independent of deformation because cross-sections are assumed rigid. The orientation of any cross-section in the deformed configuration is quantified by the set of orthogonal Cosserat triad called directors $\{\boldsymbol{d}_i(\xi_1)\}$ such that

$$\boldsymbol{d}_i = d_{ij} \boldsymbol{E}_j. \quad (2)$$

Any point on the beam is defined by the material coordinates (ξ_1, ξ_2, ξ_3) that are independent of the configuration of the beam. The position vector $\boldsymbol{\varphi}(\xi_1)$ is sufficient to define the mid-curve but not the orientation of the cross-section that is affected by shear and torsion. The directors take care of this. The director $\boldsymbol{d}_1(\xi_1)$ is perpendicular to the cross-section and the directors $\boldsymbol{d}_2(\xi_1)$ and $\boldsymbol{d}_3(\xi_1)$ span the cross-section $\blacksquare(\xi_1)$. Any point P on the cross-section is defined with respect to the point G on the midcurve at $\blacksquare(\xi_1)$ by the position vector $\boldsymbol{r}_{PG} = \xi_2 \boldsymbol{d}_2(\xi_1) + \xi_3 \boldsymbol{d}_3(\xi_1)$, as shown

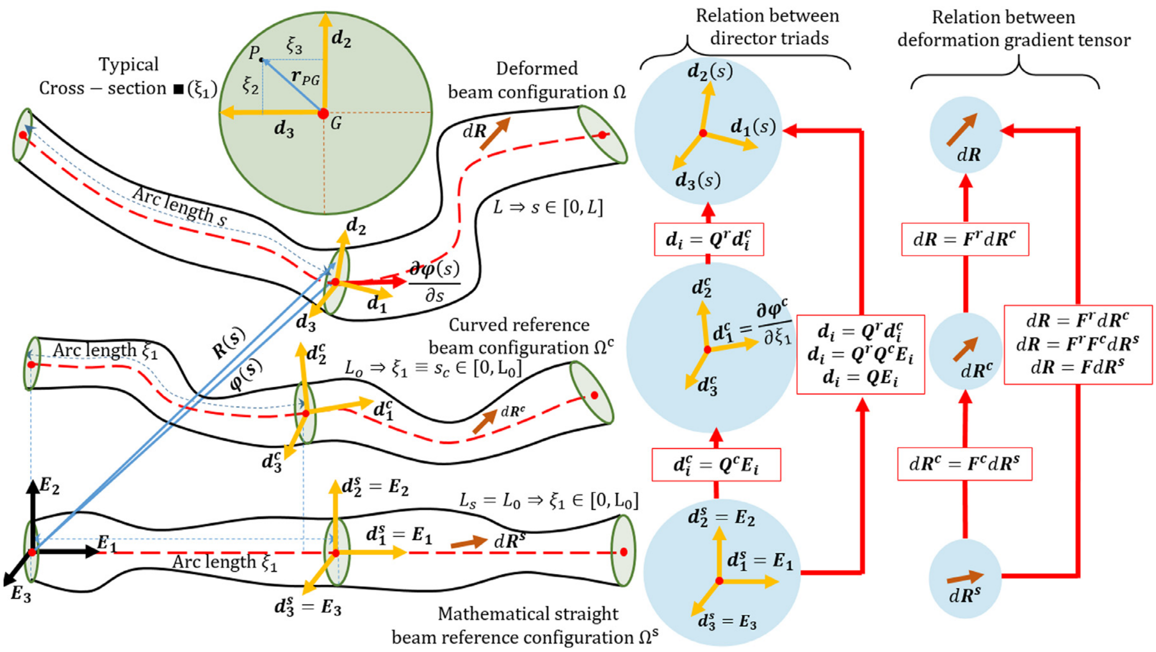

Fig. 1. Deformed and undeformed configurations of Cosserat rod, material adapted frames and deformation gradient tensors.

in Fig. 1. Therefore, any point P in the structure is given by the position vector

$$\boldsymbol{R}(\xi_1,\xi_2,\xi_3)=\boldsymbol{\varphi}(\xi_1)+\xi_2\boldsymbol{d}_2(\xi_1)+\xi_3\boldsymbol{d}_3(\xi_1)=\boldsymbol{\varphi}(\xi_1)+\boldsymbol{r}_{PG}(\xi_2,\xi_3). \tag{3}$$

The initially curved reference beam configuration is defined by $\boldsymbol{d}_i^c(\xi_1)=d_{ij}^c(\xi_1)\boldsymbol{E}_j$, $\boldsymbol{\varphi}^c(\xi_1)=\varphi_i^c(\xi_1)\boldsymbol{E}_i$ and any point on the cross-section is given by the vector $\boldsymbol{R}^c(\xi_1,\xi_2,\xi_3)=\boldsymbol{\varphi}^c(\xi_1)+\xi_2\boldsymbol{d}_2^c(\xi_1)+\xi_3\boldsymbol{d}_3^c(\xi_1)$. It is convenient to mathematically define a straight beam configuration Ω^s such that the directors are defined by $\{\boldsymbol{E}_i\}$, the position vector of the midcurve is given by $\boldsymbol{\varphi}^s(\xi_1)=\xi_1\boldsymbol{E}_1$ and any point in the beam is defined by $\boldsymbol{R}^s(\xi_1,\xi_2,\xi_3)=\boldsymbol{\varphi}^s(\xi_1)+\xi_2\boldsymbol{E}_2(\xi_1)+\xi_3\boldsymbol{E}_3(\xi_1)$. The triads $\{\boldsymbol{E}_i\}$, $\{\boldsymbol{d}_i^c\}$ and $\{\boldsymbol{d}_i\}$ are related to each other by means of orthogonal directional cosine tensors as shown in Fig. 1, such that

$$\boldsymbol{d}_i=\boldsymbol{Q}\boldsymbol{E}_i;\;\;\boldsymbol{d}_i^c=\boldsymbol{Q}^c\boldsymbol{E}_i;\;\;\boldsymbol{d}_i=\boldsymbol{Q}^r\boldsymbol{d}_i^c. \tag{4}$$

Therefore, any general vector $\boldsymbol{g}(\xi_1)$ can be expressed in the fixed frame $\{\boldsymbol{E}_i\}$ or the local frame $\{\boldsymbol{d}_i\}$ such that $\boldsymbol{g}=g_i\boldsymbol{E}_i=\bar{g}_i\boldsymbol{d}_i$. It can be established from Eq. (4) that

$$\boldsymbol{Q}=\boldsymbol{Q}^r\boldsymbol{Q}^c, \tag{5}$$

$$\boldsymbol{Q}=\boldsymbol{d}_i\otimes\boldsymbol{E}_i;\;\;\boldsymbol{Q}^r=\boldsymbol{d}_i\otimes\boldsymbol{d}_i^c;\;\;\boldsymbol{Q}^c=\boldsymbol{d}_i^c\otimes\boldsymbol{E}_i. \tag{6}$$

In general, a Cosserat beam can capture the effect of elongation, shear, and multiple curvatures. Defining the deformed arc length as s, axial strain as $e(\xi_1)$ the three shear angles as $\gamma_{11}(\xi_1)$, $\frac{\pi}{2}-\gamma_{12}(\xi_1)$ and $\frac{\pi}{2}-\gamma_{13}(\xi_1)$ subtended by the directors $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ and $\boldsymbol{d}_3$ with the tangent vector $\frac{\partial\boldsymbol{\varphi}}{\partial s}$ as in Chadha and Todd (2017), the following relations may be established:

$$\frac{d\xi_1}{ds}=\frac{1}{1+e}; \tag{7}$$

$$\frac{\partial\boldsymbol{\varphi}}{\partial s}.\boldsymbol{d}_1=\cos\gamma_{11};\quad\frac{\partial\boldsymbol{\varphi}}{\partial s}.\boldsymbol{d}_2=\sin\gamma_{12};\quad\frac{\partial\boldsymbol{\varphi}}{\partial s}.\boldsymbol{d}_3=\sin\gamma_{13}. \tag{8}$$

Therefore,

$$\boldsymbol{\varphi}_{,\xi_1}=(1+e)\{\cos\gamma_{11}\boldsymbol{d}_1+\sin\gamma_{12}\boldsymbol{d}_2+\sin\gamma_{13}\boldsymbol{d}_3\} \tag{9}$$

The above equation is not enough to uniquely define the shear angles. Section 3.2 addresses a way to uniquely define them.

2.2. Mathematical tools

2.2.1. Derivative of the moving frame

The derivative of the director $\boldsymbol{d}_i$ with respect to a general parameter x is obtained using Eq. (4) as

$$\boldsymbol{d}_{i,x}=\boldsymbol{Q}_{,x}\boldsymbol{E}_i=\boldsymbol{Q}_{,x}\boldsymbol{Q}^T\boldsymbol{d}_i=\boldsymbol{q}_x\times\boldsymbol{d}_i. \tag{10}$$

It may be proven that $\boldsymbol{Q}_{,x}\boldsymbol{Q}^T$ is antisymmetric from the fact that $\boldsymbol{Q}$ is orthogonal. Therefore, there exists a corresponding axial vector $\boldsymbol{q}_x$ such that Eq. (10) holds. For a deforming beam, the director frame $\{\boldsymbol{d}_i(t,\xi_1)\}$ is function of time t and arc-length ξ_1. The axial vector corresponding to the time derivative $\boldsymbol{d}_{i,t}$ and the derivative with respect to arc length $\boldsymbol{d}_{i,\xi_1}$ is given by the angular velocity vector $\boldsymbol{\omega}=\omega_i\boldsymbol{E}_i=\overline{\omega}_i\boldsymbol{d}_i$ and the Darboux vector $\boldsymbol{\kappa}=\kappa_i\boldsymbol{E}_i=\overline{\kappa}_i\boldsymbol{d}_i$ respectively, as shown in Eqs. (11) and (12). The component of the Darboux vector gives the curvature about the corresponding director. The first component $\overline{\kappa}_1$ represents torsional deformation, whereas $\overline{\kappa}_2$ and $\overline{\kappa}_3$ represent bending curvature about $\boldsymbol{d}_2$ and $\boldsymbol{d}_3$, respectively.

$$\boldsymbol{d}_{i,t}=\boldsymbol{Q}_{,t}\boldsymbol{Q}^T\boldsymbol{d}_i=\boldsymbol{W}\boldsymbol{d}_i=\boldsymbol{\omega}\times\boldsymbol{d}_i, \tag{11}$$

$$\boldsymbol{d}_{i,\xi_1}=\boldsymbol{Q}_{,\xi_1}\boldsymbol{Q}^T\boldsymbol{d}_i=\boldsymbol{K}\boldsymbol{d}_i=\boldsymbol{\kappa}\times\boldsymbol{d}_i. \tag{12}$$

Consider the orthogonal rotation tensor, for example $\boldsymbol{Q}(\xi_1)$. It represents the family of orthogonal tensors that belong to the $SO(3)$ rotational group. Therefore, they satisfy $\boldsymbol{Q}(\xi_1)\boldsymbol{Q}^T(\xi_1)=\boldsymbol{I}_3$ and det $[\boldsymbol{Q}]=1$. The rotation tensor $\boldsymbol{Q}(\xi_1)$, being a curve in the manifold $SO(3)$, $\boldsymbol{Q}_{,\xi_1}$ represents the tangent vector to this curve in $SO(3)$. Therefore, $\boldsymbol{Q}_{,\xi_1}\boldsymbol{Q}^T=\boldsymbol{K}(\xi_1)$ is the linear space of skew symmetric matrices that has $\boldsymbol{\kappa}(\xi_1)$ as the corresponding axial vector.

2.2.2. Parametrization of the rotation tensor

Argyris (1982) describes various methods to describe large vector rotations. We choose Rodrigues formula to describe the rotation of director frame.

Consider a general vector $\boldsymbol{g^a}$ that is rotated to $\boldsymbol{g^b}$ by an orthogonal tensor $\mathfrak{R}$ such that, $\boldsymbol{g^b}=\mathfrak{R}\boldsymbol{g^a}$. The orthogonal tensor has 3 independent entries because of the restriction $\mathfrak{R}^T\mathfrak{R}=\boldsymbol{I}_3$. Therefore, $\mathfrak{R}$ can be parametrized by three parameters. The rotation

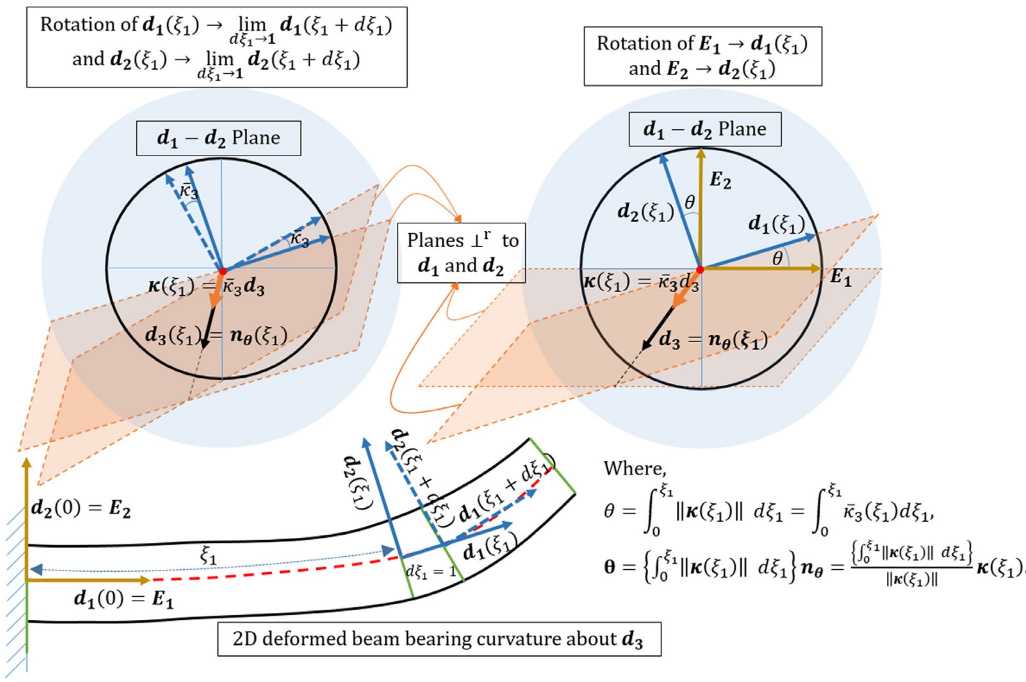

Fig. 2. Geometric interpretation of solution to Eq. (15) for a 2D plane beam with curvature about director $\boldsymbol{d}_3$.

described by $\mathfrak{R}$ can be thought of as the rotation of the vector $\boldsymbol{g^a}$ about the unit vector $\boldsymbol{n}_\theta$ by an angle θ. Therefore the vector $\boldsymbol{\theta} = \theta \boldsymbol{n}_\theta$ completely describes the rotation. If $\boldsymbol{\Theta}$ represent anti-symmetric tensor for the axial vector $\boldsymbol{\theta}$, then by Rodrigues formula

$$\begin{aligned} \boldsymbol{g^b} &= [\boldsymbol{g^a} + \boldsymbol{n}_\theta \times \boldsymbol{n}_\theta \times \boldsymbol{g^a}] + [\boldsymbol{n}_\theta \times \boldsymbol{g^a}] \; \sin\theta \\ &\quad - [\boldsymbol{n}_\theta \times \boldsymbol{n}_\theta \times \boldsymbol{g^a}] \; \cos\theta = \mathfrak{R}(\boldsymbol{\theta})\boldsymbol{g^a}, \end{aligned} \tag{13}$$

where

$$\mathfrak{R}(\boldsymbol{\theta}) = \boldsymbol{I}_3 + \frac{\sin\theta}{\theta} \; \boldsymbol{\Theta} + \frac{(1-\cos\theta)}{\theta^2} \boldsymbol{\Theta}^2 = e^{\boldsymbol{\Theta}}. \tag{14}$$

From Eq. (4), the orthogonal tensor $\boldsymbol{Q}(\xi_1)$ can be parametrized by the rotation vector $\boldsymbol{\theta}(\xi_1)$ such that the vector triad $\{\boldsymbol{E}_i\}$ is rotated to the director triad $\{\boldsymbol{d}_i\}$ by an angle θ about the unit vector $\boldsymbol{n}_\theta$.

Solution of the director triad $\{\boldsymbol{d}_i\}$ for a beam with a fixed left end serves as a good example to appreciate above discussion. Eq. (12) represents set of three differential equations that can be written in the matrix form as

$$\begin{bmatrix} \boldsymbol{d}_{1,\xi_1} \\ \boldsymbol{d}_{2,\xi_1} \\ \boldsymbol{d}_{3,\xi_1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \bar{\kappa}_3 & -\bar{\kappa}_2 \\ -\bar{\kappa}_3 & 0 & \bar{\kappa}_1 \\ \bar{\kappa}_2 & -\bar{\kappa}_1 & 0 \end{bmatrix}}_{\boldsymbol{K}^T} \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \boldsymbol{d}_3 \end{bmatrix}. \tag{15}$$

Assume that the left end of the beam is fixed, implying $\boldsymbol{d}_i(0) = \boldsymbol{E}_i$ and $\boldsymbol{\theta}(0) = \boldsymbol{0}$. These also serve as the three vector boundary conditions to solve Eq. (15). The Darboux vector, $\boldsymbol{\kappa} = \bar{\kappa}_i \boldsymbol{d}_i$, may be interpreted as the rotation of the director frame per unit arc length at ξ_1 by an angle $\|\boldsymbol{\kappa}\| = \sqrt{\bar{\kappa}_1^2 + \bar{\kappa}_2^2 + \bar{\kappa}_3^2}$. Since the left end of the beam is fixed, the director frame $\{\boldsymbol{d}_i(\xi_1)\}$ can be obtained by rotating the vectors $\boldsymbol{E}_i$ by an angle $\theta(\xi_1) = \int_0^{\xi_1} \|\boldsymbol{\kappa}(\xi_1)\| d\xi_1$ about the unit vector $\boldsymbol{n}_\theta(\xi_1) = \frac{\boldsymbol{\kappa}(\xi_1)}{\|\boldsymbol{\kappa}(\xi_1)\|}$.

Fig. 2 geometrically explains the concept described above using a simplified 2D beam fixed at left end. The director $\boldsymbol{d}_3(\xi_1) = \boldsymbol{E}_3$ remains same throughout the midcurve for the problem being planar in nature. Since the torsion is assumed to be zero, $\boldsymbol{\kappa}(\xi_1) = \bar{\kappa}_3 \boldsymbol{d}_3$. This scenario simplifies the unit vector about which rotation occurs at any arc-length as $\boldsymbol{n}_\theta(\xi_1) = \boldsymbol{d}_3(\xi_1) = \boldsymbol{E}_3$ and the angle of rotation of directors $\boldsymbol{d}_1(\xi_1)$ and $\boldsymbol{d}_2(\xi_1)$ with respect to the directors (in a straight configuration) $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ respectively as, $\theta(\xi_1) = \int_0^{\xi_1} \bar{\kappa}_3(\xi_1) d\xi_1$. Note that this is a special case where the vector $\boldsymbol{n}_\theta(\xi_1) = \boldsymbol{E}_3$ is constant for all ξ_1. Therefore, a general rotation tensor $\boldsymbol{Q}$, such that $\boldsymbol{d}_i(\xi_1) = \boldsymbol{Q}(\xi_1)\boldsymbol{E}_i$, for a beam fixed at left end, is then expressed in terms of the curvatures as

$$\boldsymbol{Q}(\xi_1) = e^{\left\{ \frac{\int_0^{\xi_1} \|\boldsymbol{\kappa}(\xi_1)\| d\xi_1}{\|\boldsymbol{\kappa}(\xi_1)\|} \right\} \boldsymbol{K}(\xi_1)}, \tag{16}$$

where, $\boldsymbol{K}(\xi_1)$ is the anti-symmetric tensor corresponding to Darboux vector $\boldsymbol{\kappa}(\xi_1)$. From the above discussion, the result of Eqs. (13) and (14) is not surprising because the solution of the first order differential equation is an exponential.

