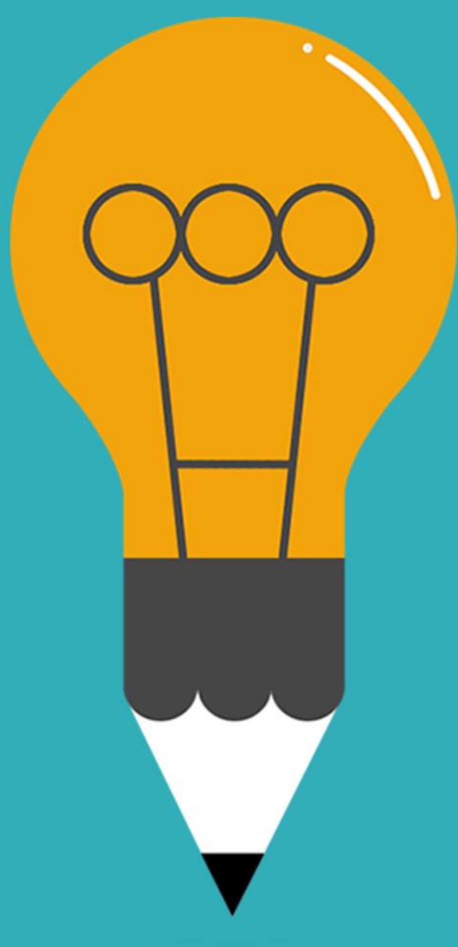
POLITEKNIK MALAYSIA
PORT DICKSON

ENGINEERING MATHEMATICS 2
VOLUME 1



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ENGINEERING MATHEMATICS 2
Volume 1

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PREFACE

Assalamualaikum & Peace be Upon You.

In the name of Allah, the Almighty who give us the truth, the knowledge, the enlightment and with regards to Prophet Muhammad S.A.W. May Allah give us the ability to continue our good deeds in this field.

Alhamdulillah, finally the e-book of Engineering Mathematics 2 For Polytechnics has been successfully published. This e-book is developed based on the latest Engineering Mathematics 2, Polytechnics Course Syllabus and is written by the lecturers from the Mathematics, Science and Computer Department, Polytechnic Port Dickson.

This e-book is a useful learning material for students who have never taken additional mathematics in secondary school and as a source of reference for newly trained lecturers in this field.

We hope this attempt will be a continuous process towards achieving our vision to become an excellent academic center. Any comments and suggestion from students or other readers is welcomed to make sure this module can be improved for further editions.

Thank you so much.

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CHAPTER I

INDEX & LOGARITHM

1.1 Law of Indices

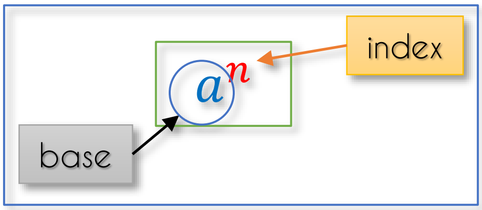
Index?

$a^n = a \times a \times a \times \ldots \times a$

n represents the number of how many times a should be multiplied , in condition $a \neq 0$ *and* $n > 0$.

Basic Law of Indices

$a^0 = 1$ (Zero Index)
$a^{-n} = \frac{1}{a^n}$ (Negative Index)
$a^{\frac{1}{n}} = \sqrt[n]{a}$ (Fractional Index)
$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ or $\left(\sqrt[n]{a}\right)^m$



Note:-

Index is also known as power and exponent.

More Law of Indices

$a^m \times a^n = a^{m+n}$ (Multiplication Rule)
$a^m \div a^n = a^{m-n}$ (Division Rule)
$(a^m)^n = a^{m \times n} = a^{mn}$ (Power of a Power Rule)
$(ab)^n = a^n b^n$ (Power of Product Rule)
$\left(a^k b^m\right)^n = a^{kn} b^{mn}$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ (Power of Fraction Rule)
$\left(\frac{a^k}{b^m}\right)^n = \frac{a^{kn}}{b^{mn}}$

1.2 Simplify Index Expression by Using Law of Indices

Example 1

Express each of the following expressions in the simplest form.

i.	$5 \times 8^{\frac{2}{3}}$	ii.	$\frac{3^2(2^2)^{-2}}{2^3}$
iii.	$(3a)^{-2}$	iv.	$\left(\frac{5a}{b^2}\right)^2$
v.	$4^{n+1} \times 4^2$		

Solution:-

i.	$5 \times 8^{\frac{2}{3}}$

$= 5 \times (2^3)^{\frac{2}{3}}$

$= 5 \times 2^{3 \times \frac{2}{3}}$

$= 5 \times 2^2$

$= 5 \times 4$

$= 20$

- Use your knowledge in indices to solve this question instead of directly using your calculator.
- Convert the base of $8^{\frac{2}{3}}$ to the lowest base of index possible. In this case, the lowest index for 8 is 2^3.
- Then, multiply the index (power) using the **Law of Power of Power Rule**.
- For the case, which both the base and index are numbers, the final answer should be a **value**. Never leave it in the Index Form.

Solution:-

ii.	$\dfrac{3^2(2^2)^{-2}}{2^3}$

$$= \frac{3^2(2^{2\times(-2)})}{2^3}$$
$$= \frac{3^2(2^{-4})}{2^3}$$
$$= 3^2 \times 2^{-4-3}$$
$$= 3^2 \times 2^{-7}$$
$$= 3^2 \times \frac{1}{2^7}$$
$$= 9 \times \frac{1}{128}$$
$$= \frac{9}{128}$$

- This question consists of three indices with two different base.
- First, apply the **Law of Power of Power Rule** for $(2^2)^{-2}$, to get a single index without repeating of its power.
- Second, apply the **Division Rule** for index with base of 2.
- Bear in mind, since both indices are numerical (index and base are numbers), state your final answer in one term of numerical value. Both decimal value and fractional value are accepted.

Solution:-

iii.	$(3a)^{-2}$
	$= 3^{1\times(-2)}a^{1\times(-2)}$ $= 3^{-2}a^{-2}$ $= \frac{1}{3^2a^2}$ $= \frac{1}{9a^2}$

- For both example **iii** and **iv**, Firstly, apply the **Law of Power of Power Rule**.
- A silly mistake that is always done by students is that they did not multiply numerical base index. Ensure to multiply all indices (including numerical values).
- Second, because we have a negative index, apply the **Negative Index Law**. Index should not be in a negative value.
- So now, since only 3^2 is a numerical index, convert it to a numerical value.
- The same concept is applied to **iv**.

iv.	$\left(\frac{5a}{b^2}\right)^2$
	$= \left(\frac{5a}{b^2}\right)^2$ $= \frac{5^{1\times2}a^{1\times2}}{b^{2\times2}}$ $= \frac{5^2a^2}{b^4}$ $= \frac{25a^2}{b^4}$

v.	$4^{n+1} \times 4^2$
	$= 4^{n+1+2}$ $= 4^{3+n}$

- For this question, just apply the **Multiplication Rule**, because both indices have a similar base.
- And you will get the answer in 1 term of index form.

Exercise Time!

Express each of the following Indices expressions in the simplest form.

a. $9^{n+1} \times 9^{\frac{n}{3}}$

b. $\left(\sqrt{4}a^{m}\right)^{-2}$

c. $16^{n-1} \div 4^{2-n}$

d. $\dfrac{2^{5-x}(3^{x})^{-2}}{2^{x}}$

e. $\left(\dfrac{n^{3}}{n^{4+n}}\right)^{2}$

f. $\dfrac{3^{2-m}\left(s^{2}\right)^{m-2}}{(3s)^{3}}$

g. $64^{n+1} \times 256^{2n}$

CHECK YOUR ANSWERS!

a. $9^{\frac{4}{3}n+1}$

b. $\frac{1}{4a^{2m}}$

c. 4^{3n-4}

d. $\frac{2^{5-2x}}{3^{2x}}$

e. $\frac{1}{n^{2+2n}}$

f. $\frac{s^{2m-7}}{3^{1+m}}$

g. 4^{3+11n}

Example 2

Express each of the following expressions in the simplest form.

i.	$\left(\frac{32x^{10}}{y^5}\right)^{\frac{3}{5}}$	ii.	$\frac{x^2}{z^3} \times (xz^2)^2$
iii.	$\frac{a^{3+4n}}{a^{3n+2} \times a^{4-n}}$	iv.	$\frac{2^{n+4} \times 4(2^{n-1})}{2^n}$
v.	$(27^4)^{\frac{3}{4}} \times 9^4 \div 81^5$		

Solution:-

i.	$\left(\frac{32x^{10}}{y^5}\right)^{\frac{3}{5}}$

$$= \left(\frac{2^5 x^{10}}{y^5}\right)^{\frac{3}{5}}$$

$$= \frac{2^{5\times\frac{3}{5}} x^{10\times\frac{3}{5}}}{y^{5\times\frac{3}{5}}}$$

$$= \frac{2^3 x^6}{y^3}$$

$$= \frac{8x^6}{y^3}$$

- As usual, avoid using your calculator to simplify index expressions. What I mean is avoid pushing your calculator to calculate $32^{\frac{3}{5}}$.
- Apply the Law of Indices is the best practice to simplify even to numerical indices.
- Convert 32 to index form, the best base of index is the lowest number it could be. So, for 32 the lowest number for base is 2.

$$32 = 2^5$$

- Then, by using the **Law of Power of Power Rule**, multiply index for each of indices available.
- Do not forget, for numerical index convert it to numerical value. So, $2^3 = 8$.

Solution:-

ii.	$\frac{x^2}{z^3} \times (xz^2)^2$

$= \frac{x^2}{z^3} \times x^{1\times2} z^{2\times2}$

$= \frac{x^2}{z^3} \times x^2 z^4$

$= \frac{x^2}{z^3} \times \frac{x^2 z^4}{1}$

$= x^{2+2}\ z^{4-3}$

$= x^4\ z$

- Same goes here, firstly, apply the **Law of Power of Power Rule** to $(xz^2)^2$.
- Then, apply the **Multiplication Rule** and **Division Rule**. Both of these rules can only be applied for index which have a similar base.
- To simplify index with base x, use the Multiplication Rule while for index with base z, use the Division Rule (*also known as Quotient Rule*) .
- Normally, we would just write z instead of z^1. This practice is applied around the globe, so do not worry if you make this mistake.

iii.	$\frac{a^{3+4n}}{a^{3n+2} \times a^{4-n}}$

$= a^{3+4n-(3n+2)-(4-n)}$

$= a^{3+4n-3n-2-4+n}$

$= a^{-3+2n}$

- For iii, all indices have the same base, therefore just apply the Multiplication Rule and Division Rule to simplify it.
- Do be extra careful to not do any silly mistake when you subtracted power. Best practice is:-

  $$-(3n+2)$$

  Once you put the sign for subtraction, the terms after should be put in the bracket. There are more steps involved, but this will lower the risk for a mistake
- Then, apply your knowledge in Algebra by expanding it first before subtracting it. Now, you will get the simplest form of index.

Solution:-

iv.	$\frac{2^{n+4} \times 4(2^{n-1})}{2^n}$

$= \frac{2^{n+4} \times 2^2(2^{n-1})}{2^n}$

$= 2^{n+4+2+(n-1)-n}$

$= 2^{n+4+2+n-1-n}$

$= 2^{5+n}$

- For iv, a frequent **mistake** done by student is $4 \times 2^{n-1} = 8^{n-1}$, which is totally **wrong**.
- For this question, convert 4 to index form with base of 2. So, $4 = 2^2$.
- Now, simplify it by using the **Multiplication Rule** and **Division Rule**.

v.	$(27^4)^{\frac{3}{4}} \times 9^4 \div 81^5$

$= \left((3^3)^4\right)^{\frac{3}{4}} \times (3^2)^4 \div (3^4)^5$

$= 3^{3\times4\times\frac{3}{4}} \times 3^{2\times4} \div 3^{4\times5}$

$= 3^9 \times 3^8 \div 3^{20}$

$= 3^{9+8-20}$

$= 3^{-3}$

$= \frac{1}{3^3}$

$= \frac{1}{27}$

- For v, this question have three indices with different base. The first step to do is, convert all those bases into index form with a similar base.

  It is much more easier to convert it to the lowest value possible for base.

$$27 = 3^3$$
$$9 = 3^2$$
$$81 = 3^4$$

- Then apply the **Law of Power of Power Rule** to simplify all indices.
- After that apply the **Multiplication Rule** and **Division Rule**, then you will get the single term of Index Form.
- For rest of the steps, do not forget to apply the relevant Law of Indices as stated in the previous explanation.

Exercise Time!

Express each of the following Indices expressions in the simplest form.

a. $27^{n} \times 9^{\frac{n}{3}}$

b. $\left(a^{m}\sqrt[3]{b^{4}}\right)^{-2}$

c. $16^{n-1} \times 4^{2-n} \div 2^{2n+3}$

d. $\dfrac{2^{5-x}(3^{x})^{2}}{8^{2-3x}}$

e. $\left(\dfrac{n^{3}}{2m^{2}n^{4+n}}\right)^{2}$

f. $\dfrac{3^{2-m}\left(s^{2}\right)^{m-2}}{\left(3^{2m}\sqrt[3]{s}\right)^{3}}$

g. $64^{n+1} \times 256^{2n} \times \dfrac{2}{4^{n+2}}$

CHECK YOUR ANSWERS!

a. $3^{\frac{11n}{3}}$

b. $\frac{1}{a^{2m}b^{\frac{8}{3}}}$

c. $\frac{1}{8}$

d. $2^{8x-1}3^{2x}$

e. $\frac{1}{4m^4n^{2+2n}}$

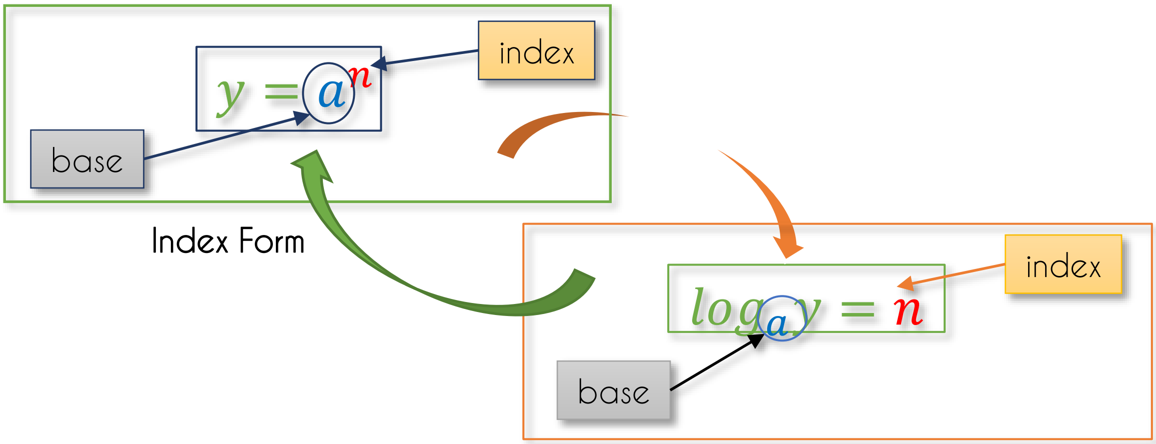
f. $3^{2-7m}s^{2m-5}$

g. 2^{3+20n}

1.3 Law of Logarithms

Logarithm?

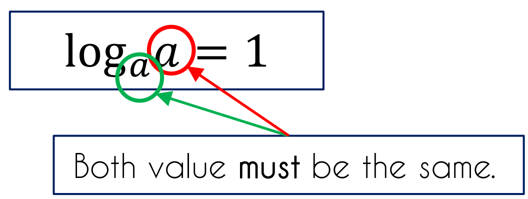
Inverse function to Exponentiation.



Bear in mind, for *Logarithmic Function*, $a > 0$ and $a \neq 1$ *while* $y > 0$.

