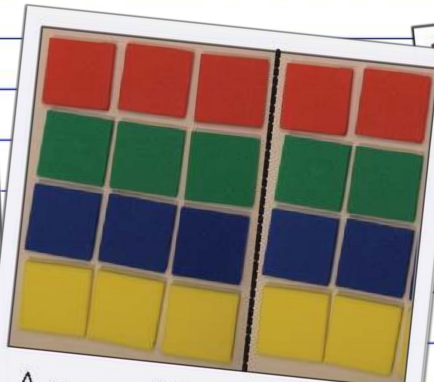
Building Fluency Flexibility Strategic Thinking

MULTIPLICATION

Designing Micro-Progressions Using Tools, Models, and Strategies

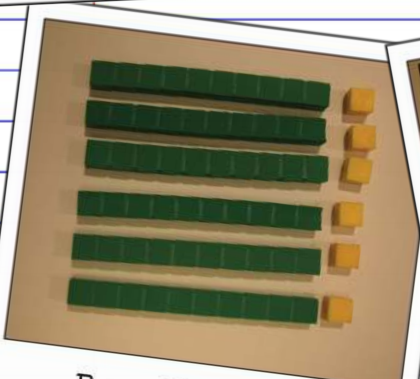
(This is the lite, stripped down version. DRAFT)