2.2.3. The material form and co-rotated derivatives of the vector

Consider a general vector $\boldsymbol{g} = \bar{g}_i \boldsymbol{d}_i$. The material form of the vector $\boldsymbol{g}$ is defined using Eq. (4) as,

$$\boldsymbol{g} = \bar{g}_i \boldsymbol{d}_i = \bar{g}_i(\boldsymbol{Q}\boldsymbol{E}_i) = \boldsymbol{Q}\bar{\boldsymbol{g}}, \quad \bar{\boldsymbol{g}} = \boldsymbol{Q}^T \boldsymbol{g} = \bar{g}_i \boldsymbol{E}_i. \tag{17}$$

We obtain the material vector $\bar{\boldsymbol{g}}$ by expressing the components of the vector $\boldsymbol{g}$ in the director frame $\{\boldsymbol{d}_i\}$, in the fixed frame $\{\boldsymbol{E}_i\}$. The total derivative of the vector $\boldsymbol{g}$, using Eq. (4), comprises of two components– first being change in the magnitude and second representing the change due to the rotation of the frame of reference (i.e., rotation of the director frame)

$$\boldsymbol{g}_{,x} = \bar{g}_{i,x}\boldsymbol{d}_i + \bar{g}_i \boldsymbol{d}_{i,x} = \bar{g}_{i,x}\boldsymbol{d}_i + \boldsymbol{Q}_{,x}\boldsymbol{Q}^T(\bar{g}_i \boldsymbol{d}_i) = \tilde{\boldsymbol{g}}_{,x} + \boldsymbol{q}_x \times \boldsymbol{g}. \tag{18}$$

The co-rotational derivative $\tilde{\boldsymbol{g}}_{,x} = \bar{g}_{i,x}\boldsymbol{d}_i$ gives the contribution due to the change in the magnitude of the vector $\boldsymbol{g}$ due to change dx in the parameter x. It may also be interpreted as the derivative of the vector $\boldsymbol{g}$ as observed in the director frame. Physically the co-rotated derivatives can be obtained by taking the total derivative of the vector $\boldsymbol{g}$ (by the observer in the fixed frame $\{\boldsymbol{E_i}\}$) followed by subtracting the rotational component $\boldsymbol{q}_x \times \boldsymbol{g}$. From Eqs. (17) and (18)

$$\tilde{\boldsymbol{g}}_{,x} = \boldsymbol{Q}\tilde{\bar{\boldsymbol{g}}}_{,x} \tag{19}$$

2.2.4. The material form and co-rotated derivatives of a tensor

Consider any two deformed state of the beam Ω^a and Ω^b. Consider two vectors $\boldsymbol{g^a}(\xi_1)$ and $\boldsymbol{g^b}(\xi_1)$ (in states Ω^a and Ω^b, respectively) spanned by the director triads $\{\boldsymbol{d_i^a}\}$ and $\{\boldsymbol{d_i^b}\}$ respectively, such that $\boldsymbol{d_i^a}(\xi_1) = \boldsymbol{Q^a}(\xi_1)\boldsymbol{E}_i$ and $\boldsymbol{d_i^b}(\xi_1) = \boldsymbol{Q^b}(\xi_1)\boldsymbol{E}_i$. Therefore, $\boldsymbol{g^a} = \bar{g}_i^a \boldsymbol{d_i^a}$ and $\boldsymbol{g^b} = \bar{g}_i^b \boldsymbol{d_i^b}$. Now assume that these vectors are related by the tensor $\boldsymbol{G}$ such that $\boldsymbol{g^b} = \boldsymbol{G}\boldsymbol{g^a}$. The material form of tensor $\overline{\boldsymbol{G}}$ is defined such that $\bar{\boldsymbol{g}}^{\boldsymbol{b}} = \overline{\boldsymbol{G}}\bar{\boldsymbol{g}}^{\boldsymbol{a}}$. The relation between $\overline{\boldsymbol{G}}$ and $\boldsymbol{G}$ can be arrived using Eq. (17) as,

$$\bar{\boldsymbol{g}}^{\boldsymbol{b}} = \overline{\boldsymbol{G}}\bar{\boldsymbol{g}}^{\boldsymbol{a}} \Rightarrow \boldsymbol{Q^b}^T \boldsymbol{g^b} = \overline{\boldsymbol{G}}\boldsymbol{Q^a}^T \boldsymbol{g^a} \Rightarrow \boldsymbol{g^b} = \left[\boldsymbol{Q^b}\overline{\boldsymbol{G}}\boldsymbol{Q^a}^T\right]\boldsymbol{g^a}.$$

Hence,

$$\boldsymbol{G} = \boldsymbol{Q^b}\overline{\boldsymbol{G}}\boldsymbol{Q^a}^T, \qquad \overline{\boldsymbol{G}} = \boldsymbol{Q^b}^T\boldsymbol{G}\boldsymbol{Q^a}. \tag{20}$$

Therefore, the derivative of the tensor $\boldsymbol{G}$ can be obtained from Eq. (20) as

$$\boldsymbol{G}_{,\xi_1} = \left[\boldsymbol{Q^b}\overline{\boldsymbol{G}}\boldsymbol{Q^a}^{\boldsymbol{T}}\right]_{,\xi_1} = \overbrace{\boldsymbol{Q}^{\boldsymbol{b}}_{,\xi_1}\overline{\boldsymbol{G}}\boldsymbol{Q^a}^{\boldsymbol{T}} + \boldsymbol{Q^b}\overline{\boldsymbol{G}}\boldsymbol{Q^a}^{\boldsymbol{T}}_{,\xi_1}}^{\text{Contribution due to change in the orientation of frame}} + \overbrace{\boldsymbol{Q^b}\overline{\boldsymbol{G}}_{,\xi_1}\boldsymbol{Q^a}^{\boldsymbol{T}}}^{\text{Change in the magnitude of components}}$$
$$\boldsymbol{G}_{,\xi_1} = (\boldsymbol{Q}^{\boldsymbol{b}}_{,\xi_1}\boldsymbol{Q^b}^{\boldsymbol{T}})\boldsymbol{G} - \boldsymbol{G}(\boldsymbol{Q}^{\boldsymbol{a}}_{,\xi_1}\boldsymbol{Q^a}^{\boldsymbol{T}}) + \tilde{G}_{,\xi_1} = \boldsymbol{K^b}\boldsymbol{G} - \boldsymbol{G}\boldsymbol{K^a} + \tilde{G}_{,\xi_1} \tag{21}$$

Hence,

$$\tilde{\boldsymbol{G}}_{,\xi_1} = \boldsymbol{G}_{,\xi_1} - \boldsymbol{K^b}\boldsymbol{G} + \boldsymbol{G}\boldsymbol{K^a} = \boldsymbol{Q^b}\overline{\boldsymbol{G}}_{,\xi_1}\boldsymbol{Q^a}^T \tag{22}$$

3. Kinematic and kinetic relations

We approach along the lines of Simo (1985) and Li (2000), in an exhaustive way, to define the kinematics of the beam such that the results can be used readily to obtain both the weak and strong forms in detail.

3.1. Deformation gradient tensor and strain vector

The initially curved configuration Ω^c is assumed to be unstrained. This is because the stresses in the current configuration Ω are defined with reference to Ω^c. The straight beam configuration Ω^s is defined for mathematical convenience. If the beam in consideration is initially straight, then $\Omega^c \equiv \Omega^s$. The deformation gradient tensor of current state ($\boldsymbol{F}$) and the curved reference state ($\boldsymbol{F}^c$) is obtained referenced to Ω^s. The deformation gradient tensors $\boldsymbol{F}$ and $\boldsymbol{F}^c$ are then used to define the deformation gradient tensor $\boldsymbol{F}^r$ of the current configuration referred to Ω^c.

3.1.1. Deformation gradient tensor and strain vector referenced to initially straight configuration

Consider an infinitesimal vector $d\boldsymbol{R}^s = d\xi_i\boldsymbol{E}_i$ in Ω^s, that deforms to $d\boldsymbol{R}$ in configuration Ω. The deformation gradient tensor $\boldsymbol{F}$ maps the vector $d\boldsymbol{R}^s$ from the straight configuration Ω^s to $d\boldsymbol{R}$ in the current configuration Ω such that

$$d\boldsymbol{R} = \boldsymbol{F}d\boldsymbol{R}^s \Rightarrow \frac{d\boldsymbol{R}}{d\xi_i} = \boldsymbol{F}\boldsymbol{E}_i, \quad \boldsymbol{F} = \frac{d\boldsymbol{R}}{d\boldsymbol{R}^s} = \boldsymbol{R}_{,\xi_i} \otimes \boldsymbol{E}_i. \tag{23}$$

Using Eq. (3), $\boldsymbol{R}_{,\xi_i} = \boldsymbol{\varphi}_{,\xi_i} + \xi_2\boldsymbol{d}_{2,\xi_i} + \xi_3\boldsymbol{d}_{3,\xi_i}$. Substituting for $\boldsymbol{R}_{,\xi_i}$ in Eq. (23) yields

$$\boldsymbol{F} = \Big(\overbrace{\underbrace{(\boldsymbol{\varphi}_{,\xi_i} - \boldsymbol{d}_1)}_{\text{axial strain } \boldsymbol{\varepsilon}} + \xi_2\boldsymbol{d}_{2,\xi_i} + \xi_3\boldsymbol{d}_{3,\xi_i}}^{\text{strain vector } \boldsymbol{\epsilon}}\Big) \otimes \boldsymbol{E}_1 + \overbrace{\boldsymbol{d}_i \otimes \boldsymbol{E}_i}^{\boldsymbol{Q}} = \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{Q}. \tag{24}$$

The material form of deformation gradient tensor can be arrived using Eqs. (20) and (24) as

$$\overline{\boldsymbol{F}} = \bar{\boldsymbol{\epsilon}} \otimes \boldsymbol{E}_1 + \boldsymbol{I}_3 = \boldsymbol{Q}^T\boldsymbol{F}\boldsymbol{I}_3. \tag{25}$$

It is worth noting that the deformation gradient tensor $\boldsymbol{F}$ that describes the motion bears two parts. The motion consists of pure rotation $\boldsymbol{Q}$ and a component associated with strain $\boldsymbol{\epsilon}\otimes\boldsymbol{E}_1$. It's clear that the first component of the vector $d\boldsymbol{R}^s$ strains whereas the other two components just experience rigid body rotation. This is because, the cross-section is assumed rigid. The vector $\boldsymbol{\epsilon}$ represents the strain vector referenced to the configuration Ω^s that includes the axial strain $\boldsymbol{\varepsilon} = \boldsymbol{\varphi}_{,\xi_1} - \boldsymbol{d}_1$, and strain due to shear and curvatures. The strain vector can also be evaluated by finding the derivative of the position vector of any point subtracted by the director $\boldsymbol{d}_1$ as in Chadha and Todd (2017). We subtract the director $\boldsymbol{d}_1$ to eliminate the contribution of pure rotation on the deformation. Therefore, using Eq. (12)

$$\begin{aligned}\boldsymbol{\epsilon} &= \bar{\epsilon}_i\boldsymbol{d}_i = \frac{\partial \boldsymbol{R}}{\partial \xi_1} - \boldsymbol{d}_1 = \boldsymbol{\varepsilon} + \xi_2\boldsymbol{d}_{2,\xi_1} + \xi_3\boldsymbol{d}_{3,\xi_1} \\ &= \boldsymbol{\varepsilon} + \boldsymbol{\kappa} \times (\xi_2\boldsymbol{d}_2 + \xi_3\boldsymbol{d}_3).\end{aligned} \tag{26}$$

Substituting for the *Darboux vector* $\boldsymbol{\kappa} = \bar{\kappa}_i\boldsymbol{d}_i$ and using Eq. (9) in Eq. (26), the complete expression for the strain is obtained as

$$\begin{aligned}\boldsymbol{\epsilon} = &\{((1+e)\cos\gamma_{11} - 1) - \xi_2\bar{\kappa}_3 + \xi_3\bar{\kappa}_2\}\boldsymbol{d}_1 \\ &+ \{(1+e)\sin\gamma_{12} - \bar{\kappa}_2\xi_3\}\boldsymbol{d}_2 + \{(1+e)\sin\gamma_{13} + \bar{\kappa}_1\xi_2\}\boldsymbol{d}_3.\end{aligned} \tag{27}$$

The material form of strain vector comes in handy to evaluate internal strain energy of the reduced beam. It is obtained using Eqs. (17) and (26) as

$$\begin{aligned}\bar{\boldsymbol{\epsilon}} &= \bar{\epsilon}_i\boldsymbol{E}_i = \boldsymbol{Q}^T[(\boldsymbol{\varphi}_{,\xi_1} - \boldsymbol{d}_1) + \boldsymbol{K}(\xi_2\boldsymbol{d}_2 + \xi_3\boldsymbol{d}_3)] \\ &= \underbrace{(\boldsymbol{Q}^T\boldsymbol{\varphi}_{,\xi_1} - \boldsymbol{E}_1)}_{\bar{\boldsymbol{\varepsilon}}} + \underbrace{\boldsymbol{Q}^T\boldsymbol{K}\boldsymbol{Q}}_{\overline{\boldsymbol{K}}}(\xi_2\boldsymbol{E}_2 + \xi_3\boldsymbol{E}_3) \\ &= \bar{\boldsymbol{\varepsilon}} + \overline{\boldsymbol{K}}(\xi_2\boldsymbol{E}_2 + \xi_3\boldsymbol{E}_3) = \bar{\boldsymbol{\varepsilon}} + \bar{\boldsymbol{\kappa}} \times (\xi_2\boldsymbol{E}_2 + \xi_3\boldsymbol{E}_3).\end{aligned} \tag{28}$$

It is clear from the discussion above that only the first component (along $\boldsymbol{E}_1$) of an infinitesimal vector $d\boldsymbol{R}^s$ is strained, with the cross-section being rigid. Therefore, it is insightful to observe the deformation of the vector $\boldsymbol{E}_1$. The flowchart (Fig. 3) demonstrates the straining of the unit vector $\boldsymbol{E}_1$ (not necessarily along the midcurve) with each deformation effect taken care separately followed by superimposition.

3.1.2. Deformation gradient tensor and strain vector of curved reference configuration referenced to initially straight configuration

Consider that the configuration Ω^c is obtained by straining the initially straight configuration Ω^s such that the total length of the midcurve remains the same and the cross-sections are perpendicular to the tangent vector at the midcurve. This deformation is mapped by the deformation gradient tensor $\boldsymbol{F}^c$ such that

$$\boldsymbol{F}^c = \frac{d\boldsymbol{R}^c}{d\boldsymbol{R}^s} = \left(\xi_2\boldsymbol{d}^c_{2,\xi_1} + \xi_3\boldsymbol{d}^c_{3,\xi_1}\right) \otimes \boldsymbol{E}_1 + \boldsymbol{d}^c_i \otimes \boldsymbol{E}_i = \boldsymbol{\epsilon}^c \otimes \boldsymbol{E}_1 + \boldsymbol{Q}^c. \tag{29}$$

Like Eq. (25), the material form of $\boldsymbol{F}^c$ is

$$\overline{\boldsymbol{F}}^c = \bar{\boldsymbol{\epsilon}}^c \otimes \boldsymbol{E}_1 + \boldsymbol{I}_3 = \boldsymbol{Q}^{cT}\boldsymbol{F}^c\boldsymbol{I}_3 \tag{30}$$

The strain vector $\boldsymbol{\epsilon}^c$ comprises of strain due to curvatures only because there is no shear $\gamma_{1i} = 0$ and elongation $e(\xi_1) = 0$ in the curved reference configuration Ω^c. This ensures the director $\boldsymbol{d}^c_1$ to be the tangent vector of the midcurve such that $\boldsymbol{\varphi}_{,\xi_1} = \boldsymbol{d}^c_1$. Therefore, the axial strain vector $\boldsymbol{\varepsilon}^c = \boldsymbol{\varphi}_{,\xi_1} - \boldsymbol{d}^c_1 = \boldsymbol{0}$. From Eqs. (29) and (30), it is observed that

$$\boldsymbol{F}^c\boldsymbol{E}_i = \begin{Bmatrix} \boldsymbol{\epsilon}^c + \boldsymbol{d}_1 = \bar{\epsilon}^c_i\boldsymbol{d}_i + \boldsymbol{d}_1, & \text{for } i = 1 \\ \boldsymbol{d}_i, & \text{for } i = 2, 3 \end{Bmatrix};$$

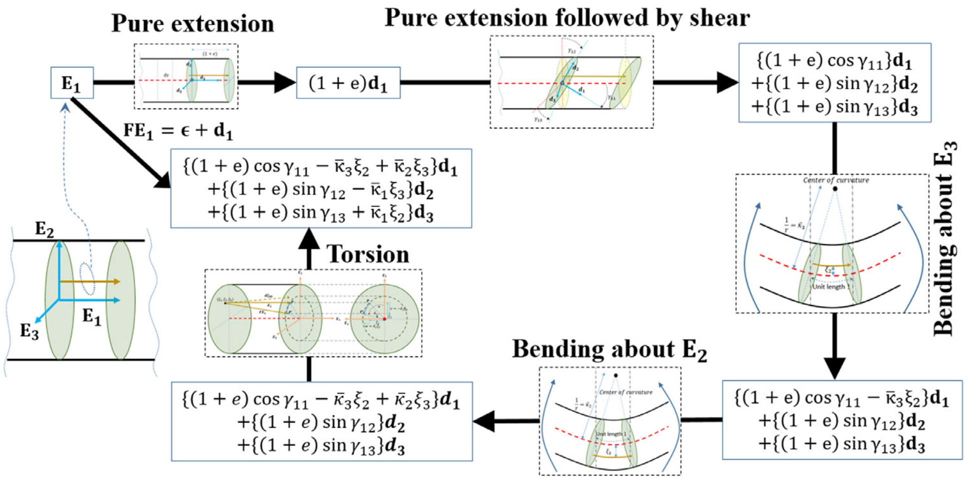

Fig. 3. Flowchart showing deformation of the unit vector $\boldsymbol{E_1}$ in the configuration Ω^s.

$$\overline{\boldsymbol{F}}^c\boldsymbol{E}_i = \begin{Bmatrix} \overline{\boldsymbol{\epsilon}}^c + \boldsymbol{E}_1 = \overline{\epsilon}^c_i\boldsymbol{E}_i + \boldsymbol{E}_1, & \text{for } i=1 \\ \boldsymbol{E}_i, & \text{for } i=2,3 \end{Bmatrix}. \quad (31)$$

From the above equation, the determinant of $\boldsymbol{F}^c$ is obtained as

$$|\overline{\boldsymbol{F}}^c| = |\boldsymbol{Q}^{c^T}| \, \| \, \boldsymbol{F}^c \, \| \, \boldsymbol{I}_3| = |\boldsymbol{F}^c| = 1 + \overline{\epsilon}^c_1. \quad (32)$$

Using Eq. (31), the first component of the vector $d\boldsymbol{R}^s$ in the straight configuration $d\xi_1\boldsymbol{E}_1$ gets strained to $\overline{\boldsymbol{F}}^c(d\xi_1\boldsymbol{E}_1) = (1+\overline{\epsilon}^c_1)d\xi_1\boldsymbol{d}^c_1$. This means that a fiber of unit length parallel to $\boldsymbol{E}_1$ in the configuration Ω^s has length of $|\overline{\boldsymbol{F}}^c|$ in the configuration Ω^c along the director $\boldsymbol{d}^c_1$. In terms of classical continuum mechanics, $|\overline{\boldsymbol{F}}^c|$ is associated with volumetric strain

$$|\boldsymbol{F}^c| = \frac{d\Omega^c}{d\Omega^s} = \frac{\rho^s}{\rho^c}, \quad (33)$$

where ρ^s and ρ^c represents the density field in the configuration Ω^s and Ω^c, respectively.

3.1.3. Deformation gradient tensor and strain vector of current configuration referenced to curved reference configuration

The deformation gradient tensor $\boldsymbol{F}^r$ is defined such that $d\boldsymbol{R} = \boldsymbol{F}^r d\boldsymbol{R}^c$. Therefore, from Eqs. (23) and (29), $\boldsymbol{F}^r = \boldsymbol{F}\boldsymbol{F}^{c^{-1}}$ and from Eq. (30), $\boldsymbol{F}^{c^{-1}} = \overline{\boldsymbol{F}}^{c^{-1}}\boldsymbol{Q}^{c^T}$. The tensor $\overline{\boldsymbol{F}}^{c^{-1}}$ can be found by using the theorem for inverse of sum of matrices (refer Miller, 1981) as

$$\overline{\boldsymbol{F}}^{c^{-1}} = \left[\overline{\boldsymbol{\epsilon}}^c \otimes \boldsymbol{E}_1 + \boldsymbol{I}_3\right]^{-1} = \boldsymbol{I}_3{}^{-1} - \frac{\boldsymbol{I}_3{}^{-1}\left(\overline{\boldsymbol{\epsilon}}^c \otimes \boldsymbol{E}_1\right)\boldsymbol{I}_3{}^{-1}}{1 + \text{trace}\left[\overline{\boldsymbol{\epsilon}}^c \otimes \boldsymbol{E}_1\right]} = \boldsymbol{I}_3 - \frac{\left(\overline{\boldsymbol{\epsilon}}^c \otimes \boldsymbol{E}_1\right)}{1+\overline{\epsilon}^c_1}$$
$$= -\frac{1}{|\boldsymbol{F}^c|}\left(\overline{\boldsymbol{\epsilon}}^c \otimes \boldsymbol{E}_1\right) + \boldsymbol{I}_3. \quad (34)$$

Therefore, the tensor $\boldsymbol{F}^{c^{-1}}$ can be found as

$$\boldsymbol{F}^{c^{-1}} = \left[-\frac{1}{|\boldsymbol{F}^c|}\left(\overline{\boldsymbol{\epsilon}}^c \otimes \boldsymbol{E}_1\right) + \boldsymbol{I}_3\right]\boldsymbol{Q}^{c^T}$$
$$= \left[-\frac{1}{|\boldsymbol{F}^c|}\left(\left(\boldsymbol{Q}^{c^T}\boldsymbol{\epsilon}^c\right) \otimes \left(\boldsymbol{Q}^{c^T}\boldsymbol{d}^c_1\right)\right) + \boldsymbol{I}_3\right]\boldsymbol{Q}^{c^T}$$
$$= \boldsymbol{Q}^{c^T}\left[\boldsymbol{I}_3 - \frac{1}{|\boldsymbol{F}^c|}\left(\boldsymbol{\epsilon}^c \otimes \boldsymbol{d}^c_1\right)\right]. \quad (35)$$

This brings us to the point of evaluating the deformation gradient tensor $\boldsymbol{F}^r$ as follows

$$\boldsymbol{F}^r = [\boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{Q}]\boldsymbol{Q}^{c^T}\left[\boldsymbol{I}_3 - \frac{1}{|\boldsymbol{F}^c|}\left(\boldsymbol{\epsilon}^c \otimes \boldsymbol{d}^c_1\right)\right]$$
$$= \boldsymbol{Q}^r - \frac{1}{|\boldsymbol{F}^c|}\left(\left(\boldsymbol{Q}^r\boldsymbol{\epsilon}^c\right) \otimes \boldsymbol{d}^c_1\right) + \left\{1 - \frac{\boldsymbol{\epsilon}^c.\boldsymbol{d}^c_1}{|\boldsymbol{F}^c|}\right\}\left(\boldsymbol{\epsilon} \otimes \boldsymbol{d}^c_1\right). \quad (36)$$

Noting that $\boldsymbol{\epsilon}^c.\boldsymbol{d}^c_1 = \epsilon^c_1 = |\boldsymbol{F}^c| - 1$, Eq. (36) can be simplified as

$$\boldsymbol{F}^r = \frac{1}{|\boldsymbol{F}^c|}\Big(\overbrace{\left(\boldsymbol{\epsilon} - \boldsymbol{Q}^r\boldsymbol{\epsilon}^c\right)}^{\boldsymbol{\epsilon}^r} \otimes \boldsymbol{d}^c_1\Big) + \boldsymbol{Q}^r = \frac{1}{|\boldsymbol{F}^c|}\left(\boldsymbol{\epsilon}^r \otimes \boldsymbol{d}^c_1\right) + \boldsymbol{Q}^r. \quad (37)$$

There are three important points to infer from Eq. (37):

1. The strain $\boldsymbol{\epsilon}^r$ represents the relative strain in the current configuration Ω with respect to the strained curved reference configuration Ω^c (strained with respect to mathematically straight configuration Ω^s).
2. The strain $\boldsymbol{\epsilon}^r$ is obtained as $(\boldsymbol{\epsilon} - \boldsymbol{Q}^r\boldsymbol{\epsilon}^c)$ and not as $(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^c)$ because the strain $\boldsymbol{\epsilon}^c$ is represented in the Ω^c configuration, whereas the strain $\boldsymbol{\epsilon}$ is represented in Ω configuration. The rotation tensor $\boldsymbol{Q}^r$ transforms the strain $\boldsymbol{\epsilon}^c$ in the current configuration space.
3. The curved configuration is strained referenced to the mathematically straight configuration. To obtain the strain vector in the current state Ω with respect to the unstrained curved configuration, the strain $\boldsymbol{\epsilon}^r$ must be normalized by $|\boldsymbol{F}^c|$.