Remember this

$\log_a a = 1$
$\log_a 1 = 0$



Law of Logarithm

$\log_a MN = \log_a M + \log_a N$	(Multiplication Rule)
$\log_a \frac{M}{N} = \log_a M - \log_a N$	(Division Rule)
$c\log_a M = \log_a M^c$	(Power Rule)
$a^{\log_a M} = M$	

Change base Logarithm

$$\log_a MN = \frac{\log_b M}{\log_b N}$$

$$\log_a M = \frac{1}{\log_M a}$$

Bear in mind,

$\log_a MN$ is not equal to $\log_a M \times \log_a N$.

$\log_a \frac{M}{N}$ is not equal to $\frac{\log_a M}{\log_a N}$.

Example 3

Express the following Indices Equation in Logarithmic Form.

i.	$y = 3^x$	ii.	$5 = 2^n$

Solution:-

i.	$y = 3^x$
$\log_3 y = x$	

ii.	$5 = 2^n$
$\log_2 5 = n$	

Example 4

Express the following Logarithmic Equation in Index Form.

i.	$\log_m 5 = 32$	ii.	$\log_3 \sqrt{t} = 9$

Solution:-

i.	$\log_m 5 = 32$
$5 = m^{32}$	

ii.	$\log_3 \sqrt{t} = 9$
$\sqrt{t} = 3^9$	

1.4 Simplify Logarithmic Expression by Using Law of Logarithms.

Example 5

Express each of the following expressions in the simplest form.

i.	$\log_5 \frac{1}{25}$	ii.	$4\log a + \frac{\log a}{2} - 2\log a$
iii.	$\log_2 a + 2 - 3\log_2 3$	iv.	$2\log_m 4 - n\log_m 2 - 1$
v.	$\log_3 m + 2\log_9 27$		

Solution:-

i.	$\log_5 \frac{1}{25}$

$= \log_5 1 - \log_5 25$

$= \log_5 1 - \log_5 5^2$

$= \log_5 1 - 2\log_5 5$

$= 0 - 2$

$= -2$

ii.	$4\log a + \frac{\log a}{2} - 2\log a$

$= \log a^4 + \log a^{\frac{1}{2}} - \log a^2$

$= \log \frac{a^4 \times a^{\frac{1}{2}}}{a^2}$

$= \log a^{4+\frac{1}{2}-2}$

$= \log a^{\frac{5}{2}}$

$= \frac{5}{2}\log a$

- Use your knowledge in logarithmic to solve **i** instead of directly using your calculator. Bear in mind, programming calculator is forbidden during examination.
- Firstly, use the **Division Rule** to split that single term of logarithm into two logarithmic terms.
- Since the base of logarithm is **5**, and **25** can be converted to index with base of **5**. So do it.

$$\log_5 25 = \log_5 5^2$$

- Then, apply the **Power Rule** to bring the index in front of the logarithm term .

$$\log_5 5^2 = 2\log_5 5$$

- Always remember this law;

$$\log_5 5 = 1; \log_5 1 = 0$$

$$2\log_5 5 = 2 \times 1$$

- For example **ii**, apply the **Power Rule**, then follow by **Multiplication Rule**.

Solution:-

iii.	$\log_2 a + 2 - 3\log_2 3$

$$= \log_2 a + 2\log_2 2 - 3\log_2 3$$
$$= \log_2 a + \log_2 2^2 - \log_2 3^3$$
$$= \log_2 a + \log_2 4 - \log_2 27$$
$$= \log_2 \frac{a \times 4}{27}$$
$$= \log_2 \frac{4a}{27}$$

- Example **iii** consists of three terms where two of it are logarithm terms and one is a number.
- First step, convert the number to a logarithmic form with a similar base with existing logarithmic terms.
- As the existing logarithmic base is **2**, so use the logarithmic base **2** which the value must equal to **2** . i.e. $\log_2 2 = 1$.
- Now you have three terms of logarithmic terms with a similar base. Firstly, apply the **Power Rule** to convert the coefficient to become index (*power*).
- Then apply both the **Multiplication Rule** and **Division Rule** to simplify the logarithmic expression.
- Remember to leave a single term of logarithmic as your final answer.

iv.	$2\log_m 4 - n\log_m 2 - 1$

$$= 2\log_m 4 - n\log_m 2 - log_m m$$
$$= \log_m 4^2 - \log_m 2^n - log_m m$$
$$= \log_m \frac{4^2}{2^n \times m}$$
$$= \log_m \frac{16}{2^n m}$$

- Same concept of solution goes to example **iv.**

Solution:-

v.	$\log_3 m + 2\log_9 27$

$$= \log_3 m + \log_9 27^2$$

$$= \log_3 m + \left(\frac{\log_3 27}{\log_3 9}\right)^2$$

$$= \log_3 m + \left(\frac{\log_3 3^3}{\log_3 3^2}\right)^2$$

$$= \log_3 m + \left(\frac{3\log_3 3}{2\log_3 3}\right)^2$$

$$= \log_3 m + \left(\frac{3}{2}\right)^2$$

$$= \log_3 m + \frac{3^2}{2^2}$$

$$= \log_3 m + \frac{9}{4}$$

$$= \log_3 m + \frac{9}{4}\log_m m$$

$$= \log_3 m + \log_m m^{\frac{9}{4}}$$

$$= \log_3 \left(m \times m^{\frac{9}{4}}\right)$$

$$= \log_3 \left(m^{1+\frac{9}{4}}\right)$$

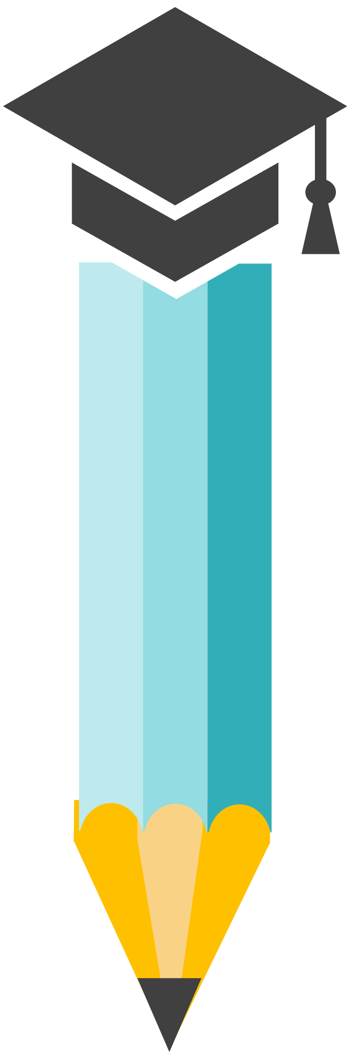
$$= \log_3 m^{\frac{13}{4}}$$

- Example **v** consists of two logarithmic terms with different bases. Without any option, you must use the **Change Base Logarithmic Rule** to make all the logarithmic bases similar.
- You may change both logarithmic terms to any new base of logarithm as you wish, but I prefer to change only one, and the best of course is the bigger value i.e. **9** which I can convert it to index form with base of **3**.

$$9 = 3^2$$

- Before applying the **Change Base Logarithmic Rule** , apply the **Power Rule** first to convert the coefficient to index.
- Then, you can apply the **Change Base Logarithmic Rule**, and I chose to use logarithm with base of **3**.
- For any logarithmic terms, that can be simplified for a value, just do it.
- Then, apply any relevant laws of logarithm as explained in the previous solution to simplify until you get a single term of logarithmic form or a numerical value.

Exercise Time!



Express each of the following Indices expressions in the simplest form.

a. $\log_7 \frac{7}{343}$

b. $4\log_2 b - \frac{2}{3}\log_2 b + 3\log_2 b$

c. $4\log_2 a + 5 - 3\log_2 5$

d. $3 - n\log_m(2x) - 2\log_m(4x)$

e. $2\log_4 p + 3\log_{16} 64$

f. $2\log_m 4 - n\log_m 2 + 2$

g. $\frac{1}{2}\log_m 9 + 3\log_m 2 - 3log_m 2^n$

CHECK YOUR ANSWERS!

a. 2

b. $\log_2 \frac{19}{3}$

c. $\log_2 \frac{32a^4}{125}$

d. $\log_2 \frac{m^3}{64x^{2+n}}$

e. $\log_4 512p^2$

f. $\log_m \frac{16m^2}{2^n}$

g. $\log_m \frac{24}{2^{3n}}$

1.5 Solve Equations Involving Indices and Logarithms Expression.

Example 6

Solve the following equations using a suitable method.

i.	$2^{5x} = \sqrt{2}$	ii.	$3^{2x-1} \times 27^{3-x} = 1$
iii.	$3^{x-2} - 9^{x+4} = 0$	iv.	$256 = 3^{x-5}$
v.	$\log_4(5x-1) = 3$	vi.	$\log_x 8 + \log_x 4 = 5$
vii.	$2^x + 2^{x+3} = 72$	viii.	$3^{x-1} + 3^x = 12$
ix.	$\log(x+3) - \log 5 = \log(1-3x) + 2$		

Solution:-

i.	$2^{5x} = \sqrt{2}$

$$2^{5x} = 2^{\frac{1}{2}}$$

$$5x = \frac{1}{2}$$

$$10x = 1$$

$$x = \frac{1}{10}$$

Left side	=	Right side

If **only** one term exists for each sides (*either indices expression or logarithms expression*), then we can use the comparison concept to ignore anything that is equal.

- Example **i** is an indices equation with a similar base.
- Convert the square root symbol to become a numerical value. i.e. $\sqrt{2} = 2^{\frac{1}{2}}$.
- Now, check how many term of indices exist both on the right and the left side.

Left side	**=**	**Right side**

- As you may see, the left side has 2^{5x} and the right side has $2^{\frac{1}{2}}$. Only one index expression for each side with similar base.
- So ignore the base, just take into consideration only index (power), and solve it by using the Algebraic approach until you get the value of unknown stated in the question.

Solution:-

ii.	$3^{2x-1} \times 27^{3-x} = 1$

$$3^{2x-1} \times \left(3^3\right)^{3-x} = 3^0$$
$$3^{2x-1} \times 3^{3\times(3-x)} = 3^0$$
$$3^{2x-1} \times 3^{9-3x} = 3^0$$
$$3^{2x-1+9-3x} = 3^0$$
$$3^{8-x} = 3^0$$
$$8 - x = 0$$
$$x = 8$$

- For example **ii**, convert all index expression and numerical value to index form.

$$27^{3-x} = \left(3^3\right)^{3-x}$$
$$1 = 3^0$$

- Now, apply the **Power of Power Rule** followed by the **Multiplication Rule** to simplify the left side until you get a single index form.
- Once you get $3^{8-x} = 3^0$, compare the left and right sides to ignore the base (*because both indices have a similar base right?*)
- Then, solve it to get the value of the unknown (*in this question is* x) by using Algebraic approach.

iii.	$3^{x-2} - 9^{x+4} = 0$

$$3^{x-2} = 9^{x+4}$$
$$3^{x-2} = \left(3^2\right)^{x+4}$$
$$3^{x-2} = 3^{2\times(x+4)}$$
$$3^{x-2} = 3^{2x+8}$$
$$x - 2 = 2x + 8$$
$$x - 2x = 8 + 2$$
$$-x = 10$$
$$x = -10$$

- For example **iii**, never think of trying to convert 0 to an index form. You should not be able to do it at all.
- Just take -9^{x+4} to the right side, and now you have an index form for both sides. Then apply the relevant law of indices to solve it.

Solution:-

iv.	$256 = 3^{x-5}$

$$\log 256 = \log 3^{x-5}$$
$$\log 256 = (x-5)\log 3$$
$$\frac{\log 256}{\log 3} = x - 5$$
$$5.0474 \approx x - 5$$
$$x \approx 5.0474 - 5$$
$$x \approx 0.474$$

- For example **iv**, it is quite possible to guess $3^{?} = 256$, we know $3^5 = 243$ and $3^6 = 729$. So ? is any value in between 5 to 6. Will you spend hours just to get the exact value?
- To save time, you just simply give logarithm with the same base to both left and right terms. **This can only be applied if only one term exists for each side.**
- I used logarithm with a base of 10. Bear in mind, logarithm with a base of 10 can be written as :-

$$\log A = \log_{10} A = \lg A$$

- Then, apply the **Power Rule** to convert index (power) for $\log 3^{x-5}$ to a coefficient.
- Now, use the Algebraic approach to solve it until you get the value of x.

v.	$\log_4(5x-1) = 3$

$$\log_4(5x-1) = 3\log_4 4$$
$$\log_4(5x-1) = \log_4 4^3$$
$$5x - 1 = 4^3$$
$$5x - 1 = 64$$
$$5x = 65$$
$$x = \frac{65}{5}$$
$$x = 13$$

- In Example **v**, there is a logarithm expression on the left side while a number on right side. So, we need to convert 3 into a logarithmic form. To avoid using the Change Base Logarithm Rule, do use the existing base in the question i.e. logarithm with base 4 with value of 1.
- Then, apply the **Power Rule** to ensure there is no coefficient for every logarithm term. Now, solve it until you get the value of x.

Solution:-

vi.	$\log_x 8 + \log_x 4 = 5$

$\log_x 8 + \log_x 4 = 5\log_x x$

$log_x(8 \times 4) = log_x x^5$

$log_x 32 = log_x x^5$

$32 = x^5$

$2^5 = x^5$

$x = 2$

- For example **vi**, there are two terms of logarithmic expression on the left and a numerical value on the right side.
- So, our aim is to ensure there is only one logarithmic expression with the same base on the left and right,
- Therefore, apply the **Multiplication Rule** on the left while on the right convert **5** to logarithmic form.
- Do not forget to apply the **Power Rule** on the right side once you convert it to logarithmic form to ensure that there is no coefficient which exists in the expression.
- Then, compare for both left and right sides and solve it.

vii.	$2^x + 2^{x+3} = 72$

$2^x + 2^x(2^3) = 72$

$2^x + 8(2^x) = 72$

$2^x(1 + 8) = 72$

$9(2^x) = 72$

$2^x = \frac{72}{9}$

$2^x = 8$

$2^x = 2^3$

$x = 3$

- Always be extra careful when you get the question as in **vii**, there is no law in indices for addition and subtraction index. The only solution is to solve it by using the Algebraic approach but before that, split any index that is possible.
- In this case, 2^{x+3} can be split to $2^x \times 2^3$, then get a value for 2^3 which is **8.**
- Now, apply the Algebraic approach to solve it.

Solution:-

viii.	$3^{x-1} + 3^x = 12$

$3^x \times 3^{-1} + 3^x = 12$

$\frac{3^x}{3} + 3^x = 12$

$3^x\left(\frac{1}{3} + 1\right) = 12$

$\frac{4}{3}(3^x) = 12$

$3^x = 12 \times \frac{3}{4}$

$3^x = 9$

$3^x = 3^2$

$x = 2$

- For example **viii**, apply the same concept as the solution for example vii.
- What is different is, 3^{x-1} is split to $3^x \times 3^{-1}$ or you may directly write it as $3^x \div 3^1$.
- Then, apply the Algebraic approach to solve it.

- For example **ix**, I am pretty sure that you are now confident enough to understand the steps for the solution.

ix.	$\log(x+3) - \log 5 = \log(1-3x) + 2$

$\log(x+3) - \log 5 = \log(1-3x) + 2\log 10$

$\log(x+3) - \log 5 = \log(1-3x) + \log 10^2$

$\log\frac{x+3}{5} = \log(1-3x) \times 10$

$\log\frac{x+3}{5} = \log 10(1-3x)$

$\frac{x+3}{5} = 10(1-3x)$

$\frac{x+3}{5} = 10 - 30x$

$x + 3 = 5(10 - 30x)$

$x + 3 = 50 - 150x$

$x + 150x = 50 - 3$

$151x = 47$

$x = \frac{47}{151}$

Exercise Time!