3.2. Closed-form expression for the orthogonal rotational tensor and defining a unique set of shear angles

The position vector of the midcurve may be defined in terms of the pitch angle $\phi_p(\xi_1)$ and the yaw angle $\phi_y(\xi_i)$. We define the tangent vector using Eq. (7) as

$$\boldsymbol{T}(\xi_1) = \frac{\partial\varphi}{\partial s} = \frac{1}{1+e}\frac{\partial\varphi}{\partial\xi_i}$$
$$= \cos\phi_p(\xi_1)\cos\phi_y(\xi_1)\boldsymbol{E}_1$$
$$+ \sin\phi_p(\xi_1)\boldsymbol{E}_2 + \cos\phi_p(\xi_1)\sin\phi_y(\xi_1)\boldsymbol{E}_3. \quad (38)$$

Therefore, using the above equation, the position vector can be obtained as

$$\boldsymbol{\varphi}(\xi_1) = \left(\int_0^{\xi_1}\cos\phi_p\cos\phi_y(1+e)d\xi_1\right)\boldsymbol{E}_1$$
$$+ \left(\int_0^{\xi_1}\sin\phi_p(1+e)d\xi_1\right)\boldsymbol{E}_2 \quad (39)$$
$$+ \left(\int_0^{\xi_1}\cos\phi_p\sin\phi_y(1+e)d\xi_1\right)\boldsymbol{E}_3.$$

To define the three shear angles uniquely in Eq. (8), we define another local orthonormal vector triad $\{\boldsymbol{T}(\xi_1), \boldsymbol{V}(\xi_1), \boldsymbol{H}(\xi_1)\}$ that originate at the midcurve as shown in Fig. 4 (same origin as the director triad $\{\boldsymbol{d}_i(\xi_1)\}$). The vector $\boldsymbol{T}(\xi_1)$ and $\boldsymbol{V}(\xi_1)$ spans the pitch

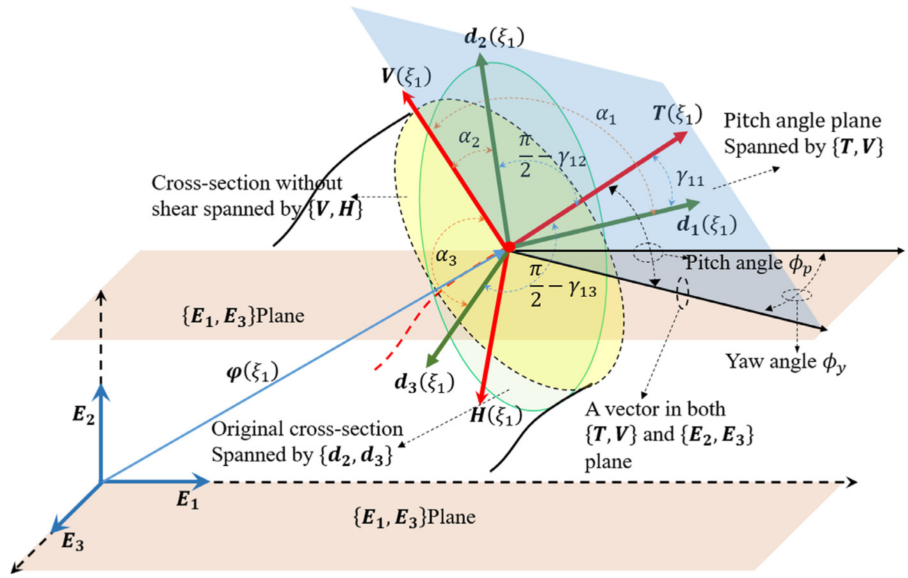

Fig. 4. Pitch angle plane and the body centered vector triad $\{\boldsymbol{T}(\xi_1), \boldsymbol{V}(\xi_1), \boldsymbol{H}(\xi_1)\}$.

angle plane. Therefore, $\boldsymbol{H}(\xi_1) = \boldsymbol{T}(\xi_1) \times \boldsymbol{V}(\xi_1)$. Hence,

$$\begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{V} \\ \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} \cos\phi_p\cos\phi_y & \sin\phi_p & \cos\phi_p\sin\phi_y \\ -\sin\phi_p\cos\phi_y & \cos\phi_p & -\sin\phi_p\sin\phi_y \\ -\sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \\ \boldsymbol{E}_3 \end{bmatrix}. \quad (40)$$

We define three angles $\alpha_1(\xi_1)$, $\alpha_2(\xi_1)$ and $\alpha_3(\xi_1)$ subtended by the directors $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ with the vector $\boldsymbol{V}(\xi_1)$. This definition serves for two purposes: firstly, it helps us to define a relationship between the triad $\{\boldsymbol{T}, \boldsymbol{V}, \boldsymbol{H}\}$ and $\{\boldsymbol{d}_i(\xi_1)\}$, secondly, it uniquely defines the shear angles. Hence,

$$\begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{V} \\ \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} \cos\gamma_{11} & \sin\gamma_{12} & \sin\gamma_{13} \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\alpha_3\sin\gamma_{12} - \cos\alpha_2\sin\gamma_{13} & \cos\alpha_3\cos\gamma_{11} - \cos\alpha_1\sin\gamma_{13} & \cos\alpha_2\sin\gamma_{11} - \cos\alpha_1\sin\gamma_{12} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \boldsymbol{d}_3 \end{bmatrix}. \quad (41)$$

Consider an orthogonal rotation matrix $\boldsymbol{\Lambda}$ that relates the director triad $\{\boldsymbol{d}_i\}$ to the fixed orthogonal Cartesian triad $\{\boldsymbol{E}_i\}$, such that $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\} = \boldsymbol{\Lambda}.\{\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$. The matrix $\boldsymbol{\Lambda}$ is related to the components of the orthogonal rotation tensor $\boldsymbol{Q}$ such that $\boldsymbol{\Lambda}^T = [\boldsymbol{Q}]_{\boldsymbol{d}_i \otimes \boldsymbol{E}_i}$. These components can be obtained using Eqs. (40) and (41). The components of the rotation matrix are shown in Appendix A.1. Note that the matrix $\boldsymbol{\Lambda}$ is orthogonal if the following constraints on $\{\alpha_1, \alpha_2, \alpha_3, \gamma_{11}, \gamma_{12}, \gamma_{13}\}$ in the above equation hold:

$$|\boldsymbol{T}| = |\boldsymbol{V}| = |\boldsymbol{H}| = 1; \qquad |\boldsymbol{T}|_{,\xi_1} = |\boldsymbol{V}|_{,\xi_1} = |\boldsymbol{H}|_{,\xi_1} = 0. \quad (42)$$

The components of the Darboux vector $\boldsymbol{\kappa} = \overline{\kappa}_i \boldsymbol{d}_i$ as in Eq. (15) using Eqs. (40) and (41) can be obtained and are shown in Appendix A.2. This is a useful result in shape reconstruction as presented in Chadha and Todd (2017) and will be used in further extension and generalization of the work in Todd et al. (2013), Chadha and Todd (2017), and experimental validation of the theory of shape reconstruction. Appendix A.3 presents an example of a deformed shape of a Cosserat rod using the description given in this section.

3.3. Variation of kinematic parameters and rotation tensor

3.3.1. Variation of the rotation tensor and directors

We need to impart an admissible variational displacement field $\delta\boldsymbol{u}$ to obtain the weak form of reduced equilibrium equation (*Principle of Virtual Work*). The variational displacement field $\delta\boldsymbol{u} = \delta(\boldsymbol{R} - \boldsymbol{R}^s) = \delta\boldsymbol{R}$ comprises of variational translation of the midcurve and rotation of the director frame. It is necessary to arrive at the variation of the rotation tensor to proceed further.

To obtain the variation in rotation tensor, assume that the tensor $\boldsymbol{Q} = \mathfrak{R}(\boldsymbol{\theta})$ rotates the vector $\boldsymbol{E}_i$ to $\boldsymbol{d}_i$ by an angle θ. As a result of the virtual displacement field $\delta\boldsymbol{u}$, the vector $\boldsymbol{d}_i$ transforms to $\boldsymbol{d}_i{}^*$. The variational rotation is parametrized by the vector $\delta\boldsymbol{\alpha} = (\delta\alpha)\boldsymbol{n}_\alpha$ such that $\boldsymbol{d}_i^* = \mathfrak{R}(\delta\boldsymbol{\alpha})\boldsymbol{d}_i$. The vector $\boldsymbol{d}_i{}^*$ can be obtained by direct rotation of $\boldsymbol{E}_i$ parametrized by the rotation vector $\boldsymbol{\theta} + \delta\boldsymbol{\theta}$ as shown in Fig. 5. Therefore, for the variational rotation of $\varepsilon\delta\boldsymbol{\alpha}$ (ε is a small number), the following relations hold

$$\begin{aligned} \boldsymbol{d}_i^* &= \mathfrak{R}(\boldsymbol{\theta} + \varepsilon\delta\boldsymbol{\theta})\boldsymbol{E}_i = \boldsymbol{Q}(\boldsymbol{\theta} + \varepsilon\delta\boldsymbol{\theta})\boldsymbol{E}_i \\ \boldsymbol{d}_i^* &= \mathfrak{R}(\varepsilon\delta\boldsymbol{\alpha})\mathfrak{R}(\boldsymbol{\theta})\boldsymbol{E}_i = e^{\varepsilon\delta\mathfrak{A}}\boldsymbol{Q}(\boldsymbol{\theta})\boldsymbol{E}_i. \end{aligned} \quad (43)$$

Note that $\delta\mathfrak{A}$ represents the skew-symmetric tensor corresponding to axial vector $\delta\boldsymbol{\alpha}$. Variation of the rotation tensor $\boldsymbol{Q}$ can then be obtained by the usual process as,

$$\delta\boldsymbol{Q}(\boldsymbol{\theta}) = \left[\frac{\partial\boldsymbol{Q}(\boldsymbol{\theta} + \varepsilon\delta\boldsymbol{\theta})}{\partial\varepsilon}\right]_{\varepsilon=0} = \left[\frac{e^{\varepsilon\delta\mathfrak{A}}\boldsymbol{Q}(\boldsymbol{\theta})}{\partial\varepsilon}\right]_{\varepsilon=0} = \delta\mathfrak{A}.\boldsymbol{Q}(\boldsymbol{\theta}). \quad (44)$$

The admissible variation in the directors can be obtained from above as

$$\delta\boldsymbol{d}_i = \delta[\boldsymbol{Q}\boldsymbol{E}_i] = \delta\boldsymbol{Q}.\boldsymbol{E}_i = \delta\mathfrak{A}.\boldsymbol{Q}.\boldsymbol{E}_i = \delta\mathfrak{A}\boldsymbol{d}_i, \quad (45)$$

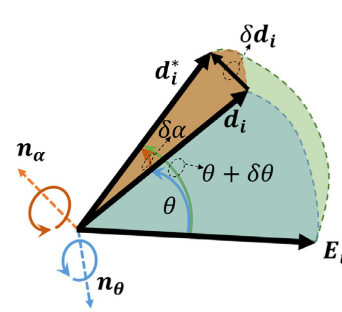

Fig. 5. Variation of the director $\boldsymbol{d}_i$.

implying

$$\delta\mathfrak{A} = \delta\mathbf{Q}.\mathbf{Q}^T.$$

The variation of the displacement field can now be obtained using Eqs. (3) and (45) as,

$$\delta\mathbf{u} = \delta\boldsymbol{\varphi} + \delta\boldsymbol{\alpha} \times \mathbf{r}_{PG}. \tag{46}$$

3.3.2. Variation and co-rotated variation of any general vector and tensor

The variation of any general vector $\mathbf{g} = \bar{g}_i\mathbf{d}_i$ consists of two parts, the first being the variation in the magnitude of components and second being the contribution due to the variation in the director frame as shown below

$$\delta\mathbf{g} = \overbrace{\delta\bar{g}_i\mathbf{d}_i}^{\tilde{\delta}\mathbf{g}} + \bar{g}_i\delta\mathbf{d}_i = \tilde{\delta}\mathbf{g} + \delta\mathfrak{A}.\mathbf{g} = \tilde{\delta}\mathbf{g} + \delta\boldsymbol{\alpha} \times \mathbf{g} \tag{47}$$

The relationship between variation of material vector $\delta\bar{\mathbf{g}}$ and the co-rotated variation $\tilde{\delta}\mathbf{g}$ can be obtained from Eqs. (4), (17) and (47) as

$$\tilde{\delta}\mathbf{g} = \delta\bar{g}_i\mathbf{d}_i = \mathbf{Q}(\delta\bar{g}_i\mathbf{E}_i) = \mathbf{Q}\delta\bar{\mathbf{g}}. \tag{48}$$

Using the description of the tensor $\mathbf{G}$ in Section (2.2.4), the co-rotated variation of the tensor can be written as

$$\tilde{\delta}\mathbf{G} = \mathbf{Q}^{\mathbf{b}}\delta\bar{\mathbf{G}}\mathbf{Q}^{\mathbf{a}T}. \tag{49}$$

3.3.3. Variation of the strain vector and deformation gradient tensor

The variation in the strain vector $\boldsymbol{\epsilon}$ can be readily obtained if the variation in axial strain vector $\boldsymbol{\varepsilon}$ and curvature tensor $\mathbf{K}$ are known. From Eqs. (24), (28) and (45),

$$\delta\boldsymbol{\varepsilon} = \delta\boldsymbol{\varphi}_{,\xi_1} - \delta\mathbf{d}_1 = \delta\boldsymbol{\varphi}_{,\xi_1} - \delta\mathfrak{A}.\mathbf{d}_1. \tag{50}$$

Similarly, recalling the relation $\delta\mathfrak{A} = \delta\mathbf{Q}.\mathbf{Q}^T$ the variation of the curvature tensor is obtained as

$$\delta\mathbf{K} = \delta[\mathbf{Q}_{,\xi_1}\mathbf{Q}^T] = (\delta\mathbf{Q})_{,\xi_1}\mathbf{Q}^T + \mathbf{Q}_{,\xi_1}\delta\mathbf{Q}^T = \delta\mathfrak{A}_{,\xi_1} + \delta\mathfrak{A}.\mathbf{K} - \mathbf{K}.\delta\mathfrak{A}. \tag{51}$$

Recognizing that $\mathbf{K}$ and $\delta\mathfrak{A}$ are skew symmetric with $\boldsymbol{\kappa}$ and $\delta\boldsymbol{\alpha}$ as the respective axial vectors, it may be readily obtained that

$$\delta\boldsymbol{\kappa} = \delta\boldsymbol{\alpha}_{,\xi_1} + \delta\boldsymbol{\alpha} \times \boldsymbol{\kappa}. \tag{52}$$

Using the result (47) on Eqs. (50)–(52), the co-rotated variation $\tilde{\delta}\boldsymbol{\varepsilon}$ and $\tilde{\delta}\boldsymbol{\kappa}$ are obtained as

$$\begin{aligned} \tilde{\delta}\boldsymbol{\varepsilon} &= \delta\boldsymbol{\varepsilon} - \delta\mathfrak{A}\boldsymbol{\varepsilon} = \delta\boldsymbol{\varphi}_{,\xi_1} - \delta\boldsymbol{\alpha} \times \boldsymbol{\varphi}_{,\xi_1}, \\ \tilde{\delta}\boldsymbol{\kappa} &= \delta\boldsymbol{\kappa} - \delta\mathfrak{A}\boldsymbol{\kappa} = \delta\boldsymbol{\alpha}_{,\xi_1}. \end{aligned} \tag{53}$$

Similarly, using the result (48) on (53), the variation of the material form $\delta\bar{\boldsymbol{\varepsilon}}$ and $\delta\bar{\boldsymbol{\kappa}}$ may be found as

$$\begin{aligned} \delta\bar{\boldsymbol{\varepsilon}} &= \mathbf{Q}^T\tilde{\delta}\boldsymbol{\varepsilon} = \mathbf{Q}^T\delta\boldsymbol{\alpha}_{,\xi_1}, \\ \delta\bar{\boldsymbol{\kappa}} &= \mathbf{Q}^T\tilde{\delta}\boldsymbol{\kappa} = \mathbf{Q}^T(\delta\boldsymbol{\varphi}_{,\xi_1} - \delta\boldsymbol{\alpha} \times \boldsymbol{\varphi}_{,\xi_1}). \end{aligned} \tag{54}$$

From Eqs. (25), (45) and (49), the variation of deformation gradient tensor is

$$\begin{aligned} \delta\mathbf{F} &= \delta(\mathbf{Q}\bar{\mathbf{F}}\mathbf{I}_3^T) = \delta\mathfrak{A}.\mathbf{F} + \tilde{\delta}\mathbf{F}, \\ \tilde{\delta}\mathbf{F} &= \mathbf{Q}\delta\bar{\mathbf{F}}\mathbf{I}_3^T = \mathbf{Q}\big[\delta(\bar{\boldsymbol{\epsilon}} \otimes \mathbf{E}_1)\big]\mathbf{I}_3^T = \tilde{\delta}\bar{\boldsymbol{\epsilon}} \otimes \mathbf{E}_1, \end{aligned} \tag{55}$$

where

$$\begin{aligned} \delta\bar{\epsilon} &= \mathbf{Q}^T\delta\boldsymbol{\varepsilon} + \mathbf{Q}^T\delta\mathbf{K}\mathbf{Q}\ [\xi_2\mathbf{E}_2 + \xi_3\mathbf{E}_3] = \delta\bar{\boldsymbol{\varepsilon}} + \delta\bar{\mathbf{K}}\ [\xi_2\mathbf{E}_2 + \xi_3\mathbf{E}_3], \\ \tilde{\delta}\boldsymbol{\epsilon} &= \mathbf{Q}\delta\bar{\boldsymbol{\epsilon}} = \tilde{\delta}\boldsymbol{\varepsilon} + \tilde{\delta}\mathbf{K}[\xi_2\mathbf{d}_2 + \xi_3\mathbf{d}_3]. \end{aligned} \tag{56}$$

3.4. Stress tensor, the reduced force and moment

Consider the Cauchy stress tensor $\boldsymbol{\sigma}$ referenced to the current configuration Ω and the first Piola Kirchhoff stress tensor $\mathbf{S}^c$ and $\mathbf{S}^s$ referenced to the configuration Ω^c and Ω^s, respectively, such that the associated stress vectors are given by

$$\begin{aligned} \boldsymbol{\sigma}_i &= \boldsymbol{\sigma}\mathbf{d}_i = \bar{\sigma}_{ji}\mathbf{d}_i; \\ \mathbf{S}_i &= \mathbf{S}^c\mathbf{d}_i^{\,c} = \mathbf{S}^s\mathbf{E}_i = \bar{S}_{ji}\mathbf{d}_i. \end{aligned} \tag{57}$$

Therefore, the stress tensor can be written in the index form as

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_i \otimes \mathbf{d}_j = \bar{\sigma}_{ji}\mathbf{d}_i \otimes \mathbf{d}_j, \\ \mathbf{S}^c &= \mathbf{S}_i \otimes \mathbf{d}_j^c = \bar{S}_{ji}\mathbf{d}_i \otimes \mathbf{d}_j^c, \\ \mathbf{S}^s &= \mathbf{S}_i \otimes \mathbf{E}_i = \bar{S}_{ji}\mathbf{d}_i \otimes \mathbf{E}_j. \end{aligned} \tag{58}$$

It is expedient to define the reduced force and moment at the mid-curve of the current configuration in the classical sense as

$$\boldsymbol{\eta}(\xi_1) = \int_{\blacksquare(\xi_1)} \boldsymbol{\sigma}_1 d\xi_2 d\xi_3 = \int_{\blacksquare(\xi_1)} \mathbf{S}_1 d\xi_2 d\xi_3 = \bar{\eta}_i\mathbf{d}_i, \tag{59}$$

$$\mathbf{m}(\xi_1) = \int_{\blacksquare(\xi_1)} \mathbf{r}_{PG} \times \boldsymbol{\sigma}_1 d\xi_2 d\xi_3 = \int_{\blacksquare(\xi_1)} \mathbf{r}_{PG} \times \mathbf{S}_1 d\xi_2 d\xi_3 = \bar{m}_i\mathbf{d}_i.$$

The first component of the force vector $\bar{\eta}_1$ represents the axial force along $\mathbf{d}_1$, whereas the components $\bar{\eta}_2$ and $\bar{\eta}_3$ represent the shear forces along the directors $\mathbf{d}_2$ and $\mathbf{d}_3$, respectively. Similarly, the component $\bar{m}_1$ of the moment vector represents the torque about the vector $\mathbf{d}_1$ whereas the components $\bar{m}_2$ and $\bar{m}_3$ represents the moments about the directors $\mathbf{d}_2$ and $\mathbf{d}_3$. Fig. 6 gives a geometric interpretation of the reduced force and couple.