Solve the following equations using a suitable method.

a. $\left(3^{2x-1}\right)^4 \times 27^{3-x} = 1$

b. $81^{2x-2} - 27^{4-5x} = 0$

c. $157 = 7^{3x-5}$

d. $\log_4(5x-1) - 3 = \log_4(2x-3)$

e. $\log_x 128 - \log_x 4 = 5$

f. $4^x + 56 = 4^{x+3}$

g. $3^{x-1} + 2(3^x) = 12 - 3^{2+x}$

CHECK YOUR ANSWERS!

a. $x = 1$

b. $x = \frac{20}{23}$

c. $x = 2.5328$

d. $x = \frac{191}{123}$

e. $x = 2$

f. $x = -0.0850$

g. $x = 0.0520$

Example 7

Given $\log_4 3 = 0.7925$ and $\log_4 6 = 1.2925$, without using a calculator, calculate the value of the following:-

i.	$\log_4 36$	ii.	$\log_4 0.5$
iii.	$\log_3 162$	iv.	$2\log_4 72$
v.	$\frac{\log_4 9}{3} - \log_3 4$	vi.	$\log_6 4 + \log_4 6$
vii.	$2\log_3 6 + \frac{1}{2}\log_4 \sqrt{6} - \log_4 9$		

Solution:-

i.	$\log_4 36$
	$= \log_4(6 \times 6)$ $= \log_4 6 + \log_4 6$ $= 1.2925 + 1.2925$ $= 2.585$

ii.	$\log_4 0.5$
	$= \log_4 \frac{3}{6}$ $= \log_4 3 - \log_4 6$ $= 0.7925 - 1.2925$ $= 2.585$

- Example 7 is totally different as compared to the previous example. $\log_4 3 = 0.7925, \quad \log_4 6 = 1.2925$
- And one more; $\log_4 4 = 1$
- So, for **i** until **vii**, you must use only these three logarithmic values for the solution.
- As you may see for **i**, $\log_4 36$ is converted to $\log_4(6 \times 6)$, then use the **Multiplication Rule** to get $\log_4 6 + \log_4 6$.
- While for ii, $\log_4 0.5$ is converted to $\log_4 \frac{3}{6}$, then use the Division Rule to get $\log_4 3 - \log_4 6$.
- For the next step, just substitute the given value accordingly and you will get the answer.

Solution:-

iii.	$\log_3 162$

$$= \frac{\log_4 162}{\log_4 3}$$

$$= \frac{\log_4(6 \times 3 \times 3 \times 3)}{\log_4 3}$$

$$= \frac{\log_4(6 \times 3^3)}{\log_4 3}$$

$$= \frac{\log_4 6 + \log_4 3^3}{\log_4 3}$$

$$= \frac{\log_4 6 + 3\log_4 3}{\log_4 3}$$

$$= \frac{1.2925 + 3(0.7925)}{0.7925}$$

$$\approx 4.6309$$

- For **iii,** use the **Change Base Logarithm Rule** to convert logarithm base of 3 to logarithm base of 4 first.
- $\log_3 162 = \frac{\log_4 162}{\log_4 3}$
- Then, determine the combination of these three numbers (3, 4 and 6) that can be multiplied to get 162.
- The step of converting $3 \times 3 \times 3$ to 3^3 is optional.
- Now you can apply the relevant **Law of Logarithm** to get the answer.

iv.	$2\log_4 72$

$$= 2\log_4(6 \times 3 \times 4)$$

$$= 2(\log_4 6 + \log_4 3 + \log_4 4)$$

$$= 2(1.2925 + 0.7925 + 1)$$

$$= 2(3.085)$$

$$= 6.17$$

- You should not have any problem to understand how to solve **iv** right?

Solution:-

v.	$\frac{\log_4 9}{3} - \log_3 4$

$$= \frac{\log_4(3 \times 3)}{3} - \frac{\log_4 4}{\log_4 3}$$

$$= \frac{\log_4 3 + \log_4 3}{3} - \frac{\log_4 4}{\log_4 3}$$

$$= \frac{0.7925 + 0.7925}{3} - \frac{1}{0.7925}$$

$$\approx 0.5283 - 1.2618$$

$$\approx -0.7335$$

- From now and on, (*I mean for v until vii*), I would not explain on the steps for the solution.
- You should recall and try to understand on your own by referring to the solution as your guide.

vi.	$\log_3 6 + \log_4 6$

$$= \frac{\log_4 6}{\log_4 3} + \log_4 6$$

$$= \frac{1.2925}{0.7925} + 1.2925$$

$$\approx 1.6309 + 1.2925$$

$$\approx 2.9234$$

Solution:-

vii.	$2\log_3 6 + \frac{1}{2}\log_4 \sqrt{6} - \log_4 9$

$$= 2\left(\frac{\log_4 6}{\log_4 3}\right) + \frac{1}{2}\log_4 6^{\frac{1}{2}} - \log_4(3 \times 3)$$

$$= 2\left(\frac{\log_4 6}{\log_4 3}\right) + \frac{1}{4}\log_4 6 - (\log_4 3 + \log_4 3)$$

$$= 2\left(\frac{1.2925}{0.7925}\right) + \frac{1}{4}(1.2925) - (0.7925 + 0.7925)$$

$$\approx 3.2618 + 0.3231 - 1.585$$

$$\approx 1.9999$$

Exercise Time!

Given $\log_5 7 = 1.2096$ and $\log_5 3 = 0.6826$, without using a calculator , calculate the value of the following:-.

a. $\frac{\log_5 7}{3} - \frac{2}{\log_5 3} + 4$

b. $\log_5 49 + \log_5 27 - \log_5 125$

c. $2\log_5 1.4 - 3\log_5 0.6$

d. $2\log_3 7 - \frac{1}{2}\log_5 \sqrt{49} - \log_5 21$

e. $2\log_3 5 + \frac{1}{2}\log_7 81$

f. $2\log_3 105 - \log_7 125 + \log_5 63$

CHECK YOUR ANSWERS!

a. 1.4731

b. 1.4659

c. 1.3703

d. 1.0463

e. 4.0591

f. 8.5654

CHAPTER 2

DIFFERENTIATION

2.1 Rules of Differentiation

Introduction

Derivatives are the result of performing a differentiation process upon a function or an expression.

Derivative notation is the way we express the derivatives mathematically.

Types of Derivatives Notation

Lagrange's notation:

$f'(x)$

Leibniz's notation:

$\frac{dy}{dx}$

Newton's notation:

$\dot{y}$

Equation/expression/function		Derivative Notation
$y = x^2$	$\rightarrow$	$\frac{dy}{dx}$
$m = n^2$	$\rightarrow$	$\frac{dm}{dn}$
$p = q^2$	$\rightarrow$	$\frac{dp}{dq}$
$y = ✿^2$	$\rightarrow$	$\frac{dy}{d✿}$
$f(x) = x^2$	$\rightarrow$	$f'(x)$
$f(☆) = ☆^2$	$\rightarrow$	$f'(☆)$
w^2	$\rightarrow$	$\frac{d}{dw}$
q^2	$\rightarrow$	$\frac{d}{dq}$

Constant rule

$y = c$, $\dfrac{dy}{dx} = 0$ OR $y = c$, $\dfrac{d}{dx}(c) = 0$

Constant multiple rule

$y = ax$, $\dfrac{dy}{dx} = a$

Power function rule

$y = x^n$, $\dfrac{dy}{dx} = nx^{n-1}$ OR $y = x^n$, $\dfrac{d}{dx}(x^n) = nx^{n-1}$

$y = ax^n$, $\dfrac{dy}{dx} = nax^{n-1}$ OR $y = ax^n$, $\dfrac{d}{dx}(ax^n) = nax^{n-1}$

Example 1

$y = 3$

$\frac{dy}{dx} = 0$

$y = \sqrt[3]{4}$

$\frac{dy}{dx} = 0$

$y = \frac{3}{5}$

$\frac{dy}{dx} = 0$

$y = 2.5\pi$

$\frac{dy}{dx} = 0$

Example 2

$y = x^5$

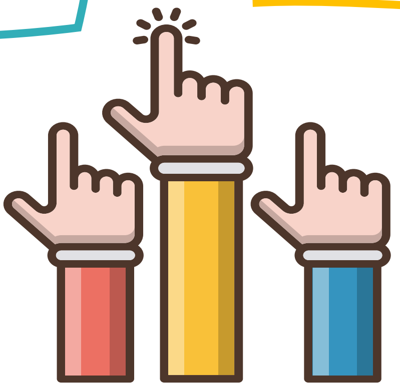
$\frac{dy}{dx} = 5x^{5-1}$

$= 5x^4$

$y = 3x^4$

$\frac{dy}{dx} = (4)3x^{4-1}$

$= 12x^3$



Example 3

$y = 3x$

$\frac{dy}{dx} = 3$

$y = -4x^3$

$\frac{dy}{dx} = (3)(-4)x^2$

$= -12x^2$

$y = -\frac{3}{x^4}$

$y = -3x^{-4}$

$\frac{dy}{dx} = 12x^{-5}$

$= \frac{12}{x^5}$

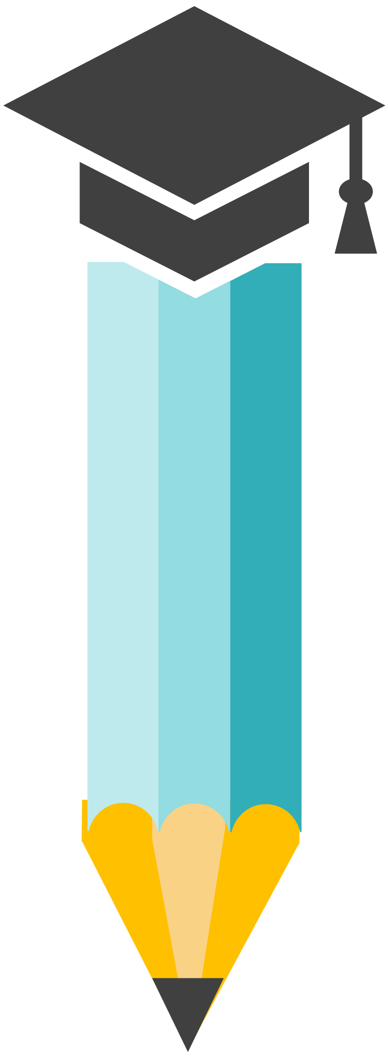
$y = \sqrt[5]{m^3}$

$y = m^{\frac{3}{5}}$

$\frac{dy}{dx} = \frac{3}{5}m^{-\frac{2}{5}}$

$= \frac{3}{5\sqrt[5]{m^2}}$

Exercise Time!



Find the derivatives of the following functions.

a. $y = -\sqrt{10}$

b. $y = \pi$

c. $f(t) = 4.132t$

d. $y = -\frac{7}{10}x$

e. $k = l^5$

f. $y = -8x^6$

g. $y = \frac{x^{\frac{1}{2}}}{2}$

h. $f(x) = \sqrt[3]{x^2}$

i. $y = -x^{-\frac{4}{7}}$

j. $y = \frac{2}{3x^5}$

k. $r = \frac{4}{5\sqrt{t}}$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = 0$

b. $\frac{dy}{dx} = 0$

c. $f'(t) = 4.132$

d. $\frac{dy}{dx} = -\frac{7}{10}$

e. $\frac{dk}{dl} = 5l^4$

f. $\frac{dy}{dx} = -48x^5$

g. $\frac{dy}{dx} = \frac{1}{4\sqrt{x}}$

h. $f'(x) = \frac{2}{3\sqrt[3]{x}}$

i. $\frac{dy}{dx} = \frac{4}{7x^{\frac{11}{7}}}$

j. $\frac{dy}{dx} = -\frac{10}{3x^6}$

k. $\frac{dr}{dt} = -\frac{2}{5\sqrt{t^3}}$

Sum & Difference Rule

If given, $y = f(x) \pm g(x)$

thus

$$y = f(x) \pm g(x)$$

$$\frac{dy}{dx} = f'(x) \pm g'(x)$$

Example 1

$$y = 8 - x^5$$

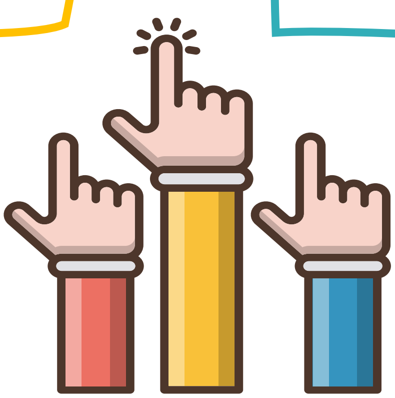
$$\frac{dy}{dx} = 0 - 5x^4$$

$$= -5x^4$$

Example 2

$$y = 3x^3 + 9x + 3$$

$$\frac{dy}{dx} = 9x^2 + 9$$



Example 3

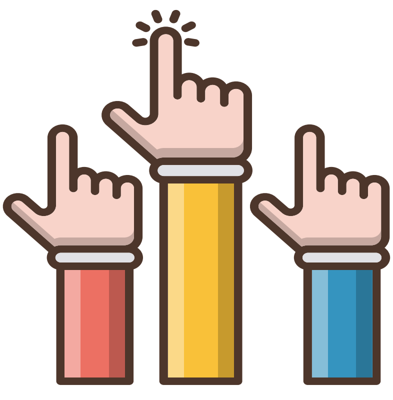
$$f(s) = \frac{2s^6 + 3s^3 - 5}{s^3}$$

$$= \frac{2s^6}{s^3} + \frac{3s^3}{s^3} - \frac{5}{s^3}$$

$$= 2s^3 + 3 - 5s^{-3}$$

$$f'(s) = 6s^2 + 15s^{-4}$$

$$= 6s^2 + \frac{15}{s^4}$$



Example 4

$$y = 6x^4 - \frac{3}{x^2} + 3x$$

$$= 6x^4 - 3x^{-2} + 3x$$

$$\frac{dy}{dx} = 24x^3 + 6x^{-3} + 3$$

$$= 24x^3 + \frac{6}{x^3} + 3$$

Example 5

$$m = (n+3)(1-2n)$$

$$= 3 - 5n - 2n^2$$

$$\frac{dm}{dn} = -5 - 4n$$

Exercise Time!

Find the derivatives of the following functions.

a. $y = \frac{2}{3}x^3 - 4\pi$

b. $y = 7 - \frac{2}{3x^5} + 5x^3$

c. $y = 3x^3 - \frac{4}{x^5} + 11$

d. $y = 8x^3 - \frac{5}{2}x - 1$

e. $y = 3x^4 - 5x + \sqrt{x}$

f. $y = -x^2 + \frac{1}{2x^2} - 7$

g. $y = 3x^3 + \frac{4}{x^2} - \sqrt[4]{x}$

h. $y = \frac{X}{2} - \frac{3}{X^2} - 2$

i. $y = 4x^2 + \frac{1}{x^2} - \frac{2}{x}$

j. $y = \frac{2x^6 + 4x^5 + 3x}{x}$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = 2x^2$

b. $\frac{dy}{dx} = \frac{10}{3x^6} + 15x^2$

c. $\frac{dy}{dx} = 9x^2 + \frac{20}{x^6}$

d. $\frac{dy}{dx} = 24x^2 - \frac{5}{2}$

e. $\frac{dy}{dx} = 12x^3 - 5 - \frac{1}{2\sqrt{x}}$

f. $\frac{dy}{dx} = -2x - \frac{1}{x^3}$

g. $\frac{dy}{dx} = 9x^2 - \frac{8}{x^3} - \frac{1}{4\sqrt[4]{x^3}}$

h. $\frac{dy}{dx} = \frac{1}{2} + \frac{6}{x^3}$

i. $\frac{dy}{dx} = 8 - \frac{2}{x^3} + \frac{2}{x^2}$

j. $\frac{dy}{dx} = 10x^4 + 16x^3$

Composite Function

If $y = (ax^n + b)^k$

thus

$$y = (ax^n + b)^k$$

$$\frac{dy}{dx} = k(ax^n + b)^{k-1} \bullet nax^{n-1}$$

TIPS!

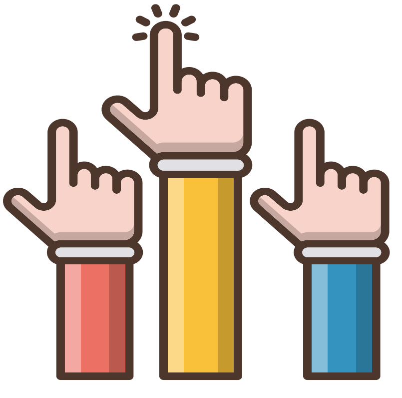
(derivative of outside) x (derivative of inside)

Example 1

$$y = \left(3 - 4x^2\right)^5$$

$$\frac{dy}{dx} = 5\left(3 - 4x^2\right)^{5-1} \cdot (-8x)$$

$$= -40x\left(3 - 4x^2\right)^4$$



Example 2

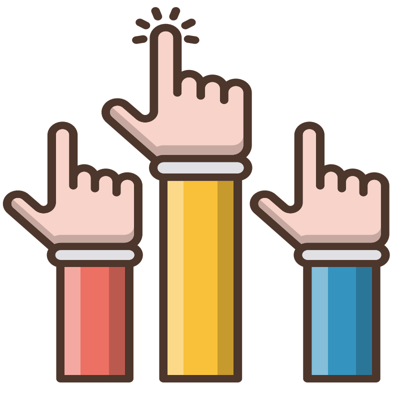
$$y = \frac{-5}{(4x+3)^3}$$

$$y = -5(4x+3)^{-3}$$

$$\frac{dy}{dx} = (-3)(-5)(4x+3)^{-3-1} \cdot (4)$$

$$= 60(4x+3)^{-4}$$

$$= \frac{60}{(4x+3)^4}$$



Example 3

$$y = \sqrt[5]{(x^5-1)^2}$$

$$y = (x^5-1)^{\frac{2}{5}}$$

$$= \frac{2}{5}(x^5-1)^{\frac{2}{5}-1} \cdot (5x^4)$$

$$= 2x^4(x^5-1)^{-\frac{3}{5}}$$

$$= \frac{2x^4}{(x^5-1)^{\frac{3}{5}}}$$

Example 4

$$y = \left(2 - \frac{3}{x^3}\right)^4$$

$$y = (2-3x^{-3})^4$$

$$\frac{dy}{dx} = 4(2-3x^{-3})^{4-1} \cdot (9x^{-4})$$

$$= \frac{36}{x^4}\left(2 - \frac{3}{x^3}\right)^3$$

Chain Rule

If given, $y = u^n$

Where

$y = f(u)$ and $u = g(x)$

thus

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 1

$$y = (3 - 4x^2)^5$$

$u = 3 - 4x^2$ $\quad$ $y = u^5$

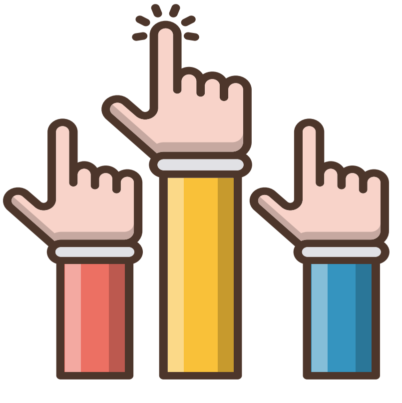
$\frac{du}{dx} = -8x$ $\quad$ $\frac{dy}{du} = 5u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5u^4 \cdot (-8x)$$

$$= -40x(u)^4$$

$$= -40x(3 - 4x^2)^4$$



Example 2

$$y = \frac{-3}{(4x+3)^3}$$

$$y = -3(4x+3)^{-3}$$

$$u = 4x + 3 \qquad y = -3u^{-3}$$

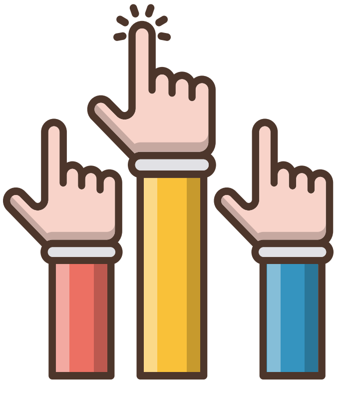
$$\frac{du}{dx} = 4 \qquad \frac{dy}{du} = 9u^{-4}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 9u^{-4} \times 4$$

$$= \frac{36}{u^4}$$

$$= \frac{36}{(4x+3)^4}$$



Example 3

$$y = \left(2 - \frac{3}{x^3}\right)^4$$

$$y = \left(2 - 3x^{-3}\right)^4$$

$$u = 2 - 3x^{-3} \qquad y = u^4$$

$$\frac{du}{dx} = 9x^{-4} \qquad \frac{dy}{du} = 4u^3$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3 \times 9x^{-4}$$

$$= \frac{36u^3}{x^4}$$

$$= \frac{36}{x^4}\left(2 - \frac{3}{x^3}\right)^3$$

Exercise Time!

Find the derivatives of the following functions by using the composite function formula and chain rule method.

a. $y = \left(4x^2 + 1\right)^3$

b. $y = \left(1 - 4x^2\right)^4$

c. $y = 2(3x + 5)^5$

d. $y = -8(5x^4 + 3x)^{-3}$

e. $f(x) = \sqrt{3x - 1}$

f. $y = \sqrt[3]{(4 - 9x)^2}$

g. $y = \frac{1}{(2-3x)^4}$

h. $y = \frac{2}{\sqrt[5]{(2x+1)}}$

i. $y = \frac{-7}{(4x^3+1)^2}$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = 24x\left(4x^2 + 1\right)^2$

b. $\frac{dy}{dx} = -32x\left(1 - 4x^2\right)^3$

c. $\frac{dy}{dx} = 30(3x + 5)^4$

d. $\frac{dy}{dx} = \frac{480x^3}{(5x^4+3x)^4}$

e. $f'(x) = \frac{3}{2\sqrt{3x-1}}$

f. $\frac{dy}{dx} = \frac{-6}{\sqrt[3]{(4-9x)}}$

g. $\frac{dy}{dx} = \frac{12}{(2-3x)^5}$

h. $\frac{dy}{dx} = \frac{4}{5\sqrt[5]{(2x+1)^4}}$

i. $\frac{dy}{dx} = \frac{168x^2}{(4x^3+1)^3}$

Product Rule

If given two functions in the form of,

$$y = f(x) \times g(x)$$

where

$u = f(x)$ and $v = g(x)$

thus

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example 1

$$y = 2x^2(3x+4)^4$$

$u = 2x^2$ $\qquad v = (3x+4)^4$

$\frac{du}{dx} = 4x$ $\qquad \frac{dv}{dx} = 4(3x+4)^3(3)$

$\qquad\qquad = 12(3x+4)^3$

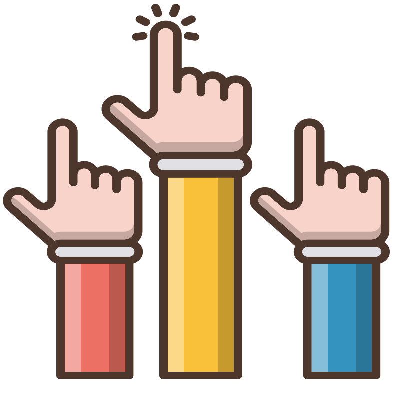
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (3x+4)^4(4x) + (2x^2)[12(3x+4)^3]$$

$$= 4x(3x+4)^3[3x+4+(2x)(3)]$$

$$= 4x(3x+4)^3[3x+4+6x]$$

$$= 4x(3x+4)^3[9x+4]$$



Example 2

$$y = \left(x^3 + 4\right)(2 + 5x)^3$$

$$u = x^3 + 4 \qquad v = (2 + 5x)^3$$

$$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = 3(2 + 5x)^2(5)$$

$$= 15(2 + 5x)^2$$

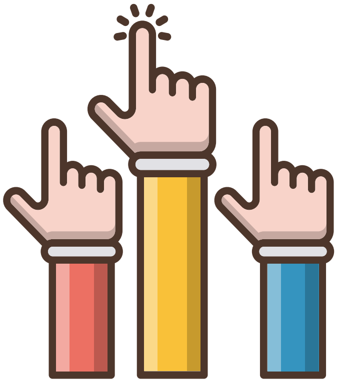
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (2 + 5x)^3\left(3x^2\right) + \left(x^3 + 4\,\right)(15)(2 + 5x)^2$$

$$= 3(2 + 5x)^2\left[x^2(2 + 5x) + 5\left(x^3 + 4\,\right)\right]$$

$$= 3(2 + 5x)^2\left(2x^2 + 5x^3 + 5x^3 + 20\right)$$

$$= 3(2 + 5x)^2\left(2x^2 + 10x^3 + 20\right)$$



Example 3

$$y = (2x - 1)^3(x + 3)^5$$

$$u = (2x - 5)^3 \qquad v = (x + 3)^5$$

$$\frac{du}{dx} = 3(2x - 5)^2(2) \qquad \frac{dv}{dx} = 5(x + 3)^4(1)$$

$$= 6(2x - 5)^2 \qquad = 5(x + 3)^4$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (x + 3)^5(6)(2x - 5)^2 + (2x - 5)^3(5)(x + 3)^4$$

$$= (x + 3)^4(2x - 5)^2[6(x + 3) + 5(2x - 5)]$$

$$= (x + 3)^4(2x - 5)^2[6x + 18 + 10x - 25]$$

$$= (x + 3)^4(2x - 5)^2[16x - 7]$$

Exercise Time!

Find the derivatives of the following functions.

a. $y = (3x+2)(4x-1)^2$

b. $y = \left(\frac{1}{x^3}\right)(2x-2)^4$

c. $y = (5x+2)^4(x-3)^2$

d. $y = x^2(2x+7)^3$

e. $y = (1-x)\left(2x^2-3\right)$

f. $y = \left(2x^3\right)(2-4x)$

g. $s = t^3\left(5+6t^2\right)^4$

h. $y = (2x+3)^6(x-5)^5$

i. $y = \sqrt{x+2}(2x-3)^4$

j. $y = (4x+3)^3(3x-2)$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = (4x-1)[36x+13]$

b. $\frac{dy}{dx} = \frac{(2x-2)^3}{x^3}\left[2+\frac{6}{x}\right]$

c. $\frac{dy}{dx} = 2(5x+2)^3(x-3)\ [15x-28]$

d. $\frac{dy}{dx} = 2x(2x+7)^2\ [5x+7]$

e. $\frac{dy}{dx} = -6x^2+4x+3$

f. $\frac{dy}{dx} = 12x^2-32x^3$

g. $\frac{ds}{dt} = 3t^2\left(5+6t^2\right)^3\ [22t^2+5]$

h. $\frac{dy}{dx} = (2x+3)^5(x-5)^4[22x-45]$

i. $y = \frac{(2x-3)^3}{\sqrt{x+2}}\left[9x+\frac{35}{2}\right]$

j. $\frac{dy}{dx} = 3(4x+3)^2\ [16x-5]$

Quotient Rule

If given two functions in the form of,

$$y = \frac{f(x)}{g(x)}$$

where

$u = f(x)$ and $v = g(x)$

thus

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1

$$y = \frac{x^5}{(2x+8)^6}$$

$$u = x^5 \qquad\qquad v = (2x+8)^6$$

$$\frac{du}{dx} = 5x^4 \qquad\qquad \frac{dv}{dx} = 6(2x+8)^5(2)$$

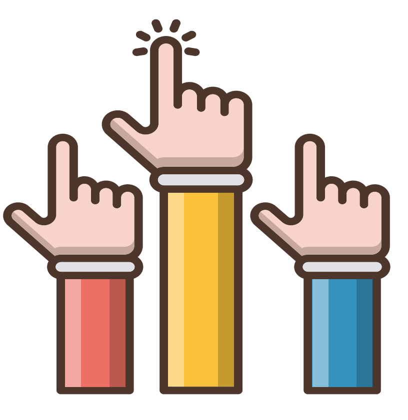
$$= 12(2x+8)^5$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(2x+8)^6(5x^4) - (x^5)(12)(2x+8)^5}{((2x+8)^6)^2}$$

$$= \frac{x^4(2x+8)^5[(5)(2x+8) - 12x]}{(2x+8)^{12}}$$

$$= \frac{x^4[10x + 40 - 12x]}{(2x+8)^7} \qquad = \frac{x^4(40-2x)}{(2x+8)^7}$$



Example 2

$$y = \frac{(2x^3 + 4)}{(1 - 5x)^3}$$

$$u = 2x^3 + 4 \qquad v = (1 - 5x)^3$$

$$\frac{du}{dx} = 6x^2 \qquad \frac{dv}{dx} = 3(1 - 5x)^2(-5)$$

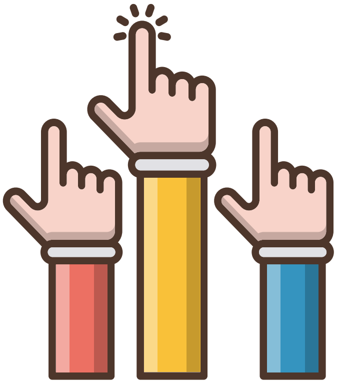
$$= -15(1 - 5x)^2$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1 - 5x)^3(6x^2) - (2x^3 + 4)(-15)(1 - 5x)^2}{\left((1 - 5x)^3\right)^2}$$

$$= \frac{(1 - 5x)^2[6x(1 - 5x) - (-15)(2x^3 + 4)]}{(1 - 5x)^6}$$

$$= \frac{(6x - 30x^2 + 30x^3 + 60)}{(1 - 5x)^4}$$



Example 3

$$y = \frac{(2x - 1)^3}{(x + 1)^5}$$

$$u = (2x - 1)^3 \qquad v = (x + 1)^5$$

$$\frac{du}{dx} = 3(2x - 1)^2(2) \qquad \frac{dv}{dx} = 5(x + 1)^4(1)$$

$$= 6(2x - 1)^2 \qquad = 5(x + 1)^4$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x + 1)^5(6)(2x - 1)^2 - (2x - 1)^3(5)(x + 1)^4}{\left((x + 1)^5\right)^2}$$

$$= \frac{(x + 1)^4(2x - 1)^2[6(x + 1) - 5(2x - 1)]}{(x + 1)^{10}}$$

$$= \frac{(2x - 1)^2(6x + 6 - 10x + 5)}{(x + 1)^6}$$

$$= \frac{(2x - 1)^2(11 - 4x)}{(x + 1)^6}$$

Exercise Time!

Find the derivatives of the following functions.

a. $y = \frac{(x-1)^3}{3x-4}$

b. $y = \frac{5x+2}{x^2-3}$

c. $y = \frac{x^3+2}{(x^2-3)^2}$

d. $y = \frac{x^3}{\sqrt{x^2+1}}$

e. $y = \frac{(2x-6)^5}{(5+3x)^6}$

f. $f(x) = \frac{2x^3-5x}{x+1}$

g. $g(t) = \frac{\sqrt{t^2+2}}{(t^4-1)^5}$

h. $m = \frac{(n^3-2)^4}{3-n^2}$

i. $y = \left(\frac{x+2}{x-2}\right)^3$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = \frac{3(x-1)^2[2x-3]}{(3x-4)^2}$

b. $\frac{dy}{dx} = \frac{-5x^2-4x-15}{(x^2-3)^2}$

c. $\frac{dy}{dx} = \frac{x[-x^3-9x-8]}{(x^2-3)^3}$

d. $\frac{dy}{dx} = \frac{x^2[2x^2+3]}{\sqrt{(x^2+1)^3}}$

e. $\frac{dy}{dx} = \frac{5(2x-6)^4[79-3x]}{(5+3x)^7}$

f. $f'(x) = \frac{4x^3+6x^2-5}{(x+1)^2}$

g. $g'(t) = \frac{t[-19t^4-40t^2-1]}{\sqrt{t^2+2}(t^4-1)^6}$

h. $\frac{dm}{dn} =$

$$\frac{2n(n^3-2)^3[-5n^3+18n-2]}{(3-n^2)^2}$$

i. $\frac{dy}{dx} = \frac{-12(x+2)^2}{(x-2)^4}$

2.2 Trigonometric, Logarithmic & Exponential Functions

Trigonometric Function

Reciprocal Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

Trigonometric Identities

$$cos^2\, x + sin^2\, x = 1$$

$$1 + tan^2\, x = sec^2\, x$$

$$cot^2\, x + 1 = cosec^2\, x$$

Basic Derivatives

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = sec^2\, x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

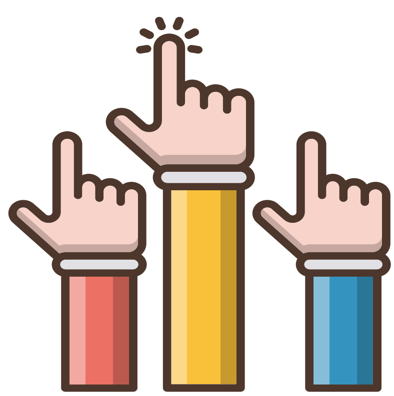
$$\frac{d}{dx}(\cot x) = -cosec^2\, x$$

Example 1

$$y = \cos 3x$$

$$\frac{dy}{dx} = -\sin 3x \cdot (3)$$

$$= -3 \sin 3x$$

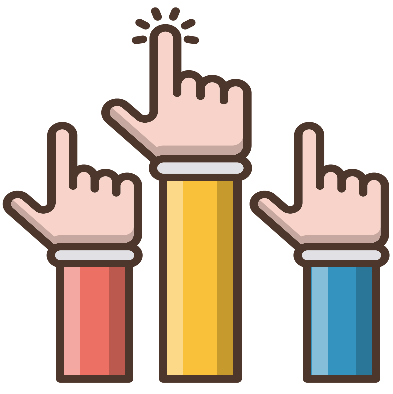


Example 2

$$y = \frac{1}{2}\tan 4x$$

$$\frac{dy}{dx} = \frac{1}{2}sec^2\ 4x \cdot (4)$$

$$= 2\ sec^2(1-4x)$$



Example 3

$$y = \sin\frac{2}{3}x$$

$$\frac{dy}{dx} = \cos\frac{2}{3}x \cdot \left(\frac{2}{3}\right)$$

$$= \frac{2}{3}\cos\frac{2}{3}x$$

Example 4

$$y = \frac{3}{8}\cos\frac{2}{x^4}$$

$$= \frac{3}{8}\cos 2x^{-4}$$

$$\frac{dy}{dx} = \frac{3}{8}(-\sin 2x^{-4}) \cdot (-8x^{-5})$$

$$= \frac{3}{x^5}\sin\frac{2}{x^4}$$

Exercise Time!