4. Strong form of the reduced balance law of Cosserat beam using Lagrangian differential equation of motion

We derive the reduced governing differential equations (*strong form*) by considering initially straight configuration Ω^s, finally obtaining the equations for initially curved (but unstrained) reference configuration Ω^c using the relations defined in the previous sections. The infinitesimal equilibrium equation for a general continuum problem referenced to the configuration Ω^s is given as in Lai et al. (2010) by

$$\nabla_{\Omega^s}.\mathbf{S}^s + \rho^s\mathbf{b} = \rho^s\ddot{\mathbf{R}} \tag{61}$$

for the material point defined by the position vector $\mathbf{R}(\xi_1, \xi_2, \xi_3)$. The operator ∇_{Ω^s} represents the gradient operator with respect to the configuration Ω^s. Therefore, $(\nabla_{\Omega^s}.\mathbf{S}^s)$ represents the divergence of the tensor $\mathbf{S}^s$ referenced to Ω^s. The quantity $\mathbf{b}(\xi_1, \xi_2, \xi_3)$ is the body force per unit mass of the body and is independent of the reference configuration. Integrating above equation over the entire undeformed domain Ω^s followed by the application of greens theorem to get the boundary terms gives the balance of linear momentum equation. Similarly, taking the cross product of the lever arm $(\mathbf{R} - \boldsymbol{\nu})$ with all the terms in Eq. (61), followed by the integration over the entire domain gives the angular momentum balance equation with respect to any fixed point p defined by the fixed vector $\boldsymbol{\nu}$ (Fig. 7), such that

$$\int_{\Gamma^s} \mathbf{S}^s\mathbf{N}^s d\Gamma^s + \int_{\Omega^s} \rho^s\mathbf{b}d\Omega^s = \int_{\Omega^s} \rho^s\ddot{\mathbf{R}}d\Omega^s, \tag{62}$$

$$\begin{aligned} &\int_{\Gamma^s} (\mathbf{R} - \boldsymbol{\nu}) \times (\mathbf{S}^s\mathbf{N}^s)d\Gamma^s + \int_{\Omega^s} \rho^s(\mathbf{R} - \boldsymbol{\nu}) \times \mathbf{b}d\Omega^s \\ &\quad = \int_{\Omega^s} \rho^s(\mathbf{R} - \boldsymbol{\nu}) \times \ddot{\mathbf{R}}d\Omega^s. \end{aligned} \tag{63}$$

Here, Γ^s and $\mathbf{N}^s$ represents the boundary and the normal vector respectively in the configuration Ω^s.

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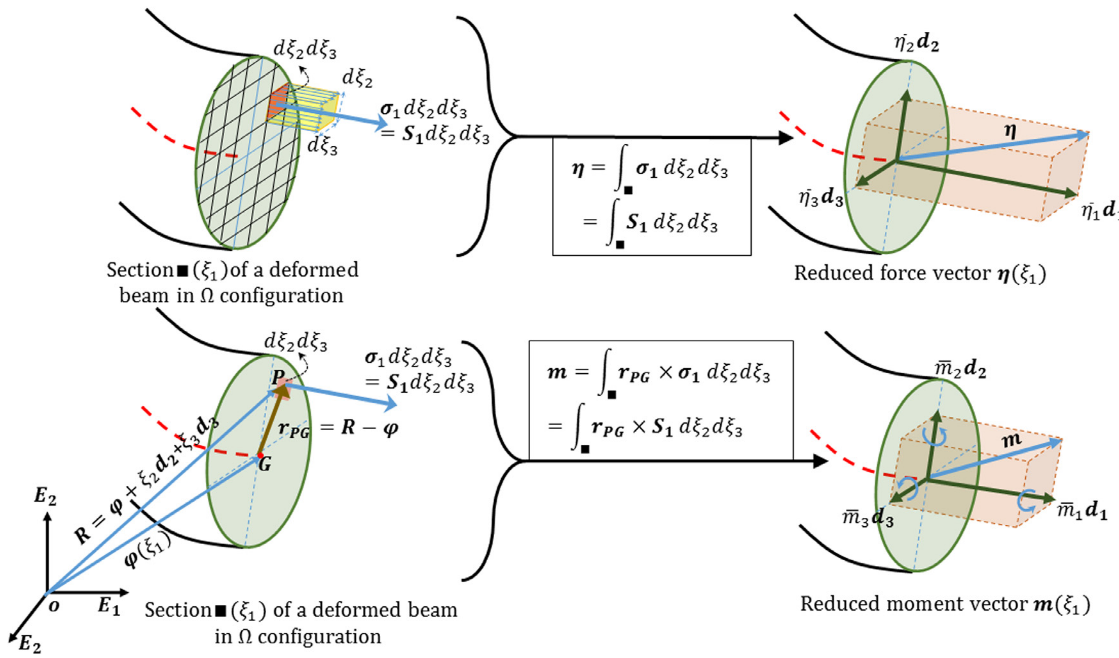

Fig. 6. Reduced force $\boldsymbol{\eta}$ and moment $\boldsymbol{m}$.

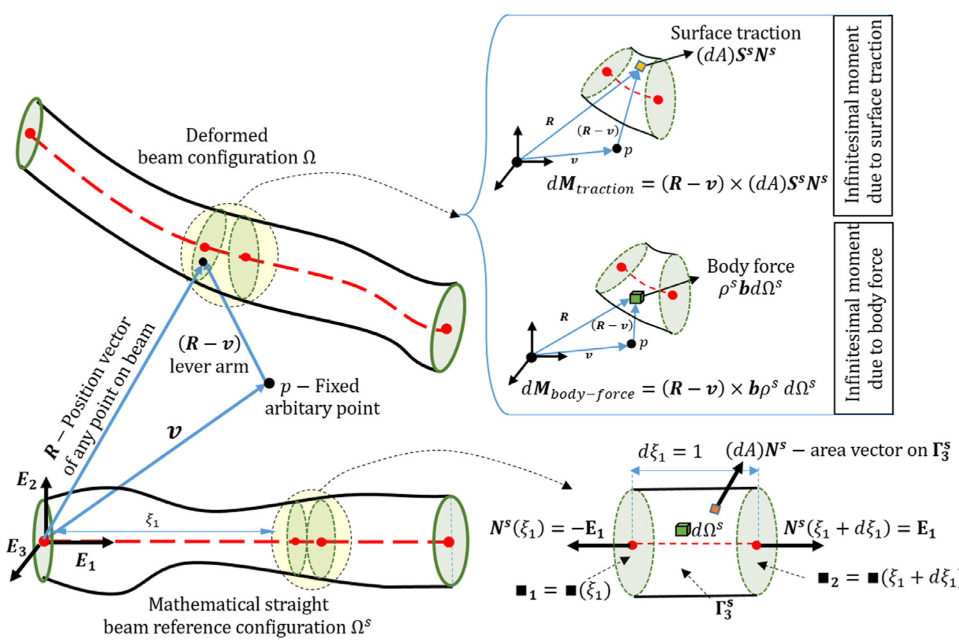

Fig. 7. Reduced element of unit arc-length of initially straight beam and incremental moment about an arbitrary fixed point-p.

Fig. 7 gives physical interpretation of terms in Eqs. (62) and (63). It also shows reduced element with $d\xi_1 = 1$ in Ω^s configuration, from which, the stress vectors at the cross-sectional boundaries $\blacksquare_1 = \blacksquare(\xi_1)$ and $\blacksquare_2 = \blacksquare(\xi_1 + d\xi_1)$ are

$$\begin{aligned} [\boldsymbol{S}^s\boldsymbol{N}^s]_{\blacksquare_1} &= \boldsymbol{S}^s\boldsymbol{N}^s(\xi_1) = -\boldsymbol{S}^s\boldsymbol{E}_1 = -\boldsymbol{S}_1, \\ [\boldsymbol{S}^s\boldsymbol{N}^s]_{\blacksquare_2} &= \boldsymbol{S}^s\boldsymbol{N}^s(\xi_1 + d\xi_1) = \boldsymbol{S}^s\boldsymbol{E}_1 = \boldsymbol{S}_1. \end{aligned} \tag{64}$$

4.1. Strong form referenced to initially straight configuration

To obtain the reduced governing differential equation that holds at every point ξ_1 on the midcurve, we exploit the fact that the conservation equations (62) and (63) obtained for the entire beam are also valid for the reduced element of the beam (Fig. 7), since equilibrium of the structure as a whole implies the equilibrium of a reduced element in Ω^s.

4.1.1. Conservation of linear momentum of the reduced beam

Like Eq. (62), the linear momentum conservation equation for the reduced unit arc-length element (Fig. 7) is obtained as

$$\overbrace{\int_{\blacksquare_1} \boldsymbol{S}^s\boldsymbol{N}^s(\xi_1) d\xi_2 d\xi_3 + \int_{\blacksquare_2} \boldsymbol{S}^s\boldsymbol{N}^s(\xi_1 + d\xi_1) d\xi_2 d\xi_3}^{\substack{\text{Term F1:} \\ \text{The reduced internal force at the cross-sectional boundary} \\ \blacksquare_1 \text{ and } \blacksquare_2 \text{ referred to unit arc-length reduced element.}}}$$
$$+ \overbrace{\int_{\Gamma_3^s} \boldsymbol{S}^s\boldsymbol{N}^s d\Gamma^s + \int_{\Omega^s} \rho^s \boldsymbol{b} d\Omega^s}^{\substack{\text{Term F2:} \\ \text{The reduced external force due to} \\ \text{body force and surface traction.}}} = \overbrace{\int_{\Omega^s} \rho^s \ddot{\boldsymbol{R}} d\Omega^s}^{\substack{\text{Term F3:} \\ \text{Inertial force term.}}}. \tag{65}$$

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For the domain of unit arc-length reduced element, the volume integral of any function $\boldsymbol{\Psi}(\xi_1, \xi_2, \xi_3)$ would become integral over the cross section $\blacksquare(\xi_1)$ as

$$\lim_{d\xi_1\to 1}\int_{\Omega^s}\boldsymbol{\Psi}(\xi_1,\xi_2,\xi_3)d\Omega^s = \int_{\blacksquare(\xi_1)}\boldsymbol{\Psi}(\xi_1,\xi_2,\xi_3)d\xi_2 d\xi_3. \tag{66}$$

Term F1 may be simplified using Eqs. (57), (59) and (64) and Term F2 may be simplified using Eq. (66) as

$$\text{Term F1} = \lim_{d\xi_1\to 1}\left[\eta(\xi + d\xi_1) - \eta(\xi_1)\right] = \eta_{,\xi_1}; \tag{67}$$

$$\text{Term F2} = \int_{\Gamma_3^s}\boldsymbol{S}^s\boldsymbol{N}^s d\Gamma^s + \int_{\blacksquare}\rho^s\boldsymbol{b}\, d\xi_2 d\xi_3 = \aleph(\xi_1). \tag{68}$$

Term F3 involves total time derivative $\ddot{\boldsymbol{R}} = \frac{D^2\boldsymbol{R}}{Dt^2}$ that may be obtained using Eqs. (3) and (11) as

$$\dot{\boldsymbol{R}}(\xi_1,\xi_2,\xi_3) = \dot{\boldsymbol{u}} = \frac{D\boldsymbol{R}}{Dt} = \dot{\boldsymbol{\varphi}}(\xi_1) + \boldsymbol{\omega}(\xi_1)\times\boldsymbol{r}_{PG},$$
$$\ddot{\boldsymbol{R}}(\xi_1,\xi_2,\xi_3) = \frac{D^2\boldsymbol{R}}{Dt^2} = \ddot{\boldsymbol{\varphi}}(\xi_1) + \dot{\boldsymbol{\omega}}(\xi_1) \times \boldsymbol{r}_{PG} + \boldsymbol{\omega}(\xi_1)\times\boldsymbol{\omega}(\xi_1)\times\boldsymbol{r}_{PG}. \tag{69}$$

Here the vector $\boldsymbol{\omega}(\xi_1)$ represent the axial vector corresponding to the anti symmetric tensor $\boldsymbol{W}(\xi_1)$ that deals with change of director with time (Eq. (11)). In other words, $\boldsymbol{\omega}(\xi_1)$ and $\dot{\boldsymbol{\omega}}(\xi_1)$ represents the rotational velocity and rotational acceleration of the beam cross-section respectively. Therefore, Term F3 can be obtained using the result (66) and (69) as,

$$\text{Term F3} = \mu^s\ddot{\boldsymbol{\varphi}} + \dot{\boldsymbol{\omega}}\times\Upsilon^s + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\Upsilon^s, \tag{70}$$

where

$$\mu^s(\xi_1) = \int_{\blacksquare}\rho^s d\xi_2 d\xi_3, \tag{71}$$

$$\boldsymbol{\Upsilon}^s(\xi_1) = \int_{\blacksquare}\rho^s\boldsymbol{r}_{PG}d\xi_2 d\xi_3 = \underbrace{\left[\int_{\blacksquare}\rho^s\xi_2 d\xi_2 d\xi_3\right]}_{\Upsilon_3^s}\boldsymbol{d_2} + \underbrace{\left[\int_{\blacksquare}\rho^s\xi_3 d\xi_2 d\xi_3\right]}_{\Upsilon_2^s}\boldsymbol{d_3}. \tag{72}$$

It is clear that the first term ($\mu^s\ddot{\boldsymbol{\varphi}}$) of Eq. (70) represents the inertial force acting at the midcurve point G (Fig. 1) on $\blacksquare(\xi_1)$. The term μ^s represents the mass density per unit arc length in the initially straight configuration Ω^s. The occurrence of second term is because of the fact that, in general the midcurve may not coincide with the mass centroid. The terms Υ_2^s and Υ_3^s represent the *first mass moment of inertia* per unit arc length of the straight beam configuration Ω^s about the director%% $\boldsymbol{d}_2$ (or $\boldsymbol{E}_2$) and $\boldsymbol{d}_3$ (or $\boldsymbol{E}_3$), respectively. These terms would vanish for the untwisted straight beam Ω^s of homogeneous material if the beam midcurve is chosen as the loci of mass centroids, which in this case would coincide with the geometric centroids. If the initial configuration of the beam were curved Ω^c, these terms would vanish only if the mass centroid were chosen as the midcurve, as in this case the loci of geometric centroids may not coincide with the mass centroids.

Combining Eqs. (65)–(72) gives the reduced linear momentum conservation equation of the moving beam at section $\blacksquare(\xi_1)$ referred to the initially straight configuration Ω^s as

$$\boldsymbol{\eta}_{,\xi_1} + \aleph(\xi_1) = \mathfrak{F}^s(\xi_1); \tag{73}$$

where

$$\mathfrak{F}^s(\xi_1) = \mu^s\ddot{\boldsymbol{\varphi}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\Upsilon}^s) + \dot{\boldsymbol{\omega}}\times\boldsymbol{\Upsilon}^s$$

represents reduced inertial force per unit arc length referenced to the straight configuration Ω^s.

4.1.2. Conservation of angular momentum of the reduced beam

Like Eq. (63), the angular momentum conservation for the unit arc-length reduced element (Fig. 7) can be written as

$$\overbrace{\int_{\blacksquare_1}(\boldsymbol{R}-\boldsymbol{v})\times\left(\boldsymbol{S}^s\boldsymbol{N}^s\right)d\xi_2 d\xi_3 + \int_{\blacksquare_2}(\boldsymbol{R}-\boldsymbol{v})\times\left(\boldsymbol{S}^s\boldsymbol{N}^s\right)d\xi_2 d\xi_3}^{\substack{\text{Term M1:}\\ \text{The reduced internal moment at the cross-sectional boundary } \blacksquare_1 \text{ and } \blacksquare_2\\ \text{referred to unit arc-length reduced element about a fixed arbitrary point } p.}}$$
$$+ \underbrace{\int_{\Gamma_3^s}(\boldsymbol{R}-\boldsymbol{v})\times\left(\boldsymbol{S}^s\boldsymbol{N}^s\right)d\Gamma^s + \int_{\Omega^s}\rho^s(\boldsymbol{R}-\boldsymbol{v})\times\boldsymbol{b}d\Omega^s}_{\substack{\text{Term M2:}\\ \text{The reduced external moment about a fixed arbitrary}\\ \text{point } p \text{ due to the body force and surface traction.}}}$$
$$= \underbrace{\int_{\Omega^s}\rho^s(\boldsymbol{R}-\boldsymbol{v})\times\ddot{\boldsymbol{R}}d\Omega^s}_{\substack{\text{Term M3:}\\ \text{Inertial term corresponding to}\\ \text{moment about point } p.}}. \tag{74}$$

It is sensible to define the moment about the midcurve such that the lever arm is $\boldsymbol{r}_{PG} = (\boldsymbol{R}-\boldsymbol{\varphi})$, for an arbitrary fixed point p in space. Therefore, from the definition of reduced force and moment as in Eqs. (59) and (60), and using the result in Eq. (64), Term M1 may be simplified as

$$\begin{aligned}\text{Term M1} &= \sum_{k=1}^{2}\left[\int_{\blacksquare_k}(\boldsymbol{R}-\boldsymbol{\varphi})\times(\boldsymbol{S}^s\boldsymbol{N}^s)d\xi_2 d\xi_3 + \int_{\blacksquare_k}(\boldsymbol{\varphi}-\boldsymbol{v})\times(\boldsymbol{S}^s\boldsymbol{N}^s)d\xi_2 d\xi_3\right]\\ &= \lim_{d\xi_1\to 1}[\boldsymbol{m}(\xi_1 + d\xi_1) - \boldsymbol{m}(\xi_1)]\\ &\quad + \lim_{d\xi_1\to 1}\left[\int_{\blacksquare_2}(\boldsymbol{\varphi}-\boldsymbol{v})\times\boldsymbol{S}_1 d\xi_2 d\xi_3 - \int_{\blacksquare_1}(\boldsymbol{\varphi}-\boldsymbol{v})\times\boldsymbol{S}_1 d\xi_2 d\xi_3\right]\\ &= \boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1}\times\boldsymbol{\eta} + (\boldsymbol{\varphi}-\boldsymbol{v})\times\boldsymbol{\eta}_{,\xi_1}.\end{aligned} \tag{75}$$

For a unit arc-length reduced element, Term M2 and Term M3 may be simplified using Eq. (66) as

$$\begin{aligned}\text{Term M2} &= \int_{\Gamma_3^s}(\boldsymbol{R}-\boldsymbol{v})\times(\boldsymbol{S}^s\boldsymbol{N}^s)d\Gamma^s + \int_{\blacksquare}\rho^s(\boldsymbol{R}-\boldsymbol{v})\times\boldsymbol{b}d\xi_2 d\xi_3\\ &= \mathfrak{M}(\xi_1) + \int_{\Gamma_3^s}(\boldsymbol{\varphi}-\boldsymbol{v})\times(\boldsymbol{S}^s\boldsymbol{N}^s)d\Gamma^s\\ &\quad + \int_{\blacksquare}\rho^s(\boldsymbol{\varphi}-\boldsymbol{v})\times\boldsymbol{b}d\xi_2 d\xi_3,\end{aligned} \tag{76}$$

where

$$\mathfrak{M}(\xi_1) = \int_{\Gamma_3^s}(\boldsymbol{R}-\boldsymbol{\varphi})\times(\boldsymbol{S}^s\boldsymbol{N}^s)d\Gamma^s + \int_{\blacksquare}\rho^s(\boldsymbol{R}-\boldsymbol{\varphi})\times\boldsymbol{b}d\xi_2 d\xi_3$$

represents the reduced moment due to surface traction on peripheral boundary Γ_3^s and body force about the midcurve point G on $\blacksquare(\xi_1)$. Similarly,

$$\text{Term M3} = \overbrace{\int_{\blacksquare}\rho^s(\boldsymbol{R}-\boldsymbol{\varphi})\times\ddot{\boldsymbol{R}}d\xi_2 d\xi_3}^{\text{Term M3a}} + \int_{\blacksquare}\rho^s(\boldsymbol{\varphi}-\boldsymbol{v})\times\ddot{\boldsymbol{R}}d\xi_2 d\xi_3. \tag{77}$$