Find the derivatives of the following functions.

a. $y = \tan 2x$

b. $y = \cos\left(4x^2 + 1\right)$ (using Chain Rule)

c. $y = 2\sin(\cos 3x)$

d. $y = 3sin^2\left(2x^2 - 1\right)$

e. $y = cos^3 3x$

f. $y = \frac{3}{4} tan^3(6x - 2)$

g. $s = \frac{\cos 2x}{\tan 2x}$

h. $y = \sin\left(x^2 + 3\right)\cos\left(x^2 + 3\right)$

i. $y = \sin\left(\frac{1}{2}x^4 - 3\right)^2$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = 2\,sec^2\,2x$

b. $\frac{dy}{dx} = -8x\ \sin\left(4x^2 + 1\right)$

c. $\frac{dy}{dx} = -6\cos(\cos 3x)\sin 3x$

d. $\frac{dy}{dx} = 24x\sin\left(2x^2 - 1\right)\cos\left(2x^2 - 1\right)$

e. $\frac{dy}{dx} = \text{- }9\,cos^2 3x\sin 3x$

f. $\frac{dy}{dx} = \frac{27}{2}tan^2(6x - 2)\,sec^2(6x - 2)$

g. $\frac{ds}{dx} = \frac{2[-\tan 2x\sin 2x - \cos 2x sec^2 2x]}{tan^2\,2x}$

h. $\frac{dy}{dx} = 2x\left[\cos^2\left(x^2 + 3\right) - sin^2\left(x^2 + 3\right)\right]$

i. $\frac{dy}{dx} = 4x^3\left(\frac{1}{2}x^4 - 3\right)\cos\left(\frac{1}{2}x^4 - 3\right)^2$

Logarithmic Function

If given, $y = \ln x$

$$y = \ln x$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Or we can write

$$y = \ln f(x)$$
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

For some problem, we need to use the logarithm laws to simplify the expression before differentiating it.

i. $\log(m)(n) = \log m + \log n$

ii. $\log\left(\frac{m}{n}\right) = \log m - \log n$

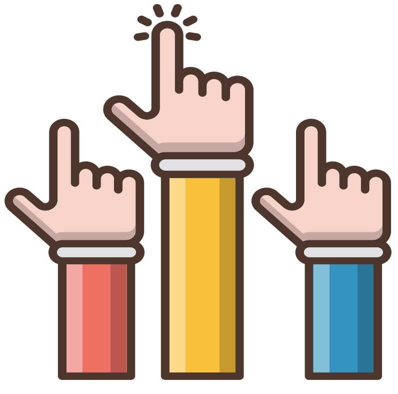
iii. $\log m^n = n \log m$

Example 1

$$y = \ln(3x-2)$$

$$\frac{dy}{dx} = \frac{1}{(3x-2)}(3)$$

$$= \frac{3}{(3x-2)}$$



Example 2

$$y = \ln\sqrt{5x+2}$$

$$= \ln\,(5x+2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\ln(5x+2)$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{(5x+2)}(5)$$

$$= \frac{5}{2(5x+2)}$$

Example 3

$$f(x) = \ln(5x-3)^2(3x^2+4)$$

$$= \ln(5x-3)^2 + \ln(3x^2+4)$$

$$= 2\ln(5x-3) + \ln(3x^2+4)$$

$$f'(x) = 2\cdot\frac{1}{(5x-3)}\cdot(5) + \frac{1}{(3x^2+4)}\cdot(6x)$$

$$= \frac{10}{(5x-3)} + \frac{6x}{(3x^2+4)}$$

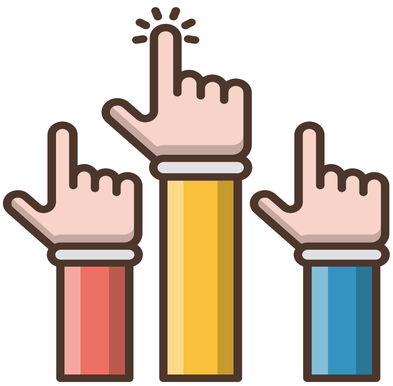
Example 4

$$f(x) = \ln(\sin 3x^2)$$

$$f'(x) = \frac{1}{(\sin 3x^2)} \cdot (\cos(3x^2)(6x))$$

$$= \frac{6x\cos(3x^2)}{(\sin 3x^2)}$$

$$= 6x\cot 3x^2$$



Example 5

$$g(x) = x^4\ln(5x+2)$$

$$u = x^4 \qquad v = \ln(5x+2)$$

$$\frac{du}{dx} = 4x^3 \qquad \frac{dv}{dx} = \frac{1}{(5x+2)}(5)$$

$$= \frac{5}{(5x+2)}$$

$$g'(x) = \ln(5x+2)(4x^3) + (x^4)\frac{5}{(5x+2)}$$

$$= x^3\left[4\ln(5x+2) + \frac{5x}{(5x+2)}\right]$$

Example 6

$$f(x) = \tan(\ln 2x)$$

$$f'(x) = sec^2(\ln 2x) \cdot \left(\frac{1}{2x}\right) \cdot (2)$$

$$= \frac{sec^2(\ln 2x)}{x}$$

Exercise Time!

Find the derivatives of the following functions.

a. $y = \ln\left(3x^2 - 4x\right)$

b. $y = \ln(x)^2$

c. $y = \frac{1}{15}\ln(5x + 1)$

d. $y = \ln\frac{x}{2x^4 - 6}$

e. $y = \ln\left(\frac{2x^3}{5x+2}\right)$

f. $f(x) = \ln\left(x^2 + 3\right)\left(x^3 + 2x\right)$

g. $y = \frac{1}{5x^5}\ln(3x - 2)$

h. $y = \ln(\sin 5x)$

i. $y = \cos(\ln 2x)$ (using Chain Rule)

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = \frac{6x-4}{3x^2-4x}$

b. $\frac{dy}{dx} = \frac{2}{x}$

c. $\frac{dy}{dx} = \frac{1}{3(5x+1)}$

d. $\frac{dy}{dx} = \frac{1}{x} - \frac{8x^3}{2x^4-6}$

e. $\frac{dy}{dx} = \frac{3}{x} - \frac{5}{5x^2+2}$

f. $f'(x) = \frac{2x}{x^2+3} + \frac{3x^2+2}{x^3+2x}$

g. $\frac{dy}{dx} = -\frac{\ln(3x-2)}{x^6} + \frac{3}{5x^5(3x-2)}$

h. $\frac{dy}{dx} = 5\cot 5x$

i. $\frac{dy}{dx} = -\sin(\ln 2x)$

Exponential Function

If given $y = e^u$ with u is the function of x,

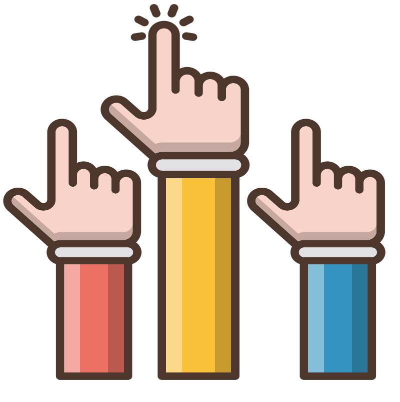
$$\frac{d}{dx}e^u = e^u \frac{du}{dx}$$

Example 1

$$y = e^{5x+2}$$

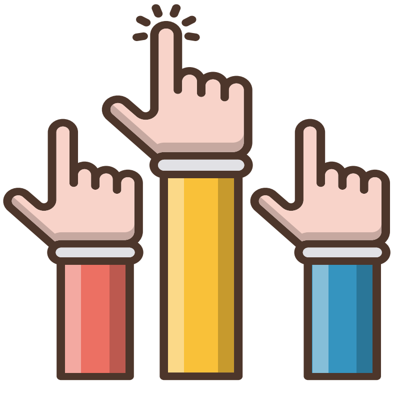
$$\frac{dy}{dx} = e^{5x+2} \cdot (5)$$

$$= 5e^{5x+2}$$



Example 2

$$y = e^{2x+3}(5e^{3x})$$
$$= 5e^{2x+3+3x}$$
$$= 5e^{5x+3}$$
$$\frac{dy}{dx} = 5e^{5x+3}(5)$$
$$= 25e^{5x+3}$$



Example 3

$$y = \sqrt[4]{16e^{8x}}$$
$$= \sqrt[4]{16} \cdot e^{\frac{8x}{4}}$$
$$= 4e^{2x}$$
$$\frac{dy}{dx} = 4e^{2x} \cdot (2)$$
$$= 8e^{2x}$$

Example 4

$$y = e^{6x} - \frac{2}{e^{2x}}$$
$$= e^{6x} - 2e^{-2x}$$
$$\frac{dy}{dx} = e^{6x} \cdot (6) - 2e^{-2x} \cdot (-2)$$
$$= 6e^{6x} + \frac{4}{e^{2x}}$$

Exercise Time!

Find the derivatives of the following functions.

a. $y = \sqrt{1 + e^{4x}}$

b. $y = e^2\left(e^{3x} + e^{-x}\right)$

c. $y = \left(5x^2 - 6\right)\left(e^{x^4+1}\right)$

d. $y = e^{2x}\left(e^{-5x} + 7\right)$

e. $y = \frac{2e^{5x}}{(x+3)^2}$

f. $y = \frac{-1}{e^{3-7x}}$

g. $s = e^{3x} \sin 2x$

h. $y = e^{\cos 2x}$

i. $y = \frac{e^{5x}}{4x+7}$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = \frac{2e^{4x}}{\sqrt{1+e^{4x}}}$

b. $\frac{dy}{dx} = 3e^{2+3x} - e^{2-x}$

c. $\frac{dy}{dx} = 2x\left(e^{x^4+1}\right)[10x^4 - 12x^2 + 5]$

d. $\frac{dy}{dx} = -3e^{-3x} + 14e^{2x}$

e. $y = \frac{2e^{5x}[5x+13]}{(x+3)^3}$

f. $\frac{dy}{dx} = \frac{-7}{e^{3-7x}}$

g. $\frac{ds}{dx} = e^{3x}[3\sin 2x + 2\cos 2x]$

h. $\frac{dy}{dx} = -2e^{\cos 2x}\sin 2x$

i. $\frac{dy}{dx} = \frac{e^{5x}[20x+31]}{(4x+7)^2}$

2.3 Second Order Differentiation

Written as

$$\frac{d^2y}{dx^2}$$

$$f''(x)$$

$$y \rightarrow \frac{dy}{dx} \rightarrow \frac{d^2y}{dx^2}$$

$$f(x) \rightarrow f'(x) \rightarrow f''(x)$$

Example 1

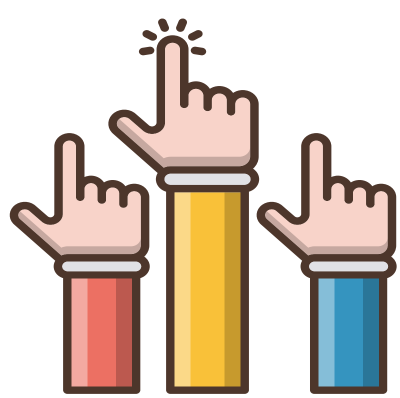
$$y = 3x^3 - 2x^2 + 7$$

$$\frac{dy}{dx} = (3)3x^2 - (2)2x$$

$$= 9x^2 - 4x$$

$$\frac{d^2y}{dx^2} = (2)9x - 4$$

$$= 18x - 4$$



Example 2

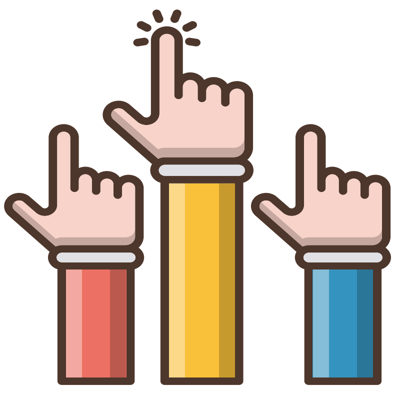
$$s = 3ln(2t+3)$$

$$\frac{ds}{dt} = 3 \cdot \frac{1}{(2t+3)} \cdot (2)$$

$$= \frac{6}{(2t+3)} \qquad = 6(2t+3)^{-1}$$

$$\frac{d^2s}{dt^2} = (-1)(6)(2t+3)^{-2}(2)$$

$$= \frac{-12}{(2t+3)^2}$$



Example 3

$$m = (3n+5)^4$$

$$\frac{dm}{dn} = 4 \cdot (3n+5)^3 \cdot (3)$$

$$= 12(3n+5)^3$$

$$\frac{d^2m}{dn^2} = (3)12(3n+5)^2(3)$$

$$= 78(3n+5)^2$$

Example 4

Find the second derivative for each of the following when $x = -2$:

$$y = (x+1)\left(3x^2-1\right)$$

$$= 3x^3 - x + 3x^2 - 1$$

$$\frac{dy}{dx} = 9x^2 - 1 + 6x$$

$$\frac{d^2y}{dx^2} = 18x + 6$$

when $x = -2$

$$\therefore \frac{d^2y}{dx^2} = 18(-2) + 6$$

$$= -30$$

Exercise Time!

Show the second derivative for the functions below.

a. $s = (4 - 3t)^3$

b. $y = \sqrt{3 - 8x}$

c. $y = sin^2 2x$

d. $y = \ln(x^2 + 3x)$

e. $s = (e^{2t} - 4)^3$

f. $y = \frac{2x-1}{2x+1}$

g. $y = (2x^3 + 3)(x^3 - 2)$

h. $f(t) = \sqrt{2t - 3}(t + 4)$

i. $s = (6t + 4)^3$ if $t = 1$

j. $f(x) = 4e^{-3x} + 5(2x + 7)^5$

CHECK YOUR ANSWERS!

a. $\frac{d^2s}{dt^2} = 54(4 - 3t)$

b. $\frac{d^2y}{dx^2} = \frac{-16}{\sqrt{(3-8x)^3}}$

c. $\frac{d^2y}{dx^2} = 8(cos^2 2x - sin^2 2x)$

d. $\frac{d^2y}{dx^2} = \frac{-2x^2-6x-9}{(x^2+3x)^2}$

e. $\frac{d^2s}{dt^2} = 12e^{2t}(e^{2t} - 4)[3e^{2t} - 4]$

f. $\frac{d^2y}{dx^2} = \frac{-16}{(2x+1)^3}$

g. $\frac{d^2y}{dx^2} = 6x[10x^3 - 1]$

h. $f''(t) = \frac{3t-16}{\sqrt{(2t-3)^3}}$

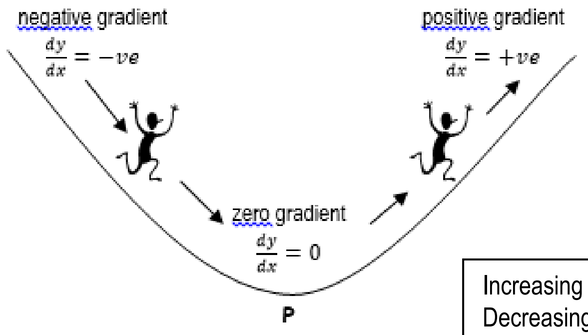
i. $\frac{d^2s}{dt^2} = 2160$

j. $f''(x) = \frac{36}{e^{3x}} + 400(2x + 7)^3$

2.4 The Application of Differentiation

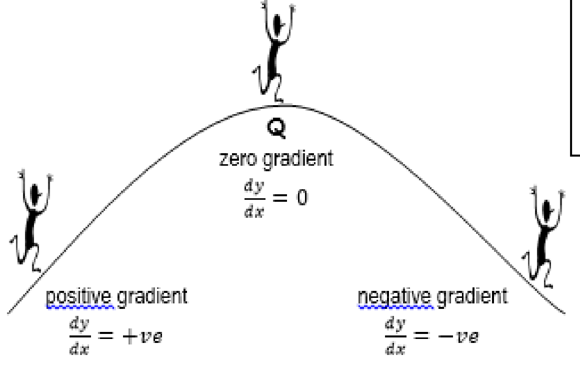
Minimum, Maximum & Inflection Point

- A point where a function changes from an increasing to a decreasing function or visa-versa is known as a turning point.
- To determine whether a function is increasing or decreasing, we use differentiation.
- If the derivative is positive, a function is increasing meanwhile if the derivative is negative, a function is decreasing.

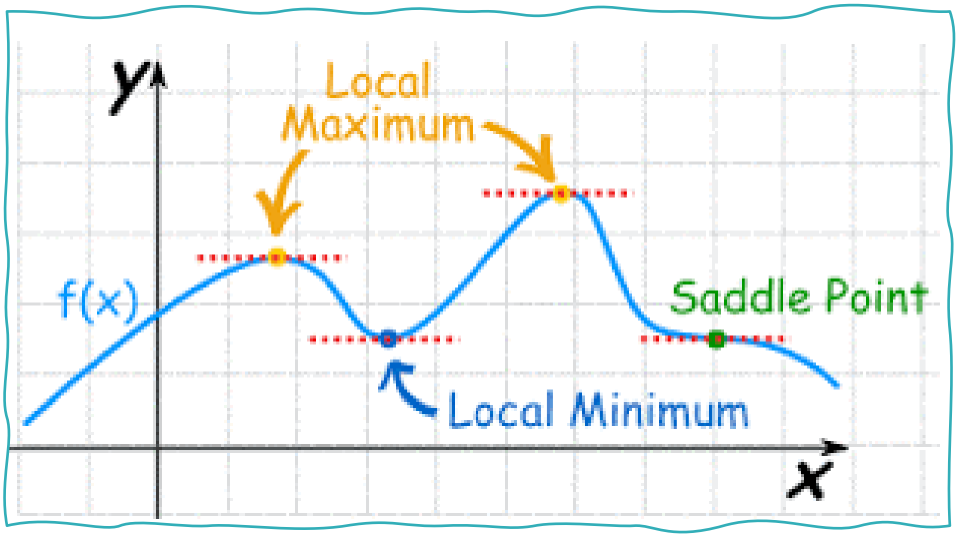


Increasing function → positive gradient
Decreasing function → negative gradient

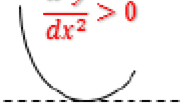
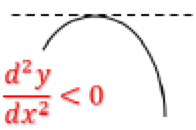
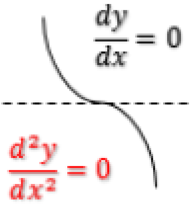
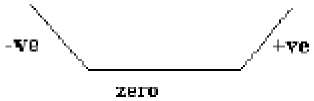
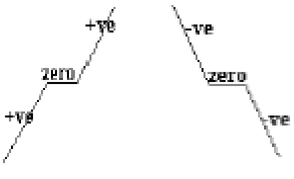
point **P** & **Q** → Stationary point/Critical point/Turning point
point **P** → **minimum** point
point **Q** → **maximum** point



- Stationary points are points on a curve where the gradient is zero.
- This is where the curve reaches a minimum or maximum.
- A maximum is a high point, and a minimum is a low point.
- Stationary point is also known as turning point or critical point.
- There are three types of stationary points: maximum, minimum and points of inflection (/inflexion).



- The gradient is zero at all of the stationary point.
- To determine whether the point is a maximum point, a minimum point or a point of inflection, second derivative will tell the nature of the curve.

The point is minimum	The point is maximum	The point is an inflection point
$\frac{d^2y}{dx^2} > 0$  $\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$  $\frac{d^2y}{dx^2} < 0$	$\frac{dy}{dx} = 0$  $\frac{d^2y}{dx^2} = 0$
		

Example 1

Find the stationary points on the graph of $y = x^2 + 4x + 3$ and state their nature.

$$y = x^2 + 4x + 3$$

$$\frac{dy}{dx} = 2x + 4$$

$$\frac{dy}{dx} = 0$$

$$2x + 4 = 0$$

$$x = -2$$

when $x = -2$

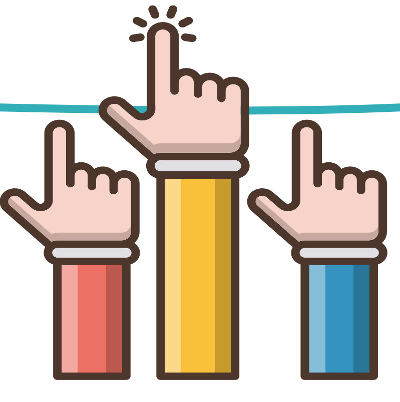
$$y = (-2)^2 + 4(-2) + 3$$

$$y = -1$$

turning point is $(-2, -1)$

$$\frac{d^2y}{dx^2} = 2 > 0$$

Point $(-2, -1)$ is a minimum point



Example 2

Find the stationary points on the graph of $y = x^3 - 2x^2 - 4x + 5$ and state their nature.

$$y = x^3 - 2x^2 - 4x + 5$$

$$\frac{dy}{dx} = 3x^2 - 4x - 4$$

$$\frac{dy}{dx} = 0$$

$$3x^2 - 4x - 4 = 0$$

$$x = 2, -\frac{2}{3}$$

$when\ x = 2$

$$y = (2)^3 - 2(2)^2 - 4(2) + 5$$

$$y = -3$$

$when\ x = -\frac{2}{3}$

$$y = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) + 5$$

$$y = \frac{175}{27}$$

$turning\ point\ are\ (2, -3)\ \&\ \left(-\frac{2}{3}, \frac{175}{27}\right)$

$$\frac{d^2y}{dx^2} = 6x - 4$$

$when\ x = 2$

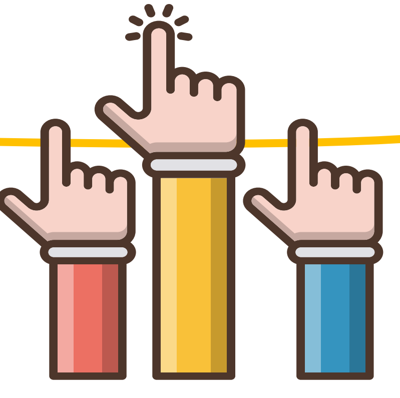
$$\frac{d^2y}{dx^2} = 6(2) - 4 = 8 \ > 0$$

$when\ x = -\frac{2}{3}$

$$\frac{d^2y}{dx^2} = 6\left(-\frac{2}{3}\right) - 4 = -8 \ < 0$$

$Point\ (2, -3)\ is\ a\ minimum\ point\ \&$

$Point\ \left(-\frac{2}{3}, \frac{175}{27}\right)\ is\ a\ maximum\ point$



Example 3

Find the stationary points on the graph of $y = 2x^2 - 8x + 8$ and state their nature.

$y = 2x^2 - 8x + 8$

$\frac{dy}{dx} = 4x - 8$

$\frac{dy}{dx} = 0$

$4x - 8 = 0$

$x = 2$

$when\ x = 2$

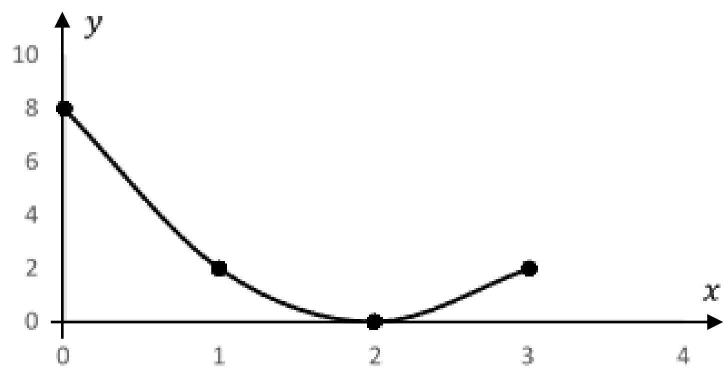
$y = (2)^2 - 8(2) + 8$

$y = 0$

$turning\ point\ is\ (2, 0)$

$\frac{d^2y}{dx^2} = 4\ > 0$

$Point\ (-2, 3)\ is\ a\ minimum\ point$



<u>How to plot the graph</u>

$y = 2x^2 - 8x + 8$

intercept x − axis, $y = 0$

$\therefore 2x^2 - 8x + 8 = 0$

$x = 2$

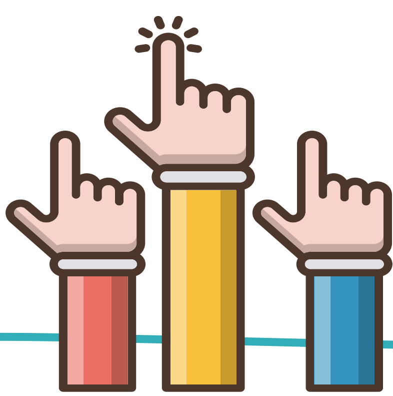
intercept y − axis, $x = 0$

$\therefore y = 2(0)^2 - 8(0) + 8$

$= 8$

points on graph

1. $(2, 0)$
2. $(0, 8)$
3. $(2, 0)$



Exercise Time!

a. Solve the stationary points for $y = x^3 + 2x^2 - 7x$ and determine the maximum and minimum points.

b. A curve has the equation $y = 2x^3 - 6x^2 + 6$. Solve the given equation to find the turning points and their nature.

c. Find the coordinates of the stationary points of the curve $y = x^3 - 6x^2 + 9x + 5$. Determine their nature and sketch the graph of the curve.

d. Given $y = 3x^2 - 6x + 2$ represents a curve line. Find:
   i. the turning point
   ii. the nature of the curve line

e. Find the coordinates of the turning points $y = x^3 - x^2$ and determine their nature.

f. Given $y = 2 + 24x + 3x^2 - x^3$ represents a curve line.
   i. Determine the coordinates of the turning points of the curve line
   ii. Determine the nature of the turning points
   iii. Sketch the graph of the turning points

CHECK YOUR ANSWERS!

a. $(1,-4)\ min, (-2.3,14.5)\ max$

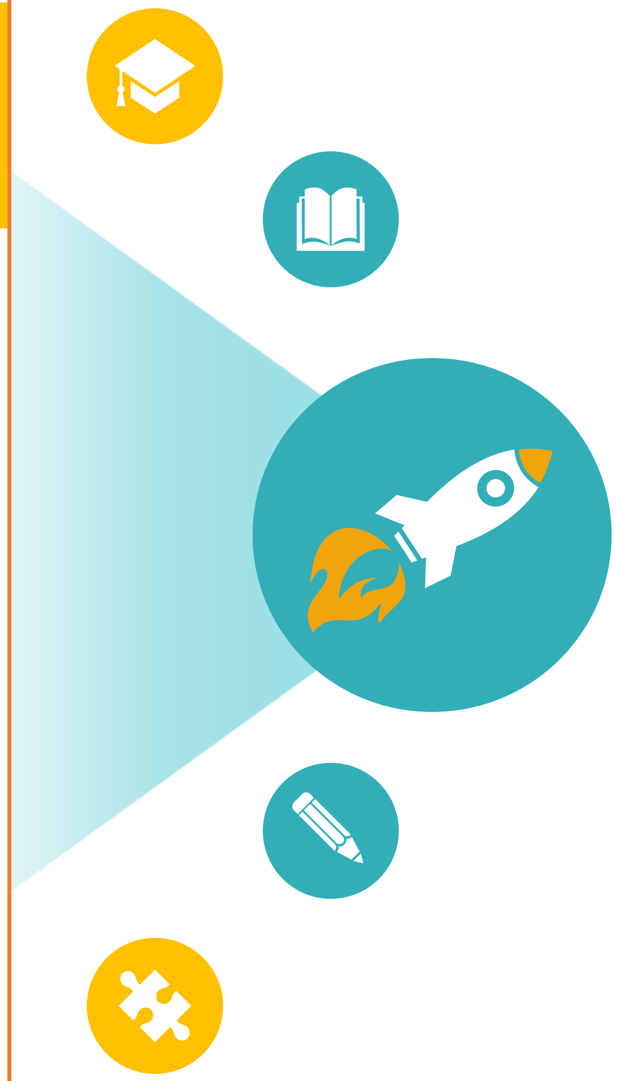
b. $(2,-2)\ min, (0,6)\ max$

c. $(3,5)\ min, (1,9)\ max$

d. $(1,-1)\ min$

e. $\left(\frac{2}{3},\frac{-4}{27}\right)\ min, (0,0)\ max$

f. $(-2,-26)\ min, (4,28)\ max$



Rate of Change

Introduction

- Differentiation is used to calculate the rate of change.
- The rate of change of a function $f(x)$ with respect to x can be found by finding the derived function $f'(x)$.
- For example, in mechanics, the rate of change of displacement (with respect to time) is the velocity. The rate of change of velocity (with respect to time) is the acceleration.

Problems related to rate of change

- An electrical engineer is interested in knowing the rate of change in an electric circuit.
- Construction engineers want to know the rate of change of concrete expansion on a bridge.
- A mechanical engineer should know the rate of shrinkage and expansion of the valve in a pump used for drilling oil.

Steps to solve problems on rate of change

i. Draw or sketch a digram (if necessary) to help in figuring out the problem and assign a letter to each of the given quantities..

ii. Identify the given rate of change and the problem of rate of change that need to be determined.

iii. Find one or more equations that are related to the given quantity and the problem of rate of change.

iv. Differentiate both sides of the equation with respect to time and solve for the unknown variable.

v. Evaluate the variable at a certain period of time.