Term M3a represents the reduced moment due to the inertial force about point G on $\blacksquare(\xi_1)$. To simplify Term M3, consider that the vector $\boldsymbol{r}_{PG}$ to be the axial vector corresponding to the anti symmetric tensor $\widetilde{\boldsymbol{R}}_{PG}$, such that for any vector $\boldsymbol{g} = \bar{g}_i\boldsymbol{d_i}$, $\widetilde{\boldsymbol{R}}_{PG}\boldsymbol{g} = \boldsymbol{r}_{PG}\times\boldsymbol{g}$. Noting the expression for $\ddot{\boldsymbol{R}}$ and $\boldsymbol{Y}^s$ as in Eqs. (69) and (72), Term

M3a becomes

$$
\begin{aligned}
\text{Term M3a} &= \int_{\blacksquare} \rho^s \big(\boldsymbol{r}_{PG} \times \ddot{\boldsymbol{R}}\big) d\xi_2 d\xi_3 \\
&= \Upsilon^s \times \ddot{\boldsymbol{\varphi}} - \int_{\blacksquare} \rho^s \boldsymbol{r}_{PG} \times (\boldsymbol{r}_{PG} \times \dot{\boldsymbol{\omega}}) d\xi_2 d\xi_3 \\
&\quad + \int_{\blacksquare} \rho^s \boldsymbol{r}_{PG} \times \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{PG}) d\xi_2 d\xi_3 \\
&= \Upsilon^s \times \ddot{\boldsymbol{\varphi}} - \int_{\blacksquare} \rho^s \widetilde{\boldsymbol{R}}_{PG} \widetilde{\boldsymbol{R}}_{PG} \dot{\boldsymbol{\omega}} d\xi_2 d\xi_3 \\
&\quad - \int_{\blacksquare} \rho^s \boldsymbol{\omega} \times \big(\widetilde{\boldsymbol{R}}_{PG} \widetilde{\boldsymbol{R}}_{PG} \boldsymbol{\omega}\big) d\xi_2 d\xi_3 \\
&= \Upsilon^s \times \ddot{\boldsymbol{\varphi}} + \left\{ \int_{\blacksquare} \rho^s \widetilde{\boldsymbol{R}}^T_{PG} \widetilde{\boldsymbol{R}}_{PG} d\xi_2 d\xi_3 \right\} \dot{\boldsymbol{\omega}} \\
&\quad + \boldsymbol{\omega} \times \left\{ \int_{\blacksquare} \rho^s \widetilde{\boldsymbol{R}}^T_{PG} \widetilde{\boldsymbol{R}}_{PG} d\xi_2 d\xi_3 \right\} \boldsymbol{\omega} \\
&= \Upsilon^s \times \ddot{\boldsymbol{\varphi}} + \boldsymbol{I}^s_{\boldsymbol{M}} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}^s_{\boldsymbol{M}} \boldsymbol{\omega},
\end{aligned} \tag{78}
$$

where

$$
\boldsymbol{I}^s_{\boldsymbol{M}}(\xi_1) = \int_{\blacksquare} \rho^s \big(\widetilde{\boldsymbol{R}}^T_{PG} \widetilde{\boldsymbol{R}}_{PG}\big) d\xi_2 d\xi_3 = \int_{\blacksquare} \rho^s \begin{bmatrix} \xi_2^2 + \xi_3^2 & 0 & 0 \\ 0 & \xi_3^2 & -\xi_2\xi_3 \\ 0 & -\xi_2\xi_3 & \xi_2^2 \end{bmatrix} d\xi_2 d\xi_3; \tag{79}
$$

$$
\widetilde{\boldsymbol{R}}_{PG} = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & 0 \\ -\xi_2 & 0 & 0 \end{bmatrix}. \tag{80}
$$

The quantity $\boldsymbol{I}^s_{\boldsymbol{M}}$ is the second mass moment of inertia tensor per unit arc length of the straight configuration Ω^s; it is associated with the distribution of mass across the cross section. The vector $(\boldsymbol{\varphi} - \boldsymbol{v})$ is independent of the parameters ξ_2 and ξ_3. Therefore, combining Eqs. (74)–(80) we get

$$
\begin{aligned}
&\overbrace{\boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1} \times \eta + \mathfrak{M} - (\boldsymbol{\Upsilon}^s \times \ddot{\boldsymbol{\varphi}} + \boldsymbol{I}^s_{\boldsymbol{M}} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}^s_{\boldsymbol{M}} \boldsymbol{\omega})}^{\text{Equation M1}} \\
&\quad + \underbrace{(\boldsymbol{\varphi} - \boldsymbol{v}) \times \left[\boldsymbol{\eta}_{,\xi_1} + \int_{\Gamma^s_3} \boldsymbol{S}^s \boldsymbol{N}^s d\Gamma^s + \int_{\blacksquare} \rho^s \boldsymbol{b} d\xi_2 d\xi_3 - \int_{\blacksquare} \rho^s \ddot{\boldsymbol{R}} d\xi_2 d\xi_3 \right]}_{\text{Equation M2}} \\
&= 0.
\end{aligned} \tag{81}
$$

It is clear that term Equation M2 contains terms consisting of $\boldsymbol{\varphi} - \boldsymbol{v}$, which must vanish in order to obtain angular momentum balance law with respect to moment taken about the point G on $\blacksquare(\xi_1)$. It is clear from the linear momentum conservation equation (73) that the term Equation M2 vanishes. Therefore, the reduced strong form of angular momentum conservation of the Cosserat beam, referenced to Ω^s is given as

$$
\boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1} \times \eta + \mathfrak{M} = \boldsymbol{\lambda}^s, \tag{82}
$$

where

$$
\boldsymbol{\lambda}^s(\xi_1) = \Upsilon^s \times \ddot{\boldsymbol{\varphi}} + \boldsymbol{I}^s_{\boldsymbol{M}} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}^s_{\boldsymbol{M}} \boldsymbol{\omega}
$$

represents reduced moment about point G on cross-section $\blacksquare(\xi_1)$ due to inertial force per unit arc length referenced to the straight configuration Ω^s.

4.2. Conservation laws of the reduced beam referenced to initially curved configuration

To derive the balance law referenced to Ω^c we need to transform the limits of the integrals in the strong form obtained in previous section to the configuration Ω^c. Consider that the unit arc-length reduced curved beam element is defined by the boundary $\Gamma^c_3 \cup \blacksquare_1 \cup \blacksquare_2$ in Ω^c configuration similar to the element defined in Fig. 7. To proceed further, it is required to establish a relation between the stress tensors $\boldsymbol{S}^s$ and $\boldsymbol{S}^c$. The relationship between $\boldsymbol{\sigma}$, $\boldsymbol{S}^s$ and $\boldsymbol{S}^c$ as referred from any standard continuum mechanics text like, Lai et al. (2010) leads to

$$
\boldsymbol{\sigma} = \frac{1}{|\boldsymbol{F}^r|} \boldsymbol{S}^c \boldsymbol{F}^{r^T} = \frac{1}{|\boldsymbol{F}|} \boldsymbol{S}^s \boldsymbol{F}^{s^T}, \tag{83}
$$

$$
\boldsymbol{S}^s = |\boldsymbol{F}^c| \boldsymbol{S}^c \boldsymbol{F}^{c^{-T}}. \tag{84}
$$

The area vector on the surface boundary $\boldsymbol{N}^s d\Gamma^s$ and $\boldsymbol{N}^c d\Gamma^c$ in the configurations Ω^s and Ω^c, respectively, is related by Nanson's relation as

$$
\boldsymbol{N}^s d\Gamma^s = \frac{1}{|\boldsymbol{F}^c|} \boldsymbol{F}^{c^T} \boldsymbol{N}^c d\Gamma^c. \tag{85}
$$

Using Eqs. (84) and (85) and the result in Eq. (33), the reduced linear momentum conservation equation referenced to the curved configuration Ω^c are obtained as,

$$
\boldsymbol{\eta}_{,\xi_1} + \boldsymbol{\aleph}(\xi_1) = \mathfrak{F}^{\boldsymbol{c}}(\xi_1), \tag{86}
$$

where,

$$
\boldsymbol{\aleph} = \int_{\Gamma^c_3} \boldsymbol{S}^c \boldsymbol{N}^c d\Gamma^c + \int_{\blacksquare} |\boldsymbol{F}^c| \rho^c \boldsymbol{b} d\xi_2 d\xi_3; \tag{87}
$$

$$
\mathfrak{F}^c(\xi_1) = \mu^c \ddot{\boldsymbol{\varphi}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\Upsilon}^c) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\Upsilon}^c; \tag{88}
$$

$$
\mu^c = \int_{\blacksquare} |\boldsymbol{F}^c| \rho^c d\xi_2 d\xi_3; \tag{89}
$$

$$
\Upsilon^c = \left\{ \int_{\blacksquare} |\boldsymbol{F}^c| \rho^c \xi_2 d\xi_2 d\xi_3 \right\} \boldsymbol{d}_2 + \left\{ \int_{\blacksquare} |\boldsymbol{F}^c| \rho^c \xi_3 d\xi_2 d\xi_3 \right\} \boldsymbol{d}_3. \tag{90}
$$

Similarly, the reduced angular momentum conservation equation referenced to Ω^c has similar form as Eq. (82), such that,

$$
\boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1} \times \boldsymbol{\eta} + \mathfrak{M} = \boldsymbol{\lambda}^c(\xi_1), \tag{91}
$$

where,

$$
\mathfrak{M} = \int_{\Gamma^c_3} \boldsymbol{r}_{PG} \times (\boldsymbol{S}^c \boldsymbol{N}^c) d\Gamma^c + \int_{\blacksquare} |\boldsymbol{F}^c| \rho^c (\boldsymbol{r}_{PG} \times \boldsymbol{b}) d\xi_2 d\xi_3; \tag{92}
$$

$$
\boldsymbol{\lambda}^c(\xi_1) = \boldsymbol{\Upsilon}^c \times \ddot{\boldsymbol{\varphi}} + \boldsymbol{I}^c_{\boldsymbol{M}} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{I}^c_{\boldsymbol{M}} \boldsymbol{\omega}); \tag{93}
$$

$$
\boldsymbol{I}^c_{\boldsymbol{M}} = \int_{\blacksquare} \rho^c \big(\widetilde{\boldsymbol{R}}^T_{PG} \widetilde{\boldsymbol{R}}_{PG}\big) d\xi_2 d\xi_3. \tag{94}
$$

The parameter $\boldsymbol{Y}^c$ defines the first mass moment vector and $\boldsymbol{I}^c_{\boldsymbol{M}}$ defines the second mass moment of inertia tensor per unit arc length of the curved reference configuration Ω^c.

5. Weak form of reduced balance law for Cosserat beam

5.1. Weak form from Lagrangian differential equation of motion

To obtain the weak form of equilibrium equation, we imparted the object in the current state Ω with an admissible but arbitrary virtual displacement field $\delta\boldsymbol{u}$ given by Eq. (46). It is clear that $\delta\boldsymbol{u}$ comprises of the virtual displacement of the midcurve (translation) $\delta\boldsymbol{\varphi}$ and a component due to virtual rotation of the frame of reference, parametrized by $\delta\boldsymbol{\alpha}$ as explained in Section 3.3. From the definition of $\boldsymbol{F}$ and $\boldsymbol{u}$ following results hold,

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{I}_3 + \nabla_{\Omega^s} \boldsymbol{u}; \\
\delta\boldsymbol{F} &= \nabla_{\Omega^s} \delta\boldsymbol{u}.
\end{aligned} \tag{95}
$$

The point-wise equilibrium equation Eq. (61) can be written in an integral (weak or scalar or residual) form as

$$\int_{\Omega^s} \delta \boldsymbol{u}.\left(\nabla_{\Omega^s}.\boldsymbol{S}^s + \rho^s \boldsymbol{b} - \rho^s \ddot{\boldsymbol{R}}\right) d\Omega^s = 0. \tag{96}$$

Using divergence theorem on the equation above, followed by substitution of Eq. (95) yields,

$$-\overbrace{\int_{\Omega^s} \delta \boldsymbol{F} : \boldsymbol{S}^s d\Omega^s}^{\text{Term A}} + \overbrace{\int_{\Gamma^s} \delta \boldsymbol{u}.(\boldsymbol{S}^s \boldsymbol{N}^s) d\Gamma^s}^{\text{Term B}} + \overbrace{\int_{\Omega^s} \delta \boldsymbol{u}.\boldsymbol{b} d\Gamma^s}^{\text{Term C}}$$
$$- \overbrace{\int_{\Omega^s} \rho^s \delta \boldsymbol{u}.\ddot{\boldsymbol{R}} d\Gamma^s}^{\text{Term D}} = 0; \tag{97}$$

Note that unlike the strong form, the weak form considers the equilibrium of the structure as a whole (in integral sense). Therefore, for any function $\boldsymbol{\Psi}(\xi_1, \xi_2, \xi_3)$ the volume integrals can be written as,

$$\int_{\Omega^s} \boldsymbol{\Psi}(\xi_1, \xi_2, \xi_3) d\Omega^s = \int_0^L \left[\int_{\blacksquare} \boldsymbol{\Psi} d\xi_2 d\xi_3 \right] d\xi_1. \tag{98}$$

5.1.1. Term A: virtual strain energy

Term A represents the virtual strain energy stored in the beam. The result of the virtual strain energy in Eq. (97) is not surprising as the stress conjugate to the first PK stress tensor is the deformation gradient tensor. Using the expression for $\delta \boldsymbol{F}$ and $\tilde{\delta} \boldsymbol{F}$ obtained in Eq. (55), *Term A* can be simplified as,

$$\text{Term A} = \int_{\Omega^s} \delta \boldsymbol{F} : \boldsymbol{S}^s d\Omega^s$$
$$= \overbrace{\int_{\Omega^s} (\delta \mathfrak{A}.\boldsymbol{F}) : \boldsymbol{S}^s d\Omega^s}^{\text{Term A1}} + \overbrace{\int_{\Omega^s} (\tilde{\delta} \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1) : \boldsymbol{S}^s d\Omega^s}^{\text{Term A2}}. \tag{99}$$

Note that *Term A1*, represents the virtual strain energy stored due to variation in the director frame, which is purely due to virtual rigid body rotation (not strain!). Hence, using Eqs. (83) and (84), and noting that the scalar product between symmetric and anti-symmetric tensor is zero $(\boldsymbol{\sigma} : \delta \mathfrak{A} = 0)$, it can be shown that *Term A1* vanishes as,

$$\begin{aligned}\text{Term A1} &= \int_{\Omega^s} (\delta \mathfrak{A}.\boldsymbol{F}) : \boldsymbol{S}^s d\Omega^s = \int_{\Omega^s} \text{trace}[\boldsymbol{S}^s (\delta \mathfrak{A}.\boldsymbol{F})^T] d\Omega^s \\ &= \int_{\Omega^s} \text{trace}[|\boldsymbol{F}| \boldsymbol{\sigma} \boldsymbol{F}^{-T} (\delta \mathfrak{A}.\boldsymbol{F})^T] d\Omega^s \\ &= |\boldsymbol{F}| \int_{\Omega^s} \text{trace}[\boldsymbol{\sigma} \boldsymbol{F}^{-T} \boldsymbol{F}.\delta \mathfrak{A}] d\Omega^s \\ &= |\boldsymbol{F}| \int_{\Omega^s} \boldsymbol{\sigma} : \delta \mathfrak{A} d\Omega^s = 0.\end{aligned} \tag{100}$$

Therefore, the virtual strain energy of the beam reduces to *Term A2*. It can be simplified using the definition of $\tilde{\delta} \boldsymbol{\varepsilon}$ as in Eq. (56) and the result in Eq. (98) as,

$$\begin{aligned}\text{Term A2} &= \int_{\Omega^s} \left(\tilde{\delta} \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1\right)_{ij} S^s_{ij} d\Omega^s = \int_{\Omega^s} \tilde{\delta} \epsilon_i S^s_{ij} E_{1j} d\Omega^s \\ &= \int_{\Omega^s} \tilde{\delta} \boldsymbol{\epsilon}.\boldsymbol{S}^s_1 d\Omega^s \\ &= \int_0^L \left[\int_{\blacksquare} \boldsymbol{S}_1.\tilde{\delta} \boldsymbol{\varepsilon}\; d\xi_2 d\xi_3 \right] d\xi_1 \\ &\quad + \int_0^L \left[\int_{\blacksquare} \boldsymbol{S}_1.\left[\tilde{\delta} \boldsymbol{\kappa} \times \boldsymbol{r}_{PG}\right] d\xi_2 d\xi_3 \right] d\xi_1.\end{aligned} \tag{101}$$

Noticing that $\tilde{\delta} \boldsymbol{\varepsilon}$ and $\tilde{\delta} \boldsymbol{\kappa}$ are independent of ξ_2 and ξ_3 and using the property in Eq. (48), *Term A2* simplifies as,

$$\begin{aligned}\text{Term A2} &= \int_0^L (\boldsymbol{\eta}.\tilde{\delta} \boldsymbol{\varepsilon} + \boldsymbol{m}.\tilde{\delta} \boldsymbol{\kappa}) d\xi_1 \\ &= \int_0^L (\boldsymbol{\eta}.(\boldsymbol{Q} \delta \overline{\boldsymbol{\varepsilon}}) + \boldsymbol{m}.(\boldsymbol{Q} \delta \overline{\boldsymbol{\kappa}})) d\xi_1 \\ &= \int_0^L \left(\overline{\boldsymbol{\eta}}.\delta \overline{\boldsymbol{\varepsilon}} + \overline{\boldsymbol{m}}.\delta \overline{\boldsymbol{\kappa}}\right) d\xi_1.\end{aligned} \tag{102}$$

It is noteworthy that the strain energy density $(\delta \boldsymbol{F} : \boldsymbol{S}^s = \boldsymbol{S}_1.\tilde{\delta} \boldsymbol{\epsilon})$, is contributed solely by the stress vector $\boldsymbol{S}_1$. This is because the cross-sections are assumed to be rigid. Secondly, the strain conjugate of reduced force vector $\boldsymbol{\eta}$ and reduced couple $\boldsymbol{m}$ is the co-rotated variance of the virtual midcurve strain $\tilde{\delta} \boldsymbol{\varepsilon}$ and the co-rotated variance of the virtual rotation of the director frame $\tilde{\delta} \boldsymbol{\kappa}$. Thus, we infer that the virtual strain energy is only contributed because of the variation in the components of the strain vector (related to co-rotated virtual quantities).

5.1.2. Term B and Term C: virtual external work due to surface tractions and body forces

Term B represents the total external virtual work due to traction on the boundary of the beam. The boundary of the entire beam in Ω^s consists of two cross-sectional boundaries $\blacksquare(0)$ and $\blacksquare(L)$ and the lateral surface of the beam. External virtual work due to traction in the reduced element of unit arc length in Fig. 7 can be summed over the entire length to give *Term B*. Referring Eq. (64),

$$\text{Term B} = \overbrace{\int_0^L \left[\int_{\blacksquare_2} \delta \boldsymbol{u}.\boldsymbol{S}^s_1 d\xi_2 d\xi_3 - \int_{\blacksquare_1} \delta \boldsymbol{u}.\boldsymbol{S}^s_1 d\xi_2 d\xi_3 \right] d\xi_1}^{\text{Term B1}}$$
$$+ \overbrace{\int_0^L \left[\int_{\Gamma^s_3} \delta \boldsymbol{u}.(\boldsymbol{S}^s \boldsymbol{N}^s) d\xi_2 d\xi_3 \right] d\xi_1}^{\text{Term B2}} \tag{103}$$

Term B1 represents the virtual work at the end boundary of beams. All the reduced element of unit arc length $(d\xi_1 \to 1)$ club together to give the entire beam, such that the external work due to traction at the cross-sectional boundary is only because of end boundaries $(\blacksquare(0)$ and $\blacksquare(L))$ of the beam. Substituting for $\delta \boldsymbol{u}$ as in Eq. (46) into *Term B1*, we get

$$\begin{aligned}\text{Term B1} &= \int_0^L \left[\delta \boldsymbol{\varphi}(\xi_1 + d\xi_1) \cdot \int_{\blacksquare_2} \boldsymbol{S}^s_1 d\xi_2 d\xi_3 \right. \\ &\qquad \left. - \delta \boldsymbol{\varphi}(\xi_1) \cdot \int_{\blacksquare_1} \boldsymbol{S}^s_1 d\xi_2 d\xi_3 \right] d\xi_1 \\ &\qquad + \int_0^L \left[\delta \boldsymbol{\alpha}(\xi_1 + d\xi_1) \times \int_{\blacksquare_2} \boldsymbol{r}_{PG} \times \boldsymbol{S}^s_1 d\xi_2 d\xi_3 \right. \\ &\qquad \left. - \delta \boldsymbol{\alpha}(\xi_1) \cdot \int_{\blacksquare_1} \boldsymbol{r}_{PG} \times \boldsymbol{S}^s_1 d\xi_2 d\xi_3 \right] d\xi_1 \\ &= \int_0^L [\delta \boldsymbol{\varphi}(\xi_1 + d\xi_1) \cdot \boldsymbol{\eta}(\xi_1 + d\xi_1) \\ &\qquad - \delta \boldsymbol{\varphi}(\xi_1 + d\xi_1) \cdot \boldsymbol{\eta}(\xi_1 + d\xi_1)] d\xi_1 \\ &\qquad + \int_0^L [\delta \boldsymbol{\alpha}(\xi_1 + d\xi_1) \cdot \boldsymbol{m}(\xi_1 + d\xi_1) \\ &\qquad - \delta \boldsymbol{\alpha}(\xi_1 + d\xi_1) \cdot \boldsymbol{m}(\xi_1 + d\xi_1)] d\xi_1 \\ &= \int_0^L (\delta \boldsymbol{\varphi} \cdot \boldsymbol{\eta} + \delta \boldsymbol{\alpha} \cdot \boldsymbol{m})_{,\xi_1} d\xi_1 \\ &= [\delta \boldsymbol{\varphi} \cdot \boldsymbol{\eta}]_{\xi_1=0}^{\xi_1=L} + [\delta \boldsymbol{\alpha} \cdot \boldsymbol{m}]_{\xi_1=0}^{\xi_1=L}.\end{aligned} \tag{104}$$

Term B2 represents the virtual work due to traction on the lateral surface of the beam. This is simplified as,

$$\begin{aligned}\text{Term B2} &= \int_0^L \int_{\Gamma_3^s} (\delta\boldsymbol{\varphi} + \delta\boldsymbol{\alpha} \times \boldsymbol{r}_{PG}) \cdot (\boldsymbol{S}^s\boldsymbol{N}^s)\ d\Gamma_3^s d\xi_1 \\ &= \int_0^L \left\{ \delta\boldsymbol{\varphi} \cdot \int_{\Gamma_3^s} \boldsymbol{S}^s\boldsymbol{N}^s d\Gamma_3^s \right\} d\xi_1 \\ &\quad + \int_0^L \left\{ \delta\boldsymbol{\alpha} \cdot \int_{\Gamma_3^s} (\boldsymbol{r}_{PG} \times \boldsymbol{S}^s\boldsymbol{N}^s) d\Gamma_3^s \right\} d\xi_1. \end{aligned} \tag{105}$$

The total external virtual work due to body force can be simplified as,

$$\begin{aligned}\text{Term C} &= \int_0^L \int_{\blacksquare} \rho^s(\delta\boldsymbol{u}.\boldsymbol{b})\ d\xi_2 d\xi_3 d\xi_1 \\ &= \int_0^L \delta\boldsymbol{\varphi} \cdot \left[\int_{\blacksquare} \rho^s\boldsymbol{b}\ d\xi_2 d\xi_3 \right] d\xi_1 \\ &\quad + \int_0^L \delta\boldsymbol{\alpha} \cdot \left[\int_{\blacksquare} \rho^s(\boldsymbol{r}_{PG} \times \boldsymbol{b})\ d\xi_2 d\xi_3 \right] d\xi_1. \end{aligned} \tag{106}$$

Term B and *Term C* combined together gives the virtual work due to the external force (body force and surface traction). Therefore, using the definition of $\boldsymbol{\aleph}$ and $\mathfrak{M}$ as defined in Eqs. (68) and (76), *Term B* and *Term C* can be clubbed as,

$$\begin{aligned}\text{Term B} + \text{Term C} &= \left[\delta\boldsymbol{\varphi}.\boldsymbol{\eta}\right]_{\xi_1=0}^{\xi_1=L} + \left[\delta\boldsymbol{\alpha}.\boldsymbol{m}\right]_{\xi_1=0}^{\xi_1=L} \\ &\quad + \int_0^L (\delta\boldsymbol{\varphi} \cdot \boldsymbol{\aleph} + \delta\boldsymbol{\alpha} \cdot \mathfrak{M})\ d\xi_1. \end{aligned} \tag{107}$$

5.1.3. Term D: inertial work due to virtual displacement

Term D gives the virtual work done due to the inertial forces. This can be simplified by making substitution for $\ddot{\boldsymbol{R}}$ and $\delta\boldsymbol{u}$ as in Eqs. (69) and (46) and realizing the fact that $\delta\boldsymbol{\varphi}$, $\delta\boldsymbol{\alpha}$, $\ddot{\boldsymbol{\varphi}}$, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are functions of ξ_1 only. Thus,

$$\begin{aligned}\text{Term D} &= \int_0^L \delta\boldsymbol{\varphi}. \left[\int_{\blacksquare} \rho^s\big(\ddot{\boldsymbol{\varphi}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{PG} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{PG})\big) d\xi_2 d\xi_3 \right] d\xi_1 \\ &\quad + \int_0^L \delta\boldsymbol{\alpha}. \left[\int_{\blacksquare} \rho^s \boldsymbol{r}_{PG} \times \big(\ddot{\boldsymbol{\varphi}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{PG} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{PG})\big) d\xi_2 d\xi_3 \right] d\xi_1 \\ &= \int_0^L \delta\boldsymbol{\varphi}. \Bigg[\overbrace{\overbrace{\left(\int_{\blacksquare} \rho^s d\blacksquare \right)}^{\mu^s} \ddot{\boldsymbol{\varphi}} + \dot{\boldsymbol{\omega}} \times \overbrace{\left(\int_{\blacksquare} \rho^s \boldsymbol{r}_{PG} d\blacksquare \right)}^{\boldsymbol{\Upsilon}^s} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \overbrace{\left(\int_{\blacksquare} \rho^s \boldsymbol{r}_{PG} d\blacksquare \right)}^{\boldsymbol{\Upsilon}^s}}^{\boldsymbol{\mathfrak{F}}^s} \Bigg] d\xi_1 \\ &\quad + \int_0^L \delta\boldsymbol{\alpha}. \Bigg[\overbrace{\overbrace{\left(\int_{\blacksquare} \rho^s \boldsymbol{r}_{PG} d\blacksquare \right)}^{\boldsymbol{\Upsilon}^s} \times \ddot{\boldsymbol{\varphi}} + \overbrace{\left(\int_{\blacksquare} \rho^s \tilde{\boldsymbol{R}}_{PG}^T \tilde{\boldsymbol{R}}_{PG} d\blacksquare \right)}^{\boldsymbol{I}_M^s} + \boldsymbol{\omega} \times \overbrace{\left(\int_{\blacksquare} \rho^s \tilde{\boldsymbol{R}}_{PG}^T \tilde{\boldsymbol{R}}_{PG} d\blacksquare \right)}^{\boldsymbol{I}_M^s} \boldsymbol{\omega}}^{\boldsymbol{\lambda}^s} \Bigg] d\xi_1 \\ &= \int_0^L (\delta\boldsymbol{\varphi}.\boldsymbol{\mathfrak{F}}^s + \delta\boldsymbol{\alpha}.\boldsymbol{\lambda}^s) d\xi_1 \end{aligned} \tag{108}$$

5.1.4. Virtual work principle for Cosserat beam

The final virtual work equation for the reduced Cosserat beam referenced to the initially straight beam configuration Ω^s can be obtained by combining Eqs. (97)–(108) as,

$$\delta U_{\text{strain}}^s + \delta W_{\text{inertial}}^s = \delta W_{\text{external}}; \tag{109}$$

where,

$$\delta U_{\text{strain}}^s = \int_0^L \left(\boldsymbol{\eta}.\tilde{\delta}\boldsymbol{\varepsilon} + \boldsymbol{m}.\tilde{\delta}\boldsymbol{\kappa} \right) d\xi_1 = \int_0^L \left(\overline{\boldsymbol{\eta}}.\delta\overline{\boldsymbol{\varepsilon}} + \overline{\boldsymbol{m}}.\delta\overline{\boldsymbol{\kappa}} \right) d\xi_1, \tag{110}$$

$$\delta W_{\text{inertial}}^s = \int_0^L (\delta\boldsymbol{\varphi}.\boldsymbol{\mathfrak{F}}^s + \delta\boldsymbol{\alpha}.\boldsymbol{\lambda}^s) d\xi_1, \tag{111}$$

$$\delta W_{\text{external}} = \left[\delta\boldsymbol{\varphi}.\boldsymbol{\eta}\right]_{\xi_1=0}^{\xi_1=L} + \left[\delta\boldsymbol{\alpha}.\boldsymbol{m}\right]_{\xi_1=0}^{\xi_1=L} + \int_0^L (\delta\boldsymbol{\varphi}.\boldsymbol{\aleph} + \delta\boldsymbol{\alpha}.\mathfrak{M})\ d\xi_1. \tag{112}$$

Eq. (109) bears a recognizable form of virtual work principle which states that if the body in dynamic equilibrium is subjected to a virtual displacement at a given instant of time, the virtual work done due to the real external forces $\delta W_{\text{external}}$ (Traction and body force) is stored as virtual strain energy $\delta U_{\text{strain}}^s$ and virtual work due to the inertial forces on the body $\delta W_{\text{inertial}}^s$. The virtual work principle, when the deformation of the beam is referenced to the curved configuration would then become,

$$\delta U_{\text{strain}}^c + \delta W_{\text{inertial}}^c = \delta W_{\text{external}}; \tag{113}$$

where,

$$\delta U_{\text{strain}}^c = \int_0^L \left(\boldsymbol{\eta} \cdot \tilde{\delta}\boldsymbol{\varepsilon}^r + \boldsymbol{m} \cdot \tilde{\delta}\boldsymbol{\kappa}^r \right) d\xi_1 = \int_0^L \left(\overline{\boldsymbol{\eta}} \cdot \delta\overline{\boldsymbol{\varepsilon}}^r + \overline{\boldsymbol{m}} \cdot \delta\overline{\boldsymbol{\kappa}}^r \right) d\xi_1, \tag{114}$$

$$\delta W_{\text{inertial}}^c = \int_0^L (\delta\boldsymbol{\varphi}.\boldsymbol{\mathfrak{F}}^c + \delta\boldsymbol{\alpha}.\boldsymbol{\lambda}^c) d\xi_1. \tag{115}$$

The terms above have usual meaning as defined in previous sections. It's worth noting that the virtual external work $\delta W_{\text{external}}$ remains the same for both the reference configuration Ω^s and Ω^c. The expression for the strain energy and the inertial work changes because the strain and the inertial effect depends on the initial configuration of the beam considered.

5.2. Equivalence of weak and strong form of equilibrium equation

The linear and angular momentum conservation principle for the reduced beam is obtained in Eqs. (73) and (82) of Section 4. The weak form of equation as derived in Section 5.1 can be obtained in pure mathematical sense from the strong form. This shows the equivalence of strong and weak form and also validate the results obtained in Section 5.1. We take the similar approach as delineated in Hughes (1987). The linear momentum equation (73) is associated with the midcurve deformation. Therefore, the weight function used to obtain residual form of reduced equilibrium equation is the virtual displacement of the midcurve $\delta\boldsymbol{\varphi}$. Similarly, the angular momentum equation (82) is associated with the curvatures of the cross-section, thus making virtual rotation $\delta\boldsymbol{\alpha}$ as the natural choice for the weight function. Note that $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\alpha}$ are admissible and are related to $\delta\boldsymbol{u}$ as shown in Eq. (46). The residual form of equilibrium equation referenced to the straight configuration Ω^s can be written as,

$$\int_0^L [\delta\boldsymbol{\varphi}\cdot(\boldsymbol{\eta}_{,\xi_1} + \boldsymbol{\aleph} - \mathfrak{F}^s) + \delta\boldsymbol{\alpha}\cdot(\boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1}\times\eta + \mathfrak{M} - \boldsymbol{\lambda}^s)]d\xi_1 = 0. \tag{116}$$

Using Green's theorem and the property of the triple product of vectors, following results hold,

$$\begin{aligned}
&\int_0^L (\delta\boldsymbol{\varphi}\cdot\boldsymbol{\eta}_{,\xi_1})d\xi_1 = [\delta\boldsymbol{\varphi}\cdot\boldsymbol{\eta}]_{\xi_1=0}^{\xi_1=L} - \int_0^L (\delta\boldsymbol{\varphi}_{,\xi_1}\cdot\boldsymbol{\eta})d\xi_1,\\
&\int_0^L (\delta\boldsymbol{\varphi}\cdot\boldsymbol{m}_{,\xi_1})d\xi_1 = \left[\delta\boldsymbol{\alpha}\cdot\boldsymbol{m}\right]_{\xi_1=0}^{\xi_1=L} - \int_0^L (\delta\boldsymbol{\alpha}_{,\xi_1}\cdot\boldsymbol{m})d\xi_1,\\
&\int_0^L \delta\boldsymbol{\kappa}\cdot(\boldsymbol{\varphi}_{,\xi_1}\times\boldsymbol{\eta})d\xi_1 = \int_0^L \boldsymbol{\eta}\cdot(\delta\boldsymbol{\alpha}\times\boldsymbol{\varphi}_{,\xi_1})d\xi_1.
\end{aligned} \tag{117}$$

Therefore, using the results in Eq. (117) with Eq. (116), the residual form of equilibrium equation simplifies as,

$$\begin{aligned}
&\int_0^L [(\delta\boldsymbol{\varphi}_{,\xi_1} - \delta\boldsymbol{\alpha}\times\boldsymbol{\varphi}_{,\xi_1})\cdot\boldsymbol{\eta} + \delta\boldsymbol{\alpha}_{,\xi_1}\cdot\boldsymbol{m}]d\xi_1\\
&\quad + \int_0^L (\delta\boldsymbol{\varphi}\cdot\mathfrak{F}^s + \delta\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}^s)d\xi_1\\
&= [\delta\boldsymbol{\varphi}\cdot\boldsymbol{\eta}]_{\xi_1=0}^{\xi_1=L} + [\delta\boldsymbol{\alpha}\cdot\boldsymbol{m}]_{\xi_1=0}^{\xi_1=L} + \int_0^L (\delta\boldsymbol{\varphi}\cdot\boldsymbol{\aleph} + \delta\boldsymbol{\alpha}\cdot\mathfrak{M})\, d\xi_1.
\end{aligned} \tag{118}$$

Noticing the expression for $\tilde{\delta\boldsymbol{\varepsilon}}$ and $\tilde{\delta\boldsymbol{\kappa}}$ in Eq. (53), the above equation becomes,

$$\begin{aligned}
&\int_0^L [\tilde{\delta\boldsymbol{\varepsilon}}\cdot\boldsymbol{\eta} + \tilde{\delta\boldsymbol{\kappa}}\cdot\boldsymbol{m}]d\xi_1 + \int_0^L (\delta\boldsymbol{\varphi}\cdot\mathfrak{F}^s + \delta\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}^s)d\xi_1\\
&= [\delta\boldsymbol{\varphi}\cdot\boldsymbol{\eta}]_{\xi_1=0}^{\xi_1=L} + [\delta\boldsymbol{\alpha}\cdot\boldsymbol{m}]_{\xi_1=0}^{\xi_1=L} + \int_0^L (\delta\boldsymbol{\varphi}\cdot\boldsymbol{\aleph} + \delta\boldsymbol{\alpha}\cdot\mathfrak{M})\, d\xi_1.
\end{aligned} \tag{119}$$

which is exactly same as the weak form (Eq. (109)) derived from the infinitesimal Lagrangian equation of motion thereby validating the former approach.

6. Strong form of equations derived from Hamilton's equation

Hamilton's Principle (refer Rao, 2007) can be used to evaluate the dynamic equation of motion. The principle assumes that the configuration of the deformed beam is exactly known at time t_1 and t_2. Therefore, the variational field $\delta\boldsymbol{u}(t_1,\xi_1,\xi_2,\xi_3) = \mathbf{0}$ and $\delta\boldsymbol{u}(t_2,\xi_1,\xi_2,\xi_3) = \mathbf{0}$. There are infinitesimal configurations that the beam can have at any time t ($t \neq t_1$ and t_2), each configuration deviating from the correct one by an arbitrary but admissible displacement field $\delta\boldsymbol{u}(t,\xi_1,\xi_2,\xi_3) = \delta\boldsymbol{\varphi}(t,\xi_1) + \delta\boldsymbol{\alpha}(t,\xi_1)\times\mathbf{r}_{PG}(\xi_1,\xi_2,\xi_3)$, where $\delta\boldsymbol{\varphi}$ defines the admissible variation in the midcurve and the vector $\delta\boldsymbol{\alpha}$ parametrizes the variation in the director frame. The exact deformed configuration at any time $t_1 < t < t_2$ is determined by making the *action A* stationary, defined as,

$$A = \int_{t_1}^{t_2} \mathfrak{L} dt = \int_{t_1}^{t_2} (T - U^s_{\text{strain}} + W_{\text{external}})dt. \tag{120}$$

where, the functional $\mathfrak{L}$ is called the *Lagrangian* of the problem. The Principle states that,

$$\delta\int_{t_1}^{t_2} (T - U_{\text{strain}} + W_{\text{external}})\ dt = \overbrace{\int_{t_1}^{t_2} \delta T\ dt}^{\text{Term 1}} - \overbrace{\int_{t_1}^{t_2} \delta U_{\text{strain}}\ dt}^{\text{Term 2}} + \overbrace{\int_{t_1}^{t_2} \delta W_{\text{external}}\ dt}^{\text{Term 3}} = 0. \tag{121}$$

6.1. Term 1: simplification of kinetic energy term

The total kinetic energy of the beam referenced to Ω^s can be written using Eq. (69) as,

$$\begin{aligned}
T &= \frac{1}{2}\int_{\Omega^s} \rho^s \dot{\mathbf{u}}.\dot{\mathbf{u}}\ d\Omega^s = \frac{1}{2}\int_{\Omega^s} \rho^s \dot{\boldsymbol{R}}.\dot{\boldsymbol{R}}\ d\Omega^s\\
&= \frac{1}{2}\int_{\Omega^s} \rho^s(\dot{\boldsymbol{\varphi}} + \dot{\mathbf{r}}_{PG}).(\dot{\boldsymbol{\varphi}} + \dot{\mathbf{r}}_{PG})\ d\Omega^s.
\end{aligned} \tag{122}$$

Therefore,

$$\delta\int_{t_1}^{t_2} T\ dt = \int_{t_1}^{t_2}\int_{\Omega^s} \rho^s[\delta\dot{\boldsymbol{\varphi}}.\dot{\boldsymbol{\varphi}} + \delta\dot{\boldsymbol{\varphi}}.\dot{\mathbf{r}}_{PG} + \dot{\boldsymbol{\varphi}}.\delta\dot{\mathbf{r}}_{PG} + \delta\dot{\mathbf{r}}_{PG}.\dot{\mathbf{r}}_{\mathbf{PG}}]d\Omega^s\ dt.$$

We subject Eq. (123) to integration by parts with respect to time and note that $\delta\boldsymbol{\varphi}(t_1) = \delta\boldsymbol{\varphi}(t_2) = \delta\boldsymbol{\alpha}(t_1) = \delta\boldsymbol{\alpha}(t_2) = \mathbf{0}$, therefore $\delta\mathbf{r}_{PG}(t_1) = \delta\boldsymbol{\alpha}(t_1)\times\mathbf{r}_{PG} = \mathbf{0}$ and $\delta\mathbf{r}_{PG}(t_2) = \mathbf{0}$. This leads to,

$$\delta\int_{t_1}^{t_2} T\ dt = -\int_{t_1}^{t_2}\int_{\Omega^s}[\delta\boldsymbol{\varphi}\cdot\ddot{\boldsymbol{\varphi}} + \delta\boldsymbol{\varphi}\cdot\ddot{\mathbf{r}}_{PG} + \ddot{\boldsymbol{\varphi}}\cdot\delta\mathbf{r}_{PG} + \delta\mathbf{r}_{PG}\cdot\ddot{\mathbf{r}}_{PG}\cdot]d\Omega^s\, dt. \tag{124}$$

We notice the following relations,

$$\ddot{\boldsymbol{\varphi}}.\delta\mathbf{r}_{PG} = \ddot{\boldsymbol{\varphi}}.[\delta\boldsymbol{\alpha}\times\mathbf{r}_{PG}] = \delta\boldsymbol{\alpha}.[\mathbf{r}_{PG}\times\ddot{\boldsymbol{\varphi}}]; \tag{125}$$

$$\delta\mathbf{r}_{PG}.\ddot{\mathbf{r}}_{PG} = \delta\boldsymbol{\alpha}.[\mathbf{r}_{PG}\times\ddot{\mathbf{r}}_{PG}]. \tag{126}$$

Substituting (125) and (126) in Eq. (124), and realizing that $\delta\boldsymbol{\varphi}$, $\delta\boldsymbol{\alpha}$, $\boldsymbol{\varphi}$, $\delta\boldsymbol{\omega}$ and $\delta\dot{\boldsymbol{\omega}}$ are function of (ξ_1, t) only, we obtain,

$$\begin{aligned}
\delta\int_{t_1}^{t_2} T\ dt = &-\int_{t_1}^{t_2}\int_0^L \delta\boldsymbol{\varphi}.\overbrace{\left[\ddot{\boldsymbol{\varphi}}.\overbrace{\left\{\int_{\blacksquare}\rho^s d\xi_2 d\xi_3\right\}}^{\mu^s} + \overbrace{\left\{\int_{\blacksquare}\rho^s\ddot{\mathbf{r}}_{PG}d\xi_2 d\xi_3\right\}}^{\dot{\boldsymbol{\omega}}\times\boldsymbol{\Upsilon}^s + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{\Upsilon}^s}\right]}^{\mathfrak{F}^s} d\xi_1\ dt\\
&-\int_{t_1}^{t_2}\int_0^L \delta\boldsymbol{\alpha}.\overbrace{\left[\overbrace{\left\{\int_{\blacksquare}\rho^s\mathbf{r}_{PG}d\xi_2 d\xi_3\right\}}^{\boldsymbol{\Upsilon}^s}\times\ddot{\boldsymbol{\varphi}} + \overbrace{\left\{\int_{\blacksquare}\rho^s(\mathbf{r}_{PG}\times\ddot{\mathbf{r}}_{PG})d\xi_2 d\xi_3\right\}}^{\boldsymbol{I}^s_M\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\boldsymbol{I}^s_M\boldsymbol{\omega}}\right]}^{\boldsymbol{\lambda}^s} d\xi_1\ dt.
\end{aligned}$$