Rates which contain $y = f(x)$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Rate which contain radius:

volume

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

area

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

Rate which contain length:

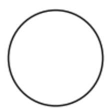
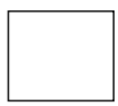
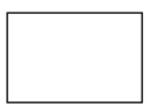
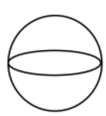
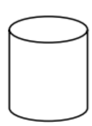
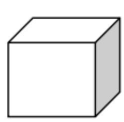
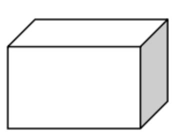
volume

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

area

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

Area and volume formula for shape and solid:

Shape @ Solid	Area @ Surface area	Perimeter @Volume
Circle 	$A = \pi r^2$	$P = 2\pi r$
Square 	$A = x^2$	$P = 4x$
Rectangle 	$A = xy$	$P = 2x + 2y$
Cone 	$A = \pi r^2 + \pi rs$	$V = \frac{1}{3}\pi r^2 h$
Sphere 	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$
Cylinder 	$A = 2\pi r^2 + 2\pi rh$	$V = \pi r^2 h$
Cube 	$A = 6x^2$	$V = x^3$
Cuboid 	$A = 2lw + 2lh + 2wh$	$V = l \times h \times w$

Example 1

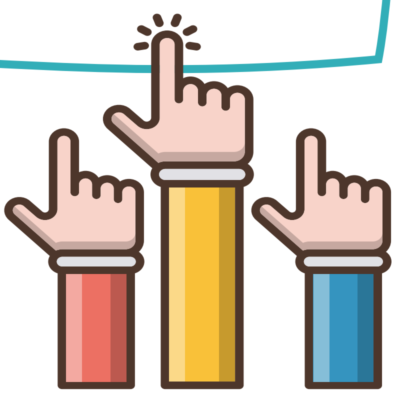
If the expression of y given by $y = x^3 + x^2 - 8$. Find the rate of change of y if x increase at $0.5\ ^{unit}/_{second}$ when $x = 5$.

$$\frac{dy}{dt} = ?$$
$$\frac{dx}{dt} = 0.5$$
$$x = 5$$

$$y = x^3 + x^2 - 8$$
$$\frac{dy}{dx} = 3x^2 + 2x$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$\frac{dy}{dt} = (3x^2 + 2x\ ) \times (0.5)$$
$$= (3(5)^2 + 2(5)\ ) \times (0.5)$$
$$= 42.5\ ^{unit}/_{sec}$$

∴ Rate of y increase at $42.5\ ^{unit}/_{second}$



Example 2

The area A of a circle is increasing at a constant rate of $1.5cm^2s^{-1}$. Calculate the rate at which the radius, r of the circle is increasing when the perimeter of the circle is $6\ cm$.

$$\frac{dr}{dt} = ?$$
$$\frac{dA}{dt} = 1.5$$
$$P = 6$$

$$l = 2\pi r$$
$$\therefore 2\pi r = 6$$

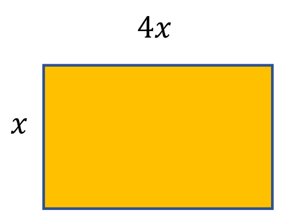
$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$1.5 = 2\pi r \times \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1.5}{2\pi r}$$
$$= \frac{1.5}{6}$$
$$= 0.25\ ^{cm^2}/_{sec}$$

∴ Rate of the area increase at $0.25\ ^{cm^2}/_{sec}$

Example 3

The length of a rectangle is always four times its width. If the width is decreasing at a rate of $0.1cms^{-1}$, find the rate of change of the area when its perimeter is $12\ cm$.



$\frac{dA}{dt} = ?$ $\quad \frac{dx}{dt} = -0.1$ $\quad P = 12$

$$l = x + x + 4x + 4x$$

$$l = 10x$$

$$\therefore 10x = 12$$

$$x = \frac{6}{5}$$

$$A = x(4x)$$

$$= 4x^2$$

$$\frac{dA}{dx} = 8x$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$\frac{dA}{dt} = 8x \times -0.1$$

$$\frac{dr}{dt} = 8\left(\frac{6}{5}\right) \times -0.1$$

$$= -0.96\ {cm^2}/{sec}$$

$\therefore$ Rate of the change of the area decreases at $-0.96\ {cm^2}/{s}$

Exercise Time!

a. Find the rate of change of square area with 8 cm long when the side length increasing at 2 cm/min.

b. The surface of a sphere is increasing at a constant rate of $10\pi cm^2/s$. Find the rate of change of the volume when the radius is 10cm. Given $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

c. The radius of a circle is decreasing at a rate of 7 cm/s. Find the rate of change of the area for circle at the instant when the radius is 4 cm.

d. A spherical balloon is inflated at a rate of $3\ cm^3/s$. Find the increment rate of the radius when the radius is 2 cm and 4 cm.

e. The side of a square is increasing at a rate of 8 cm/s. Find the rate of change of the square's area when the length of the side is 13 cm.

CHECK YOUR ANSWERS!

a. $\frac{dA}{dt} = 32\ cm^2/min$

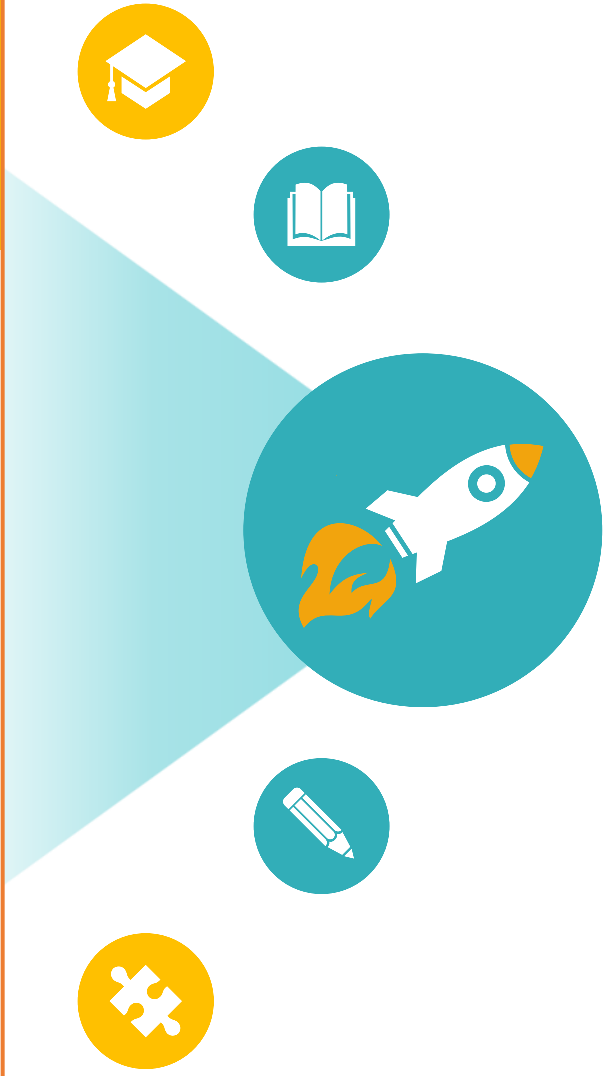
b. $\frac{dV}{dt} = 50\pi\ cm^3/s$

c. $\frac{dA}{dt} = -56\pi\ cm^2/s$

d. $r = 2, \frac{dr}{dt} = 0.0597\ cm/s$

$r = 4, \frac{dr}{dt} = 0.0149\ cm/s$

e. $\frac{dA}{dt} = 208\ cm^2/s$



2.5 Parametric Equation

- A parametric equation is where the x and y coordinates are both written in terms of another letter.
- The third variable is called a parameter and is usually given the letter t or θ.
- The differentiation of functions given in parametric form is carried out using the Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}, \quad \text{where} \quad \frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$$

Example 1

Find $\frac{dy}{dx}$ when $x = t^2 + t$ and $y = 2t - 1$

$$x = t^2 + t$$

$$\frac{dx}{dt} = 2t + 1$$

$$\frac{dt}{dx} = \frac{1}{2t + 1}$$

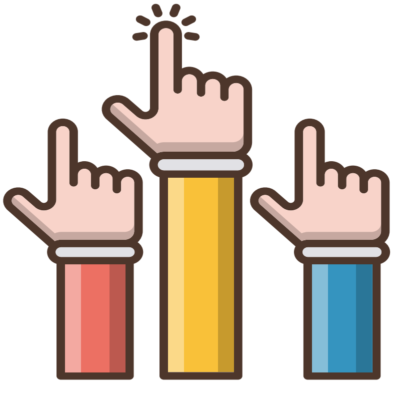
$$y = 2t - 1$$

$$\frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = (2) \times \left(\frac{1}{2t + 1}\right)$$

$$= \frac{2}{2t + 1}$$



Example 2

Given $x = 2sin\theta$ and $y = 4cos\theta$, then find $\frac{dy}{dx}$.

$$x = 2sin\theta$$

$$\frac{dx}{d\theta} = 2cos\theta$$

$$\frac{d\theta}{dx} = \frac{1}{2cos\theta}$$

$$y = 4cos\theta$$

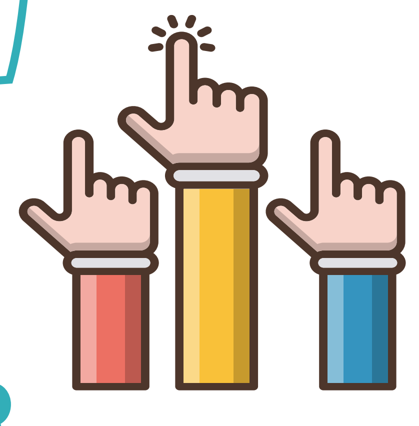
$$\frac{dy}{d\theta} = -4sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = (-4sin\theta) \times \left(\frac{1}{2cos\theta}\right)$$

$$= \frac{-4sin\theta}{2cos\theta}$$

$$= -2tan\theta$$



Example 3

The parametric equations of a curve are $x = \frac{2}{t^4}$ and $y = \frac{3}{t} - t^3$, find $\frac{dy}{dx}$ terms of t.

$$x = 2t^{-4}$$

$$\frac{dx}{dt} = -8t^{-5}$$

$$= -\frac{8}{t^5}$$

$$y = 3t^{-1} - t^3$$

$$\frac{dy}{dt} = -3t^{-2} - 3t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \left(-\frac{3}{t^2} - 3t^2\right) \times \left(-\frac{t^5}{8}\right)$$

$$= \left(\frac{-3 - 3t^4}{t^2}\right) \times \left(-\frac{t^5}{8}\right)$$

$$= -\frac{3t^3}{8}(1 + t^4)$$

Exercise Time!

a. Find $\frac{dy}{dx}$ for the following equation:

$$x = 2t^4 - 2 \;, \qquad y = 3\sin 4t$$

b. The parametric equations of a curve are $x = \frac{t^2-2}{1+t}$ and $y = \frac{1}{1+t}$.

Find $\frac{dy}{dx}$ in terms of t.

c. The parametric equation function is given as $x = 3t^2 + 4$ and $y = 5\ln(2t - 3)$. Compute $\frac{dy}{dx}$.

d. Determine $\frac{dy}{dx}$ for:

$$x = 2\sin t\,, \qquad y = t^2 - 3t$$

e. Find $\frac{dy}{dx}$ for the following equation:

$$y = 4e^{5t+3} - 2 \;, \qquad x = 8 - 6t^3$$

f. Find $\frac{dy}{dx}$ for $x = 2\sin\theta$ and $y = 3\cos 2\theta$.

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = \frac{3\cos 4t}{2t^3}$

b. $\frac{dy}{dx} = \frac{-1}{t^2+2t+2}$

c. $\frac{dy}{dx} = \frac{5}{3t(2t-3)}$

d. $\frac{dy}{dx} = \frac{2t-3}{2\cos t}$

e. $\frac{dy}{dx} = -\frac{10e^{5t+3}}{9t^2}$

f. $\frac{dy}{dx} = \frac{-3\sin 2\theta}{\cos\theta}$

2.6 Implicit Differentiation

- Implicit differentiation is the process of finding the derivative of a dependent variable in an implicit function.

- The technique of implicit differentiation allows you to find the derivative of y with respect to x without having to solve the given equation for y.

Steps to solve problem on implicit differentiation

i. Differentiate x and y with respect to x [differentiate y followed by $\frac{dy}{dx}$].

ii. Put all $\frac{dy}{dx}$ expression on the left side of the equal sign, while the other expression on the right side of the equal sign.

iii. Factorize $\frac{dy}{dx}$ expression on the left side.

iv. Find $\frac{dy}{dx}$ [move all the expression on the left to the right side].

Basic Implicit Differentiation

Function	Differentiation
x	1
y	$\frac{dy}{dx}$
$x+y$	$1+\frac{dy}{dx}$
xy	$y+x\frac{dy}{dx}$

Example 1

Find $\frac{dy}{dx}$ for all the following:

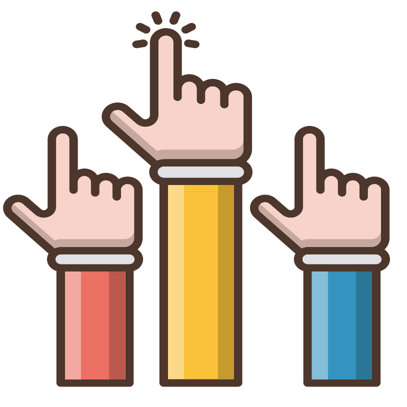
$$3x^2+5y=8-y^4+3x$$

$$6x+5\frac{dy}{dx}=0-4y^3\frac{dy}{dx}+3$$

$$5\frac{dy}{dx}+4y^3\frac{dy}{dx}=3-6x$$

$$\frac{dy}{dx}\left(5+4y^3\right)=3-6x$$

$$\frac{dy}{dx}=\frac{3-6x}{5+4y^3}$$



Example 2

Find $\frac{dy}{dx}$ for all the following:

$$x^4 - 3e^{2x+2y} = 4y + 5$$

$$4x^3 - 3e^{2x+2y}\left[2 + 2\frac{dy}{dx}\right] = 4\frac{dy}{dx} + 0$$

$$4x^3 - 6e^{2x+2y} - 6e^{2x+2y}\frac{dy}{dx} = 4\frac{dy}{dx}$$

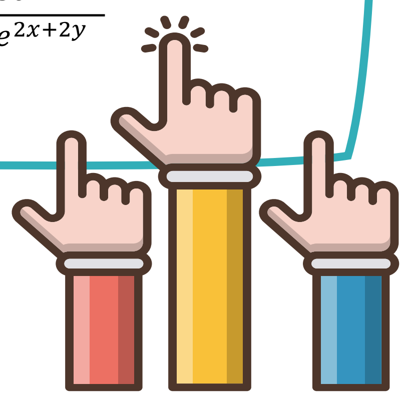
$$4\frac{dy}{dx} + 6e^{2x+2y}\frac{dy}{dx} = 4x^3 - 6e^{2x+2y}$$

$$\frac{dy}{dx}\left(4 + 6e^{2x+2y}\right) = 4x^3 - 6e^{2x+2y}$$

$$\frac{dy}{dx} = \frac{4x^3 - 6e^{2x+2y}}{4 + 6e^{2x+2y}}$$

$$\frac{dy}{dx} = \frac{2\left(2x^3 - 3e^{2x+2y}\right)}{2\left(2 + 3e^{2x+2y}\right)}$$

$$= \frac{2x^3 - 3e^{2x+2y}}{2 + 3e^{2x+2y}}$$



Example 3

Find $\frac{dy}{dx}$ for all the following:

$$x^4 - 10 = \frac{2x}{4y} + 5y$$

$$4x^3 = \left(\frac{8xy - 8x\frac{dy}{dx}}{(4y)^2}\right) + 5\frac{dy}{dx}$$

$$4x^3 = \frac{8xy - 8x\frac{dy}{dx} + 80y^2\frac{dy}{dx}}{16y^2}$$

$$64x^3y^2 = 8xy - 8x\frac{dy}{dx} + 80y^2\frac{dy}{dx}$$

$$8x\frac{dy}{dx} + 80y^2\frac{dy}{dx} = 8xy - 64x^3y^2$$

$$\frac{dy}{dx}\left(8x - 80y^2\right) = 8xy - 64x^3y^2$$

$$\frac{dy}{dx} = \frac{8xy - 64x^3y^2}{8x - 80y^2}$$

Exercise Time!

1. Determine $\frac{dy}{dx}$ by using implicit differentiation method for:

   a. $6x^2 - y^3 = 1$

   b. $4x^2 + 3xy^3 - y^2 = 6$

   c. $3x + 3y - 2x^3 = 4$

   d. $4x + 2y - 2xy = 15$

   e. $xy - \sin 5y = 9$

   f. $y^2 - 7x = \cos 2y$

   g. $4x - y^2 + 10xy = 2$

   h. $y + xy = y^2 + 8x - 5$

2. If $x^2 + y^2 - 2x - 6y + 5 = 0$, determine the value of $\frac{dy}{dx}$ when $x = 3$ and $y = 2$

3. If $3e^{-4x} - 3y^2 = 4xy$, determine the value of $\frac{dy}{dx}$ when $x = 0$ and $y =$-1

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = \frac{4x}{y^2}$

b. $\frac{dy}{dx} = \frac{-8x-3y^3}{9xy^2-2y}$

c. $\frac{dy}{dx} = 2x^2 - 1$

d. $\frac{dy}{dx} = \frac{y-2}{1-x}$

e. $\frac{dy}{dx} = \frac{-y}{x-5\cos 5y}$

f. $\frac{dy}{dx} = \frac{7}{2y+2\sin 2y}$

g. $\frac{dy}{dx} = \frac{-2-5y}{5x-y}$

h. $\frac{dy}{dx} = \frac{8-y}{1+x-2y}$

2. $\frac{\mathrm{dy}}{\mathrm{dx}} = 2$

3. $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{4}{3}$

2.7 Partial Differentiation

Suppose that $z = f(x, y)$ is a function, where x and y are the independent variables.