Therefore,

$$\delta\int_{t_1}^{t_2} T\ dt = -\int_{t_1}^{t_2}\int_0^L \left[\delta\boldsymbol{\varphi}.\mathfrak{F}^s + \delta\boldsymbol{\alpha}.\boldsymbol{\lambda}^s\right]d\xi_1\ dt. \tag{127}$$

6.2. Term 2: simplification of potential energy term

The virtual strain energy term in Hamilton's equation can be obtained from Eq. (102) and using the results from Eqs. (53) and (56) as,

$$\int_{t_1}^{t_2} \delta U_{\text{strain}}\, dt = \int_{t_1}^{t_2}\int_0^L [(\delta\boldsymbol{\varphi}_{,\xi_1} - \delta\boldsymbol{\alpha}\times\boldsymbol{\varphi}_{,\xi_1})\cdot\boldsymbol{\eta} + \delta\boldsymbol{\alpha}_{,\xi_1}\cdot\boldsymbol{m}]d\xi_1\, dt. \tag{128}$$

Rearranging the terms and carrying out integration by parts with respect to ξ_1, we obtain,

$$\int_{t_1}^{t_2} \delta U_{\text{strain}}\, dt = -\int_{t_1}^{t_2}\int_0^L \delta\boldsymbol{\varphi}\cdot\boldsymbol{\eta}_{,\xi_1} + \delta\boldsymbol{\alpha}\cdot(\boldsymbol{\varphi}_{,\xi_1}\times\boldsymbol{\eta} + \boldsymbol{m}_{,\xi_1})d\xi_1\, dt + \left[\int_{t_1}^{t_2}[\delta\boldsymbol{\varphi}\cdot\boldsymbol{\eta} + \delta\boldsymbol{\alpha}\cdot\boldsymbol{m}]dt\right]_{\xi_1=0}^{\xi_1=L} \tag{129}$$

6.3. Term 3: simplification of external work term

The body force field $\boldsymbol{b}$ and the surface traction are the external forces in the body. The external force term in Hamilton's equation can be written as,

$$\int_{t_1}^{t_2} \delta W_{\text{external}}\, dt = \overbrace{\int_{t_1}^{t_2}\int_{\Omega^s} \rho^s(\delta\boldsymbol{u}.\boldsymbol{b})d\Omega^s\, dt}^{\text{Term 3.1}} + \overbrace{\int_{t_1}^{t_2}\int_0^L\int_{\Gamma_3^s}\left(\delta\boldsymbol{u}.(\boldsymbol{S}^s\boldsymbol{N}^s)\right)d\Gamma_3^s\, d\xi_1\, dt}^{\text{Term 3.2}} \tag{130}$$

where Γ_3^s represents the surface boundary for an element of unit arc length in Ω^s configuration (refer Fig. 7). *Term 3.1* and *Term 3.2* can be simplified using the expression for $\delta\boldsymbol{u}$ Eq. (46) as,

$$\int_{t_1}^{t_2}\int_{\Omega^s} \rho^s(\delta\boldsymbol{u}\cdot\boldsymbol{b})d\xi_1\, dt = \int_{t_1}^{t_2}\int_0^L \delta\boldsymbol{\varphi}\cdot\left[\int_{\blacksquare}\rho^s\boldsymbol{b}d\xi_2 d\xi_3\right] + \delta\boldsymbol{\alpha}\cdot\left[\int_{\blacksquare}\rho^s(\boldsymbol{r}_{PG}\times\boldsymbol{b})d\xi_2 d\xi_3\right]d\xi_1\, dt; \tag{131}$$

$$\int_{t_1}^{t_2}\int_0^L\int_{\Gamma_3^s}(\delta\boldsymbol{u}.(\boldsymbol{S}^s\boldsymbol{N}^s))d\Gamma_3^s\, d\xi_1\, dt = \int_{t_1}^{t_2}\int_0^L \delta\boldsymbol{\varphi}\cdot\left[\int_{\Gamma_3^s}\boldsymbol{S}^s\boldsymbol{N}^s d\Gamma_3^s\right] + \delta\boldsymbol{\alpha}.\left[\int_{\Gamma_3^s}\boldsymbol{r}_{PG}\times(\boldsymbol{S}^s\boldsymbol{N}^s)d\Gamma_3^s\right]d\xi_1\, dt. \tag{132}$$

Combing Eqs. (130)–(132) and noting the definition of reduced external force $\aleph$ and moment $\mathfrak{M}$ in Eqs. (68) and (76), respectively, we get,

$$\int_{t_1}^{t_2} \delta W_{\text{external}}\, dt = \int_{t_1}^{t_2}\int_0^L \left[\delta\boldsymbol{\varphi}.\aleph + \delta\boldsymbol{\alpha}.\mathfrak{M}\right]d\xi_1\, dt. \tag{133}$$

6.4. Governing equations of motion and boundary terms

The Hamilton's equation for the Cosserat beam can be realized by combining Eqs. (121), (127), (129) and (133) as,

$$\int_{t_1}^{t_2}\int_0^L [\delta\boldsymbol{\varphi}\cdot\{\boldsymbol{\eta}_{,\xi_1} + \aleph - \mathfrak{F}^s\} + \delta\boldsymbol{\alpha}\cdot\{\boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1}\times\eta + \mathfrak{M} - \boldsymbol{\lambda}^s\}]d\xi_1\, dt + \left[\int_{t_1}^{t_2}[\delta\boldsymbol{\varphi}\cdot\boldsymbol{\eta} + \delta\boldsymbol{\alpha}\cdot\boldsymbol{m}]dt\right]_{\xi_1=0}^{\xi_1=L} = 0. \tag{134}$$

Realizing that $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\alpha}$ are arbitrary virtual quantities at time t, for Eq. (134) to hold good for all $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\alpha}$, following must be true,

$$\boldsymbol{\eta}_{,\xi_1} + \aleph - \mathfrak{F}^s = 0, \tag{135}$$

$$\boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1}\times\eta + \mathfrak{M} - \boldsymbol{\lambda}^s = 0, \tag{136}$$

$$\left[\delta\boldsymbol{\varphi}.\boldsymbol{\eta}\right]_{\xi_1=0}^{\xi_1=L} = 0, \tag{137}$$

$$\left[\delta\boldsymbol{\alpha}.\boldsymbol{m}\right]_{\xi_1=0}^{\xi_1=L} = 0. \tag{138}$$

Eqs. (135) and (136) represent linear momentum conservation and angular momentum conservation law referenced to straight configuration Ω^s respectively. It is not surprising that the result is same as obtained from infinitesimal equilibrium equation in Section 4 as in Eqs. (73) and (82). Secondly, Eqs. (137) and (138) represent the general boundary condition at $\xi_1 = 0$ and $\xi_1 = L$. For instance, if the left boundary is fixed and the right boundary is free, $\boldsymbol{\varphi}(0) = \boldsymbol{\theta}(0) = \boldsymbol{0}$ and $\boldsymbol{\eta}(L) = \boldsymbol{m}(L) = \boldsymbol{0}$. Note that $\delta\boldsymbol{\alpha}$ parametrizes the variational rotation of director frame that has rotation of $\boldsymbol{Q(\theta)}$ in equilibrium state. Therefore, for the fixed end, $\delta\boldsymbol{\alpha}(0) = \boldsymbol{0}$ implies $\boldsymbol{\theta}(0) = \boldsymbol{0}$ at all time t.

6.5. Interpretation of equation of motion from D'Alembert's principle-motion viewed from the director frame

To interpret motion from the non-inertial frame in general, we define the *impressed forces* as the forces that are imposed on the system due to external effects and due to the configuration of the system. In the case of Cosserat beam, the body force, traction (external forces), and the internal stresses (due to deformed configuration) are the sources of the *impressed forces*. We define the *forces of inertia referenced to a frame in consideration* as the resisting forces by the structure, as observed from the frame considered. Lastly the *Einstein forces* or the *apparent forces* are defined as the forces experienced by the object due to non-inertial nature of the frame of reference. The object satisfies the state of equilibrium if the effect of *impressed forces, Einstein forces*, and the *forces of inertia referenced to a frame in consideration* are considered simultaneously. This law is referred to as the D'Alembert's Principle.

Owing to the single manifold nature of the problem, the motion of the Cosserat beam is simplified to motion of the midcurve. Each point of the midcurve has a rigid section attached to it. Therefore, the equation of motions developed in Section 4.1 can be thought of as the equilibrium equation of a unit arc length element with the mass μ^s idealized as a rigid section $\blacksquare(\xi_1)$, with the mass μ^s distributed homogeneously throughout the section.

We have assumed that the midcurve may not necessarily be the locus of the center of mass. For the section $\blacksquare(\xi_1)$, the point G represents the intersection of the midcurve at the section and the point M represents the mass centroid. The director frame $\{\boldsymbol{d_i(\xi_1)}\}$ is attached to the section $\blacksquare(\xi_1)$ with origin at G. The point M is located by the vector $\boldsymbol{r}_{MG} = \frac{\boldsymbol{\Upsilon^s}}{\mu^s}$. Fig. 8 describes the details.

The conservation of linear momentum equation (73) represents the translational equilibrium of the mass μ^s. The mass μ^s is static with respect to the frame $\{\boldsymbol{d}_i\}$ because the section is rigid. The frame $\{\boldsymbol{d}_i\}$ is translating with the translational acceleration of $\ddot{\boldsymbol{\varphi}}$ and is rotating with the angular acceleration $\dot{\boldsymbol{\omega}}$ referenced to the inertial frame of reference $\boldsymbol{E}_i$. The mass μ^s experiences the following forces,

1. The *impressed force* $= \boldsymbol{\eta}_{,\xi_1} + \aleph(\xi_1)$.
2. The *force of inertia* w.r.t the frame $\{\boldsymbol{d}_i\} = -\mu^s\tilde{\ddot{\boldsymbol{r}}}_{MG} = \boldsymbol{0}$.
3. The *Einstein force* due to translation $= -\mu^s\ddot{\boldsymbol{\varphi}}$.

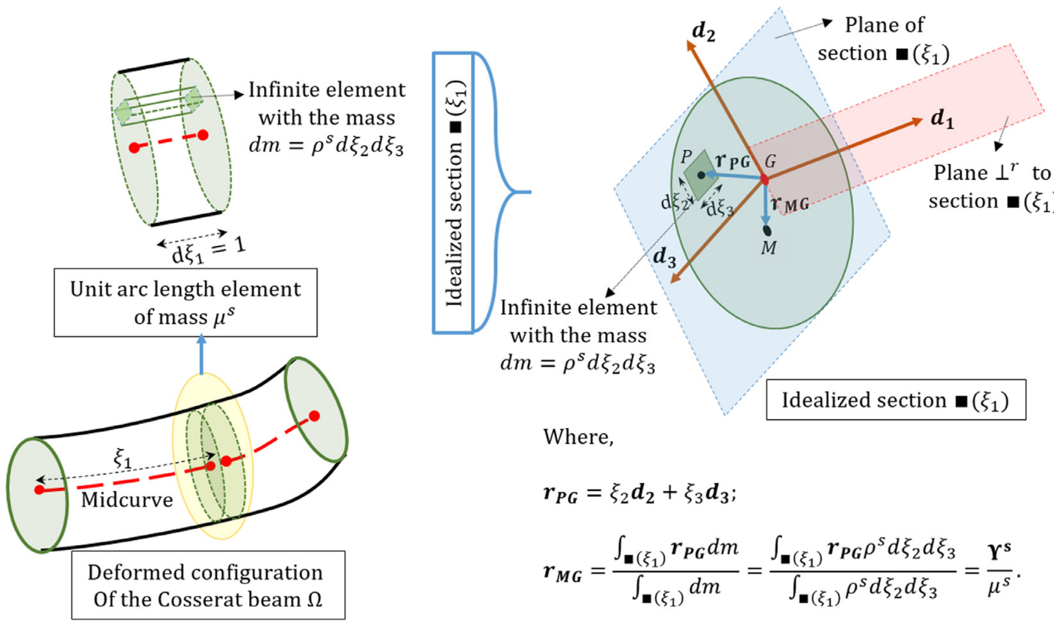

Fig. 8. Reduced element of unit arc-length idealized as a rigid section with mass μ^s.

4. The *centrifugal force* $= -\boldsymbol{\omega} \times \boldsymbol{\omega} \times (\mu^s \boldsymbol{r}_{MG}) = -\boldsymbol{\omega} \times \boldsymbol{\omega} \times (\Upsilon^s)$.
5. The *Euler force* $= -\dot{\boldsymbol{\omega}} \times (\mu^s \boldsymbol{r}_{MG}) = -\dot{\boldsymbol{\omega}} \times (\Upsilon^s)$.
6. The *Coriolis force* $= -2\dot{\boldsymbol{\omega}} \times (\mu^s \dot{\tilde{\boldsymbol{r}}}_{MG}) = \boldsymbol{0}$.

The conservation of angular momentum Eq. (82) represents the moment balance of the section $\blacksquare(\xi_1)$. If the force on the elemental mass $\rho^s d\xi_2 d\xi_3$ located at point P of the section, positioned by the vector $\boldsymbol{r}_{PG}$, is $d\boldsymbol{F}$, then the total reduced moment of the section is $\int_{\blacksquare(\xi_1)} \boldsymbol{r}_{PG} \times d\boldsymbol{F}$. Therefore,

1. The reduced moment due to the *impressed forces* $= \boldsymbol{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1} \times \eta + \mathfrak{M}$.
2. The reduced moment due to *force of inertia* w.r.t the frame $\{\boldsymbol{d}_i\}$ $= -\int_{\blacksquare(\xi_1)} \rho^s \boldsymbol{r}_{PG} \times \ddot{\tilde{\boldsymbol{r}}}_{PG} d\xi_2 d\xi_3 = \boldsymbol{0}$. Note that the parameter $\ddot{\tilde{\boldsymbol{r}}}_{PG}$ represents the acceleration of the point P w.r.t the frame $\{\boldsymbol{d}_i\}$; it vanishes since the section is assumed rigid.
3. The reduced moment due to the translational *Einstein force* $= -\int_{\blacksquare(\xi_1)} \rho^s \boldsymbol{r}_{PG} \times \ddot{\boldsymbol{\varphi}} d\xi_2 d\xi_3 = -\Upsilon^s \times \ddot{\boldsymbol{\varphi}}$.
4. The reduced moment due to the *centrifugal force* $= -\int_{\blacksquare(\xi_1)} \boldsymbol{r}_{PG} \times (\boldsymbol{\omega} \times \boldsymbol{\omega} \times (\boldsymbol{r}_{PG} \rho^s d\xi_2 d\xi_3))$ $= -\boldsymbol{\omega} \times \boldsymbol{I}_M \boldsymbol{\omega}$.
5. The reduced moment due to the *Euler force* $= -\int_{\blacksquare(\xi_1)} \boldsymbol{r}_{PG} \times (\dot{\boldsymbol{\omega}} \times (\boldsymbol{r}_{PG} \rho^s d\xi_2 d\xi_3)) = \boldsymbol{I}_M \dot{\boldsymbol{\omega}}$.
6. The moment due to the *Coriolis force* is 0 because $\dot{\tilde{\boldsymbol{r}}}_{PG} = \boldsymbol{0}$.

It is noteworthy that the *Coriolis force* and the *force of inertia w.r.t* $\{\boldsymbol{d}_i\}$ (and the respective moments) vanishes because we have ignored the *Poisson's* and the *warping effect*. If the section is not assumed to be rigid, we will have these two forces (and the respective moments) and an additional force term in the *impressed force* on account of addition stresses developed. Secondly, if the mass centroid was considered as the midcurve, the mass μ^s would not experience *centrifugal force* and *Euler force*.

7. Comments on constitutive relations

The equations of motion derived in the previous sections are completely general. The internal forces $\boldsymbol{\eta}$ and the moment $\boldsymbol{m}$ are related to the finite strain vectors $\boldsymbol{\epsilon}^r$ and $\boldsymbol{\kappa}^r$ through constitutive relations of the modeler's choice. As a matter of example, we demonstrate a hyperelastic linear constitutive model (as in Iura and Atluri, 1989), considering the Ω^c as initial configuration. As is observed in Eqs. (59) and (60), the reduced force and the moment depends on the stress vector $\boldsymbol{S}_1$. Therefore, we linearly relate the stress vector $\boldsymbol{S}_1$ to the strain vector $\boldsymbol{\epsilon}^r$ as,

$$\boldsymbol{S}_1 = \boldsymbol{C} \frac{\boldsymbol{\epsilon}^r}{|\boldsymbol{F}^c|} \tag{139}$$

From the definition of $\boldsymbol{\epsilon}^r = \boldsymbol{\epsilon} - \boldsymbol{Q}^r \boldsymbol{\epsilon}^c = \overline{\epsilon}_i^r \boldsymbol{d}_i$, we can write

$$\boldsymbol{\epsilon}^r = \boldsymbol{\varepsilon}^r + \boldsymbol{\kappa}^r \times (\xi_2 \boldsymbol{d}_2 + \xi_3 \boldsymbol{d}_3)$$

where,

$$\boldsymbol{\varepsilon}^r = \boldsymbol{\varepsilon} - \boldsymbol{Q}^r \boldsymbol{\varepsilon}^c = \overline{\varepsilon}_i^r \boldsymbol{d}_i; \quad \boldsymbol{\kappa}^r = \boldsymbol{\kappa} - \boldsymbol{Q}^r \boldsymbol{\kappa}^c = \overline{\kappa}_i^r \boldsymbol{d}_i.$$

Note that the curved reference configuration has same length as the mathematically straight configuration and zero shear angles. Therefore, $\boldsymbol{\varepsilon}^c = \boldsymbol{0}$. Using all these results and plugging Eq. (139) into Eqs. (59) and (60), we obtain a constitutive law of the form shown below.

$$\begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{m} \end{bmatrix} = \mathfrak{C} \begin{bmatrix} \boldsymbol{\varepsilon}^r \\ \boldsymbol{\kappa}^r \end{bmatrix}. \tag{140}$$

The coefficients $\boldsymbol{C}$ and $\mathfrak{C}$ are detailed for the homogeneous and isotropic case in the Appendix A.4.

8. Conservation of energy and time invariance

We know that the Hamilton's formulation of least action holds if the *impressed forces* are *monogenic* in nature (refer Lanczos, 1970). Therefore, work functions for the forces can be defined. The work function need not necessarily be conservative for the applicability of Hamilton's principle. Table 1 lists the work function for all the forces considering the straight beam as the undeformed state.

In Table 1, U^s represents the strain energy density. Secondly, the work function for external force used in Eq. (121) can be written as $W_{\text{external}} = W_b + W_t$.

We may arrive at the Energy conservation laws and the conditions associated with it by considering the real infinitesimal displacement $d\boldsymbol{u} = \dot{\boldsymbol{u}} dt$ as the variational field in the Hamilton's equation (121). This unique consideration no longer guarantees the

Table 1
Forces and their respective work functions.

Forces	**Work function**
Body force $\boldsymbol{b}$	$W_b = \int_{\Omega^s} \rho^s (\boldsymbol{u}.\boldsymbol{b}) d\Omega^s$
Surface traction	$W_t = \int_0^L \int_{\Gamma_3^s} (\boldsymbol{u}.(\boldsymbol{S}^s \boldsymbol{N}^s)) d\Gamma_3^s \, d\xi_1$
Internal stress	$U^s_{\text{strain}} = \int_{\Omega^s} F^s_{ij} S^s_{ij} d\Omega^s = \int_{\Omega^s} U^s d\Omega^s$
Inertial force	$T = \frac{1}{2}\int_{\Omega^s} \rho^s \dot{\boldsymbol{R}}.\dot{\boldsymbol{R}} d\Omega^s = \frac{1}{2}\int_{\Omega^s} \rho^s \dot{\boldsymbol{u}}.\dot{\boldsymbol{u}} d\Omega^s$

virtual displacement at time t_1 and t_2 to vanish. Therefore, for $\delta\boldsymbol{u} \to d\boldsymbol{u}$, the Hamilton's principle modifies to,

$$\delta \int_{t_1}^{t_2} \mathfrak{L} \, dt = \int_{\Omega^s} \rho^s [\dot{\boldsymbol{u}} \delta \boldsymbol{u}]_{t=t_1}^{t=t_2} d\Omega^s. \tag{141}$$

Using Table 1, the left hand side of the above equation can be simplified for $\delta\boldsymbol{u} \to d\boldsymbol{u}$ as,

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathfrak{L} \, dt &= \int_{t_1}^{t_2} \left\{ \int_{\Omega^s} (\rho^s \dot{\boldsymbol{u}} \cdot \delta\dot{\boldsymbol{u}} - \delta U^s + \rho^s \delta\boldsymbol{u} \cdot \boldsymbol{b}) d\Omega^s \right. \\ &\quad \left. + \int_0^L \int_{\Gamma_3^s} (\delta\boldsymbol{u} \cdot (\boldsymbol{S}^s \boldsymbol{N}^s)) d\Gamma_3^s \, d\xi_1 \right\} dt \\ &= \left[\int_{t_1}^{t_2} \left\{ \int_{\Omega^s} (\rho^s \dot{\boldsymbol{u}} \cdot \ddot{\boldsymbol{u}} - dU^s + \rho^s \dot{\boldsymbol{u}} \cdot \boldsymbol{b}) d\Omega^s \right. \right. \\ &\quad \left. \left. + \int_0^L \int_{\Gamma_3^s} (\dot{\boldsymbol{u}} \cdot (\boldsymbol{S}^s \boldsymbol{N}^s)) d\Gamma_3^s \, d\xi_1 \right\} dt \right] dt \\ &= \left[\int_{t_1}^{t_2} \left(\frac{dT}{dt} - \frac{dU_{\text{strain}}}{dt} + \frac{dW_{\text{b}}}{dt} + \frac{dW_{\text{t}}}{dt} \right) dt \right] dt \\ &= [T - U_{\text{strain}} + W_{\text{external}}]_{t_1}^{t_2} dt. \end{aligned} \tag{142}$$

It was possible to simplify Eq. (142) by assuming the traction and body forces to be constant with time. This was done to obtain a particular and simplified form of energy as $(T - U_{\text{strain}} + W_{\text{external}})$. The second step of (142) shows the general energy conservation law. We can evaluate the right hand side of Eq. (141) for $\delta\boldsymbol{u} \to d\boldsymbol{u}$ as,

$$\int_{\Omega^s} \rho^s [\dot{\boldsymbol{u}} \cdot \delta\boldsymbol{u}]_{t=t_1}^{t=t_2} d\Omega^s = \left[\int_{\Omega^s} \rho^s \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} d\Omega^s \right]_{t=t_1}^{t=t_2} dt = [2T]_{t=t_1}^{t=t_2} dt. \tag{143}$$

Therefore, from Eqs. (141)–(143), we have

$$\left[\int_{\Omega^s} \rho^s \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} d\Omega^s - \mathfrak{L} \right]_{t=t_1}^{t=t_2} = [T - W_{\text{external}} + U_{\text{strain}}]_{t=t_1}^{t=t_2} = 0. \tag{144}$$

This implies that the quantity $(T - W_{\text{external}} + U_{\text{strain}})$ is conserved. This quantity is energy H (or Hamiltonian). It is clear that the external work W_{external} adds energy to the system. This energy is used to deform the beam (stored as strain energy U_{strain}) and to bring the motion in the beam (stored as kinetic energy T), implying $W_{\text{external}} = U_{\text{strain}} + T$. Therefore, a relationship between the Lagrangian and the Hamilton can be established for Continuum problem as,

$$\int_{\Omega^s} \rho^s \dot{\boldsymbol{u}}.\dot{\boldsymbol{u}} d\Omega^s - \mathfrak{L} = H. \tag{145}$$

The above equation establishes a relationship between the Lagrangian and Hamiltonian functional. It is well known from the classical mechanics of discrete bodies that both the functionals are related by Legendre transformation (Lanczos, 1970). The continuum is an infinite degree of freedom system. If we assume the beam to be composed of infinite particle each of mass $m_i = \rho^s \Delta\Omega_i^s$, located by $\boldsymbol{u}_i$, the Lagrangian takes the form,

$$\mathfrak{L} = \sum_{i=1}^{\infty} \frac{1}{2} m_i \dot{\boldsymbol{u}}_i.\dot{\boldsymbol{u}}_i - U_{\text{strain}} + W_{\text{external}}. \tag{146}$$

Note that only the kinetic energy is function of velocity. We define the *generalized momentum* of the ith particle as $\boldsymbol{p}_i = (\rho^s \Delta\Omega_i^s)\dot{\boldsymbol{u}}_i = \frac{\partial \mathfrak{L}}{\partial \dot{\boldsymbol{u}}_i}$. The Legendre transformation applied to the Lagrangian is therefore, written as,

$$\sum_{i=1}^{\infty} \frac{\partial \mathfrak{L}}{\partial \dot{\boldsymbol{u}}_i}.\dot{\boldsymbol{u}}_i - \mathfrak{L} = \sum_{i=1}^{\infty} \boldsymbol{p}_i.\dot{\boldsymbol{u}}_i - \mathfrak{L} = H. \tag{147}$$

For the continuum case,

$$\sum_{i=1}^{\infty} \boldsymbol{p}_i.\dot{\boldsymbol{u}}_i = \lim_{\substack{n\to\infty \\ \Delta\Omega_i^s \to d\Omega_i^s}} \sum_{i=1}^{n} \rho^s \dot{\boldsymbol{u}}_i.\dot{\boldsymbol{u}}_i \Delta\Omega_i^s = \int_{\Omega^s} \rho^s \dot{\boldsymbol{u}}.\dot{\boldsymbol{u}} d\Omega^s. \tag{148}$$

Therefore, for continuum case, Eq. (147) is same as Eq. (145).

We were able to obtain the Energy conservation law from the Hamilton's Principle by considering the differential displacement as the virtual displacement. We can choose this special case of variation only if the Lagrangian does not have explicit time dependence. If the Lagrangian has explicit time dependence, then the variation in Lagrangian occurs at a specific time t, whereas the differential change in Lagrangian occurs in a duration of dt. Therefore, for the Energy of the system to be conserved, the system must be *scleronomic* and the forces must be *conservative* in addition to *monogenic*. If the external forces are time dependent, it would imply the presence of external source of energy which is not taken into account, leading to the addition of unaccounted energy in the system. In fact, the Energy conservation law arises from the time invariance symmetry of the nature. Therefore, our understanding are in accordance with Noether's theorem.

9. Conclusion and summary

This paper presents the reduced balance laws of a Cosserat beam in an exhaustive fashion focusing on all relevant details and the interpretation of results. The Cosserat theory of rods defines the configuration of the beam by the midcurve and the cross-section attached to the midcurve. Therefore, the technical discussion begins with the description of the geometry, deformation parameters, and mathematical tools. This sets the ground to define the deformation gradient tensor and strain vector of the Cosserat beam referenced to initially straight or curved configurations of the beam.

The deformation gradient tensor for the curved reference configuration and the current configuration is developed from the mathematically straight reference beam. The result is then used to obtain the deformation gradient tensor of the current configuration referenced to the curved referenced configuration. A complete derivation to obtain the inverse of deformation gradient tensor is shown. It is observed that, for the cross-section being rigid, only the first component of any infinitesimal vector is strained whereas, the second and the third component of the vector merely rotates. This fact is clearly reflected in the expression for the deformation gradient tensor. We also define the arbitrary but admissible variational displacement field $\delta\boldsymbol{u}$ and obtain the expression for the co-rotated variation of the axial strain and curvature vector that is used to obtain the weak form of the equilibrium equation (Virtual work principle). The virtual displacement comprises of the virtual translation given by $\delta\boldsymbol{\varphi}$ and the virtual rotation of the director frame parametrized by the rotation vector $\delta\boldsymbol{\alpha}$. A detailed description of the parametrization of the rotation tensor $\boldsymbol{Q}$ using Rodrigues' formula is presented drawing physical interpretation of virtual rotation of director $\boldsymbol{d}_i$.

The discussion in Sections 2 and 3 provides the reader with a concrete platform to obtain the strong and weak forms of reduced balance laws. The strong form in general incorporates the linear momentum and angular momentum conservation laws. The single manifold nature (defined by ξ_1) of the problem allows us to arrive at the reduced strong form (from the infinitesimal equilibrium equation that is valid at every point of the body). The reduced strong form of the Cosserat beam is the set of differential equations that governs the mechanics of the beam.

We obtained the reduced linear and angular momentum balance equation using infinitesimal equilibrium equation and Lagrangian–Hamilton's equation independently. It may be inferred that the inertial term in the strong form of equations has the terms associated with both, the first moment of inertia and the second moment of inertia. This is because we did not assume that the midcurve passes through the mass centroid of the beam. The interpretation of forces from the director frame points out that the absence of a *Coriolis force* (and respective moment) is due to assumption of Bernoulli's rigid cross-section. Hence, we anticipate the presence of a *Coriolis force, force of inertia referenced to the director frame* and additional *impressed forces* due to additional stresses developed when we consider the *Poisson's* and *warping* effects.

The integral or weak form of the equation represents the principle of virtual work for the Cosserat beam. The integral form of equilibrium equations is also obtained in two ways. In the first approach, we obtain the weak form using the infinitesimal equilibrium equation. Mathematically, strong and the weak form of the equilibrium equations are equivalent. Therefore, the second approach involves obtaining the weak form from the strong form in a complete mathematical sense.

It is also observed that the conservation of energy principle holds if the forces are *monogenic* and conservative and the Lagrangian functional is *scleronomic* as expected. The Lagrangian and Hamilton functionals are linked by Legendre transformation, in an integral sense.

Each of the derivations is performed rigorously to fully describe the mechanics of the beam. The understanding presented in this paper sets the framework to develop/understand finite element formulation of the Cosserat beam. An interesting study on the application of the Noether's Theorem for the Cosserat beam, and an extension to the formulation including Poisson's and warping effects (by developing deformation adaptive optimized warping functions) is something Authors are looking forward to.

Appendix A

A1. The component of rotation matrix

$$
\begin{aligned}
\Lambda_{11} &= \cos\phi_y\big(\cos\gamma_{11}\cos\phi_p - \cos\alpha_1\sin\phi_p\big)\\
&\quad + \sin\phi_y\big(\cos\alpha_2\sin\gamma_{13} - \cos\alpha_3\sin\gamma_{12}\big)\\
\Lambda_{12} &= \cos\alpha_1\cos\phi_p + \cos\gamma_{11}\sin\phi_p\\
\Lambda_{13} &= \cos\phi_y\big(\sin\gamma_{12}\cos\alpha_3 - \cos\alpha_2\sin\gamma_{13}\big)\\
&\quad + \sin\phi_y\big(\cos\gamma_{11}\cos\phi_p - \cos\alpha_1\sin\phi_y\big)\\
\Lambda_{21} &= \cos\phi_y\big(\sin\gamma_{12}\cos\phi_p - \cos\alpha_2\sin\phi_p\big)\\
&\quad + \sin\phi_y\big(\cos\alpha_3\cos\gamma_{11} - \cos\alpha_1\sin\gamma_{13}\big)\\
\Lambda_{22} &= \cos\alpha_2\cos\phi_p + \sin\gamma_{12}\sin\phi_p\\
\Lambda_{23} &= \cos\phi_y\big(\sin\gamma_{13}\cos\alpha_1 - \cos\alpha_3\cos\gamma_{11}\big)\\
&\quad + \sin\phi_y\big(\cos\phi_p\sin\gamma_{12} - \cos\alpha_2\sin\phi_p\big)\\
\Lambda_{31} &= \cos\phi_y\big(\sin\gamma_{13}\cos\phi_p - \cos\alpha_3\sin\phi_p\big)\\
&\quad + \sin\phi_y\big(\cos\alpha_1\sin\phi_p - \cos\alpha_2\sin\gamma_{11}\big)\\
\Lambda_{32} &= \cos\alpha_3\cos\phi_p + \sin\gamma_{13}\sin\phi_p\\
\Lambda_{33} &= \cos\phi_y\big(\cos\alpha_2\cos\gamma_{11} - \cos\alpha_1\sin\gamma_{12}\big)\\
&\quad + \sin\phi_y\big(\cos\phi_p\sin\gamma_{13} - \cos\alpha_3\sin\phi_p\big)
\end{aligned}
$$

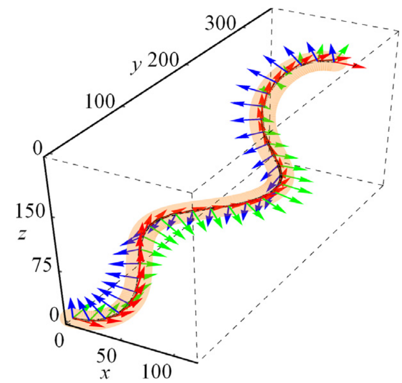

Fig. A1. Illustration of geometric description of finite deformation of the beam.

A2. The component of Darboux vector-the curvature terms

$$
\begin{aligned}
\overline{\kappa}_1 &= -\alpha_{2,\xi_1}\cos\alpha_3\sin\alpha_2 + \alpha_{3,\xi_1}\cos\alpha_2\sin\alpha_3\\
&\quad + \gamma_{11,\xi_1}\cos\alpha_1\sin\gamma_{11}(-\cos\alpha_3\sin\gamma_{12} + \cos\alpha_2\sin\gamma_{13})\\
&\quad - \gamma_{12,\xi_1}\cos\alpha_2^2\cos\gamma_{12}\sin\gamma_{13} - \gamma_{13,\xi_1}\cos\gamma_{13}\sin\gamma_{13}\\
&\quad + \gamma_{13,\xi_1}\cos\alpha_3^2\cos\gamma_{13}\sin\gamma_{12}\\
&\quad - \gamma_{13,\xi_1}\cos\alpha_2\cos\alpha_3\cos\gamma_{13}\sin\gamma_{13}\\
&\quad + \phi_{y,\xi_1}(\cos\alpha_2\sin\gamma_{12} - \cos\alpha_2\sin\gamma_{13})\\
&\quad - \phi_{y,\xi_1}(\cos\alpha_1\cos\phi_p + \cos\gamma_{11}\sin\phi_p)\\
\overline{\kappa}_2 &= \frac{1}{2}(2\alpha_{1,\xi_1}\cos\alpha_3\sin\alpha_1 - 2\alpha_{3,\xi_1}\cos\alpha_1\sin\alpha_3\\
&\quad + \gamma_{11,\xi_1}(2 + \cos 2\alpha_2 + \cos 2\alpha_3)\sin\gamma_{13}\\
&\quad + \gamma_{12,\xi_1}(\cos\alpha_1\cos\alpha_3\sin 2\gamma_{13}\\
&\quad + 2\cos\alpha_2\cos\gamma_{12}(-cos\alpha_3\cos\gamma_{11} + \cos\alpha_1\sin\gamma_{13}))\\
&\quad + 2\big(\cos\gamma_{11}\cos\gamma_{13}\sin\alpha_3^2 + \cos\alpha_1\cos\alpha_3\sin 2\gamma_{13}\big)\\
&\quad + 2\phi_{p,\xi_1}(-\cos\alpha_3\cos\gamma_{11} + \cos\alpha_1\sin\gamma_{13})\\
&\quad - 2\phi_{y,\xi_1}(\cos\alpha_2\cos\phi_p + \sin\gamma_{12}\sin\phi_p))\\
\overline{\kappa}_3 &= -\alpha_{1,\xi_1}\cos\alpha_2\sin\alpha_1 + \alpha_{2,\xi_1}\cos\alpha_1\sin\alpha_2\\
&\quad + \gamma_{12,\xi_1}(\cos\alpha_1\cos\alpha_3\cos\gamma_{12}\sin\gamma_{13} - \cos\alpha_3^2\cos\gamma_{11}\cos_{\gamma_{12}})\\
&\quad + \gamma_{13,\xi_1}(\cos\alpha_2\cos\alpha_3\cos\gamma_{11}\cos\gamma_{13}\\
&\quad - \cos\alpha_1\cos\alpha_3\cos\gamma_{13}\sin\gamma_{12})\\
&\quad + \phi_{p,\xi_1}(\cos\alpha_2\cos\gamma_{11} - \cos\alpha_1\sin\gamma_{12})\\
&\quad + \gamma_{11,\xi_1}\cos\alpha_3\sin\gamma_{11}(\cos\alpha_2\sin\gamma_{13}\\
&\quad - \cos\alpha_3\sin\gamma_{12}) - \phi_{y,\xi_1}(\cos\alpha_3\cos\phi_p\\
&\quad + \sin\gamma_{13}\sin\phi_p)
\end{aligned}
$$

A3. Illustration of a deformed shape of the beam

We present an example of geometric description of the deformed shape of a slender rod obtained by using the methodology detailed in the Section 3.2 (refer Fig. A1). The components of the directors and the Darboux vector can be obtained using the results in Appendices A.1 and A.2, respectively. The rod has an undeformed length $L_o = 500\,\text{m}$ and a circular cross-section with radius 0.15 m. The initial configuration of the rod is assumed to be straight along x-axis and the rod is fixed at $x = 0$. We ignore the shear deformation ($\gamma_{1i} = 0$) in this example. The beam is subjected to elongation and curvatures (including torsion). Therefore, the director triad $\{\boldsymbol{d}_i\}$ is same as the triad $\{\boldsymbol{T}, \boldsymbol{V}, \boldsymbol{H}\}$. The red, blue and green vectors represents the directors $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ and $\boldsymbol{d}_3$, respectively. The black curve shows the midcurve of the rod. Note that the

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directors are scaled up for clear representation. This deformation assumes following parameters satisfying the boundary conditions,

$$\phi_p(\xi_1) = \left(\frac{\pi}{2}\sin\frac{\pi\xi_1}{L_o}\right)\left(1-0.5\sin\frac{3.5\pi\xi_1}{L_o}\right);$$

$$\phi_y(\xi_1) = \pi\sin\frac{\pi\xi_1}{L_o};$$

$$e(\xi_1) = \frac{5\pi}{L_o}\sin\frac{\pi\xi_1}{2L_o}; \quad \alpha_1(\xi_1) = \frac{\pi}{2}; \quad \alpha_2(\xi_1) = 10\pi\left(\frac{\xi_1}{L_o}\right);$$

$$\alpha_3(\xi_1) = \frac{\pi}{2} + \alpha_2(\xi_1);$$

A4. The coefficients of the constitutive law

$$\boldsymbol{C} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix};$$

$$\mathfrak{C} = \begin{bmatrix} EA_1 & 0 & 0 & 0 & EA_2 & -EA_3 \\ 0 & Gk_sA_1 & 0 & -GA_2 & 0 & 0 \\ 0 & 0 & Gk_sA_1 & GA_3 & 0 & 0 \\ 0 & -GA_2 & GA_3 & Gk_tA_4 & 0 & 0 \\ EA_2 & 0 & 0 & 0 & EA_5 & -EA_7 \\ -EA_3 & 0 & 0 & 0 & -EA_7 & EA_6 \end{bmatrix}$$

Note that here k_s and k_t are the standard shape factor for shear and torsion and the geometric constants A_i are given below.

$$A_1 = \int_{\blacksquare} \frac{1}{|\boldsymbol{F}^c|} d\xi_2 d\xi_3$$

$$A_2 = \int_{\blacksquare} \frac{1}{|\boldsymbol{F}^c|} \xi_3 d\xi_2 d\xi_3$$

$$A_3 = \int_{\blacksquare} \frac{1}{|\boldsymbol{F}^c|} \xi_2 d\xi_2 d\xi_3$$

$$A_4 = \int_{\blacksquare} \frac{1}{|\boldsymbol{F}^c|} (\xi_2^2 + \xi_3^2) d\xi_2 d\xi_3$$

$$A_5 = \int_{\blacksquare} \frac{1}{|\boldsymbol{F}^c|} \xi_3^2 d\xi_2 d\xi_3$$

$$A_6 = \int_{\blacksquare} \frac{1}{|\boldsymbol{F}^c|} \xi_2^2 d\xi_2 d\xi_3$$

$$A_7 = \int_{\blacksquare} \frac{1}{|\boldsymbol{F}^c|} \xi_3\xi_2 d\xi_2 d\xi_3$$

A5. Comments on the cross section rigidity assumption and the validity of the theory

We made an assumption of a rigid cross-section in 2.1 and all the results obtained incorporated this assumption. The absence of Coriolis forces and the absence of Poisson's effect in the constitutive laws are direct consequences of this assumption, for example. Therefore, it is beneficial to look into the limitations of the results presented. Let us momentarily consider warping and Poisson's effect. If we had these effects, the position vector of any point in Eq. (3) would take the form

$$\boldsymbol{R}^* = \boldsymbol{\varphi}(\xi_1) + (\xi_2 - \nu e\xi_2)\boldsymbol{d}_2 + (\xi_3 - \nu e\xi_3)\boldsymbol{d}_3 + \Xi(\xi_2, \xi_3)\overline{\kappa}_1\boldsymbol{d}_1.$$

Note that here, $\Xi(\xi_2, \xi_3)$ represents the warping function of the problem obtained from St. Venant's theory, and ν is Poisson's ratio. To ignore warping, we must assume the section is circular (or "sufficiently" circular that warping is negligible) such that $\Xi(\xi_2, \xi_3) \to 0$. Now we are left with Poisson's effect. The expression for $\boldsymbol{R}^*$ can be rearranged as

$$\boldsymbol{R}^* = \boldsymbol{R} - \nu e\xi_2\boldsymbol{d}_2 - \nu e\xi_3\boldsymbol{d}_3.$$

Using $\boldsymbol{R}^*$ to develop the kinematics of the Cosserat beam would change the strain vector and the deformation gradient tensor. The strain vector would then become

$$\boldsymbol{\epsilon} = \sum_{i=1}^{3}\left(\frac{\partial \boldsymbol{R}^*}{\partial \xi_i} - \boldsymbol{d}_i\right).$$

For slender structures, the lateral cross-sectional strain vector components are negligible. Thus, it is acceptable to write $(\frac{\partial \boldsymbol{R}^*}{\partial \xi_i} - \boldsymbol{d}_i) \approx \boldsymbol{0}$ for $i = 2, 3$. The direct implication of this approximation is that the strain vector reduces to that in Eq. (26) as

$$\boldsymbol{\epsilon} \approx \frac{\partial \boldsymbol{R}^*}{\partial \xi_1} - \boldsymbol{d}_1 = \frac{\partial \boldsymbol{R}}{\partial \xi_1} - \boldsymbol{d}_1.$$

Thus, Bernoulli's rigid cross-section assumption makes this theory acceptable for slender structure where the total length of beam is sufficiently long compared to the lateral dimensions of the beam, and for the structure with cross-sectional shapes such that the effect of warping is negligible.

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