First Order

- Two first order partial derivatives can be derived from the function z.
- The partial derivative of z with respect to x, is denoted by $\frac{\partial z}{\partial x}$.
- The partial derivative of z with respect to y, is denoted by $\frac{\partial z}{\partial y}$.

Second Order

- For the case of a function of two variables, there will be a total of four possible second order derivatives.

- Here are the notations that we use to denote them:

Partial derivative	Notation
First order: respecting to x Second order: Respecting to x	$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}$ or f_{xx}
First order: respecting to y Second order: Respecting to y	$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}$ or f_{yy}
First order: respecting to x Second order: Respecting to y	$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y \partial x}$ or f_{xy}
First order: respecting to y Second order: Respecting to x	$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x \partial y}$ or f_{yx}

***** **Notes:** The answer for $\frac{\partial^2 z}{\partial y \partial x}$ should be equal to $\frac{\partial^2 z}{\partial x \partial y}$

Example 1

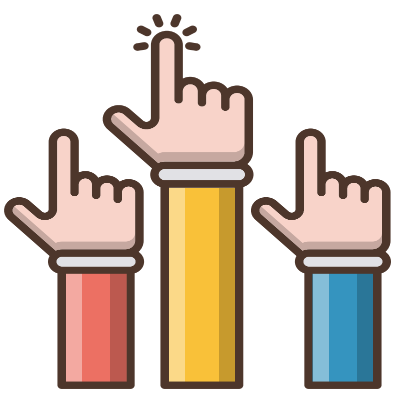
$$z = x^4 + 5 - 3y^2$$

$$\frac{\partial z}{\partial x} = 4x^3 + 0 + 0$$

$$= 4x^3$$

$$\frac{\partial z}{\partial y} = 0 + 0 - 6y$$

$$= -6y$$



Example 2

$$z = (x^2 + 2)(1 - 3y)$$

$$= x^2 - 3x^2y + 2 - 6y$$

$$\frac{\partial z}{\partial x} = 2x - 6xy$$

$$\frac{\partial z}{\partial y} = -3x^2 - 6$$

Example 3

$$z = ln(x^2 + y^2) + 4x - 3y$$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + y^2} \cdot (2x) + 4$$

$$= \frac{2x}{x^2 + y^2} + 4$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2 + y^2} \cdot 2y - 3$$

$$= \frac{2y}{x^2 + y^2} - 3$$

Example 4

$$z = 5x + x^5y^6 + 4y + 25$$

$$\frac{\partial z}{\partial x} = 5 + 5x^4y^6$$

$$\frac{\partial^2 z}{\partial x^2} = 20x^3y^6$$

$$\frac{\partial^2 z}{\partial y \partial x} = 30y^5x^4$$

$$= 30x^4y^5$$

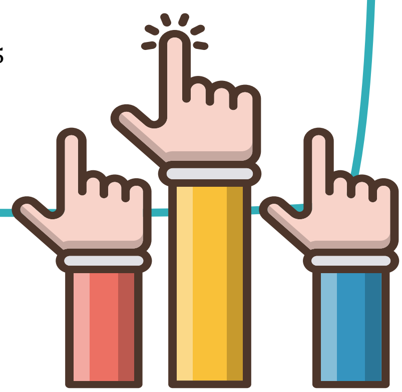
$$\frac{\partial z}{\partial y} = 6y^5x^5 + 4$$

$$= 6x^5y^5 + 4$$

$$\frac{\partial^2 z}{\partial y^2} = 30y^4x^5$$

$$= 30x^5y^4$$

$$\frac{\partial^2 z}{\partial x \partial y} = 30x^4y^5$$



Example 5

$$z = x^2 \, ln(3 - y^4)$$

$$\frac{\partial z}{\partial x} = 2x \, ln(3 - y^4)$$

$$\frac{\partial^2 z}{\partial x^2} = 2 \, ln(3 - y^4)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2x \cdot \frac{1}{(3 - y^4)} \cdot (-4y^3)$$

$$= \frac{-8xy^3}{(3 - y^4)}$$

$$\frac{\partial z}{\partial y} = x^2 \, \frac{1}{(3 - y^4)} \cdot (-4y^3)$$

$$= \frac{-4x^2y^3}{(3 - y^4)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(3 - y^4)(-12x^2y^2) - (-4x^2y^3)(-4y^3)}{(3 - y^4)^2}$$

$$= \frac{-4x^2y^2(9 + y^4)}{(3 - y^4)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-8xy^3}{(3 - y^4)}$$

Exercise Time!

a. Given $z = (x^3y^2 + x + y)(x^5 - y^4)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

b. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ for the function

$$z = (8x + 3y)(7x + 5y)$$

b. Given $z = 3x^3y^2 + x \sin 2y$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$.

c. Determine $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ of $z = 2x^2 + xy^2 - 3y^2$.

d. Given $z = x^2 \sin y + y^2 \cos x$. Find $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$.

e. Given $z = 5x^3 + 3x^2y^4 - 2y^2$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

f. Given $z = \sqrt{y} - \sin(xy) + 5x^2$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

g. Given $z = 5x^3 + 3x^2y^4 - 2y^2$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$

CHECK YOUR ANSWERS!

a. $\frac{\partial z}{\partial x} = 8x^7y^2 - 3x^2y^6 + 6x^5 - y^4 + 5x^4y$, $\frac{\partial z}{\partial y} = 2x^8y - 6x^3y^5 - 4xy^3 + x^5 - 5 - 5y^4$

b. $\frac{\partial z}{\partial x} = 112x + 61y$, $\frac{\partial z}{\partial y} = 61x + 30y$, $\frac{\partial^2 z}{\partial x \partial y} = 61$, $\frac{\partial^2 z}{\partial y \partial x} = 61$

c. $\frac{\partial z}{\partial x} = 9x^2y^2 + \sin 2y$, $\frac{\partial z}{\partial y} = 6x^3y + 2x\cos 2y$, $\frac{\partial^2 z}{\partial x^2} = 18xy^2$,

$\frac{\partial^2 z}{\partial x \partial y} = 18x^2y + 2\cos 2y$.

d. $\frac{\partial^2 z}{\partial x^2} = 4$, $\frac{\partial^2 z}{\partial y^2} = 2x - 6$, $\frac{\partial^2 z}{\partial x \partial y} = 2y$, $\frac{\partial^2 z}{\partial y \partial x} = 2y$

e. $\frac{\partial^2 z}{\partial x \partial y} = 2x\cos y - 2y\sin x$, $\frac{\partial^2 z}{\partial y \partial x} = 2x\cos y - 2y\sin x$.

f. $\frac{\partial z}{\partial x} = 15x^2 + 6xy^4$, $\frac{\partial z}{\partial y} = 12x^2y^3 - 4y$, $\frac{\partial^2 z}{\partial x^2} = 30x + 6y^4$,

$\frac{\partial^2 z}{\partial x \partial y} = 24xy^3$

g. $\frac{\partial z}{\partial x} = -y\cos xy$, $\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y}} - x\cos xy$, $\frac{\partial^2 z}{\partial x^2} = y^2 \sin xy + 10$

h. $\frac{\partial z}{\partial x} = 15x^2 + 6xy^4$, $\frac{\partial z}{\partial y} = 12x^2y^3 - 4y$, $\frac{\partial^2 z}{\partial x^2} = 30x + 6y^4$,

$\frac{\partial^2 z}{\partial y^2} = 36x^2y^2 - 4$, $\frac{\partial^2 z}{\partial y \partial x} = 24xy^3$ $\frac{\partial^2 z}{\partial x \partial y} = 24xy^3$

2.8 The Technique of Total Differentiation

- Suppose that $z = f(x, y)$ be a function with continuous first partial derivatives.
- The total differential of *z*, denoted by *dz* is defined by

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

- A function $w = f(x, y, z)$and t with x, y, z being functions of t.
- Then the total derivatives of w with respect to t is given by

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

Example 1

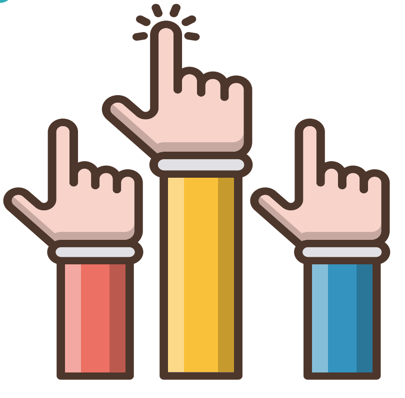
$$s = xy + siny^2 - e^{3x}$$

$$\frac{\partial s}{\partial x} = y - 3e^{3x}$$

$$\frac{\partial s}{\partial y} = x + 2y\, cosy^2$$

$$ds = \frac{\partial s}{\partial x} \cdot dx + \frac{\partial s}{\partial y} \cdot dy$$

$$= \left(y - 3e^{3x}\right) \cdot dx + \left(x + 2y\, cosy^2\right) \cdot dy$$



Example 2

The radius and height of a cylinder are both $2\ cm$. The radius is decreased at $1\ ^{cm}/_{sec}$ and the height is increasing at $2\ ^{cm}/_{sec}$. What is the change in volume with respect to time at this instant?

given info from the question

$$\frac{dr}{dt} = -1$$

$$\frac{dh}{dt} = 2$$

$$r = h = 2$$

The volume of cylinder is

$$V = \pi r^2 h$$

$$\frac{\partial V}{\partial r} = 2\pi r h$$

$$\frac{\partial V}{\partial h} = \pi r^2$$

The total derivative of this with respect to time is

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

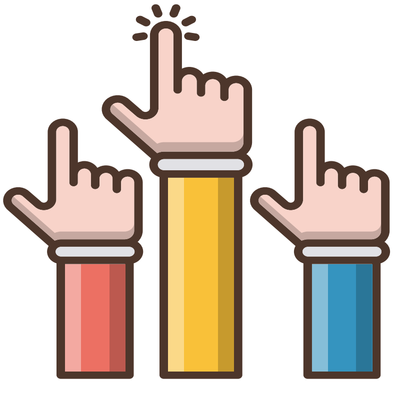
$$\frac{dV}{dt} = (2\pi r h)\cdot\frac{dr}{dt} + (\pi r^2)\cdot\frac{dh}{dt}$$

$$= 2\pi(2)(2)\cdot(-1) + \pi(2)^2\cdot(2)$$

$$= -8\pi + 8\pi$$

$$= 0$$

∴ there is no changes in volume of the cylinder



Example 3

If $z = 1 - 2x^2 + 3xy + y^3$, find the total derivative of z when (x, y) changes from $(1, 2)$ to $(0.8, 2.3)$.

given info from the question

$dx = -0.2$

$dy = 0.3$

change of x $(1, 2)$ change of y

$(0.8, 2.3)$

$(-0.2, 0.3)$

$$\frac{\partial z}{\partial x} = -4x + 3y$$

$$\frac{\partial z}{\partial y} = -3x + 3y^2$$

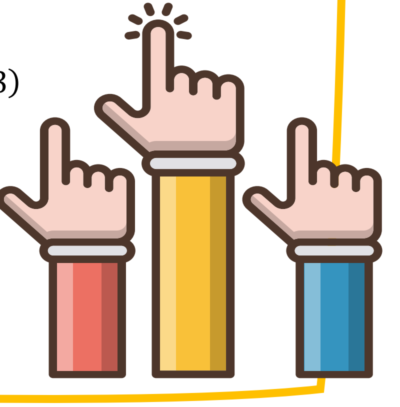
$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

$$= \big(-4(1) + 3(2)\big) \cdot (-0.2) + +(-3(1) + 3(2)^2) \cdot (0.3)$$

$$dz = (-4x + 3y\ ) \cdot dx + (-3x + 3y^2) \cdot dy$$

$$= 2.9\ unit$$

$\therefore$ there is $2.9\ unit$ changes in z



Exercise Time!

a. Given $z = 3x^2y + e^{2x}$. Determine the total differential of z.

b. The height of a cylinder is 250 cm and increasing at a rate of 0.4 m/s. The radius of its base is 100 cm and decreasing at a rate of 0.5 m/s. Find the rate of change for its volume.

c. The height of a cone is 10 mm and increases at the rate of 0.4 mm/s. The radius of the base is 8.5 mm and decrease at the rate of 0.2 mm/s. Calculate the rate of change for the volume of the cone where $V = \frac{1}{3}\pi r^2 h$

d. The height of a cylinder is 10 cm and increases at the rate of 1.2 cm/s. The radius of the cylinder is 5.7 cm and decrease at the rate of 0.8 cm/s. Calculate the rate of change for the volume of the cylinder where $V = \pi r^2 h$

e. A right circular cone radius increase at the rate of 3 cm/min. Calculate how fast is the cone's volume changing when the radius is 16 cm and the height is 21 cm.

$v_{cone} = \frac{1}{3}\pi r^2 h$

CHECK YOUR ANSWERS!

a. $\frac{dy}{dx} = \left(6xy + 2e^{2x}\right)dx + 3x^2 dy$

b. $\frac{dV}{dt} = -6.597\, m^3/s$

c. $\frac{dV}{dt} = -5.34\, mm^3/s$

d. $\frac{dV}{dt} = -164.03\, cm^3/s$

e. $\frac{dV}{dt} = 2111.15\, cm^3/min$

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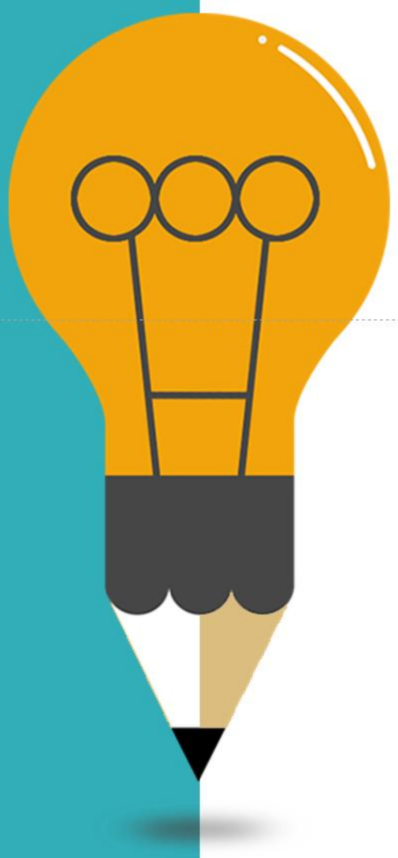

ATTRIBUTION

01

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02

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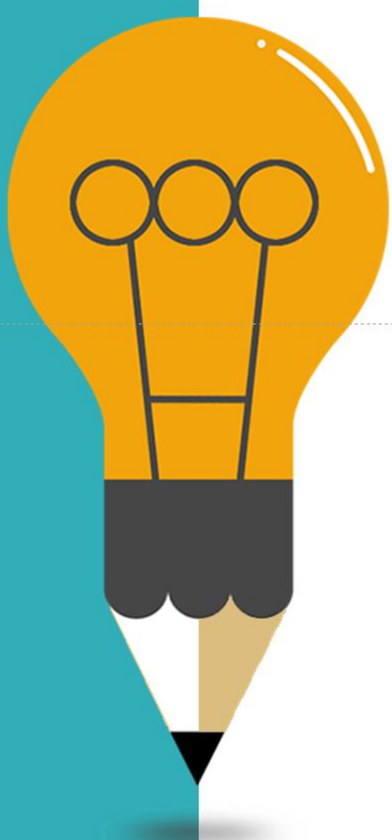

THANK YOU…

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