2 groups of 2 things

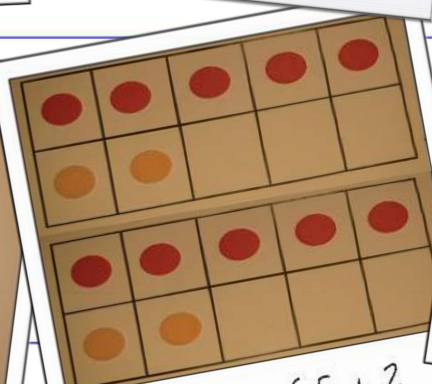


Arrays with Square Tiles

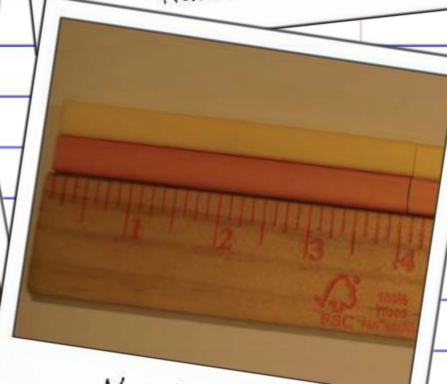
Ratio Table



Base-Ten Blocks



2 groups of 5 + 2



Number Lines

CKingEducation

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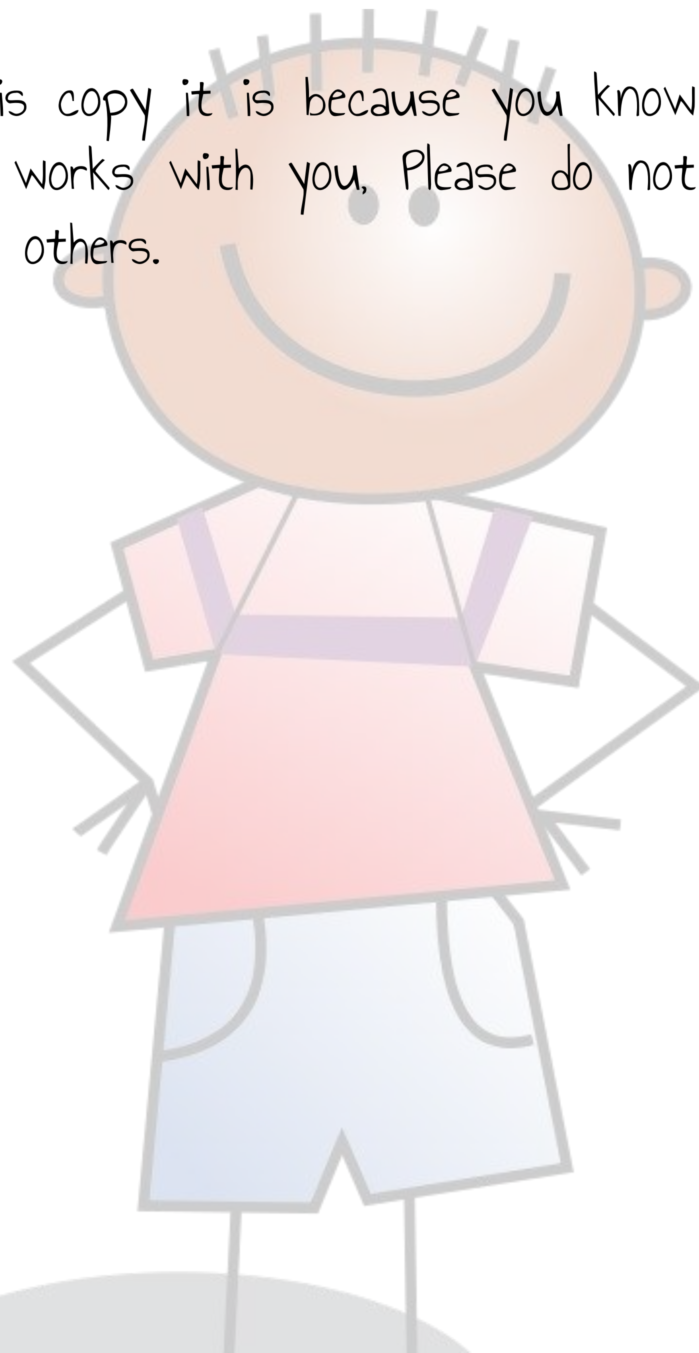
DRAFT PREVIEW for Teachers

This is a "**DRAFT PREVIEW**" of **Multiplication: Designing Micro-Progressions Using Tools, Models, and Strategies**. This is not the final version. This is a stripped down version of the book. This is a preview version.

If you have received this copy it is because you know Christine King and she works with you, Please do not share this resource with others.

Thank You!

Christine King



SECTION I:

Tools, Models and Strategies for Developing a Conceptual Understanding of Multiplication

STAGE 1: SUBITIZING **2 X 3**

STAGE 2: FRIENDLY FACTS **5 X 6**

STAGE 3: CHALLENGING FACTS **7 X 8**

STAGE 4: DECOMPOSING FACTORS **7 X 12**

STAGE 5: APPLYING PROPERTIES TO BUILD MENTAL MATH SKILLS **6 X 24**

STAGE 6: MULTI-DIGIT MULTIPLICATION AND REASONING **25 X 34**

Tony has 2 keyrings. One set is for his home and the other set for his office. Each keyring has 3 keys on it.

2 groups of 3 things

3 + 3

2 x 3

2 groups of 3 things

Expression:
The expressions support student's abilities to subitize and use her knowledge of doubles facts.

Tools:
real-life objects, e.g., keys, beans, candy, packs of gum, pairs of shoes, pennies, $1

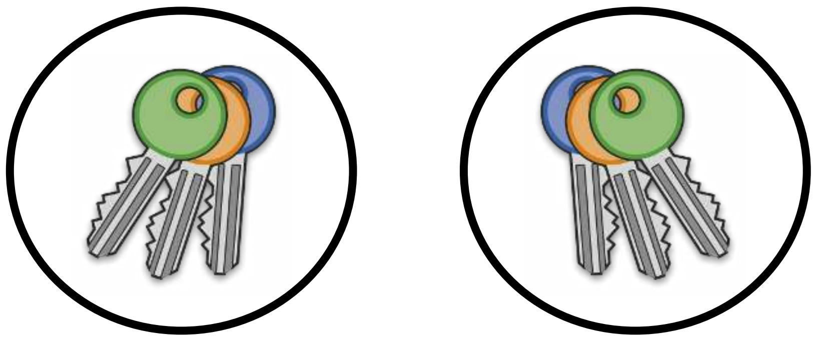
two-colored counters, unit cubes, Unifix cubes, pattern blocks, etc.

Models:
equal groups, set model, Arrays, ten frame

Strategies:
count all, subitize, doubles facts, repeated addition

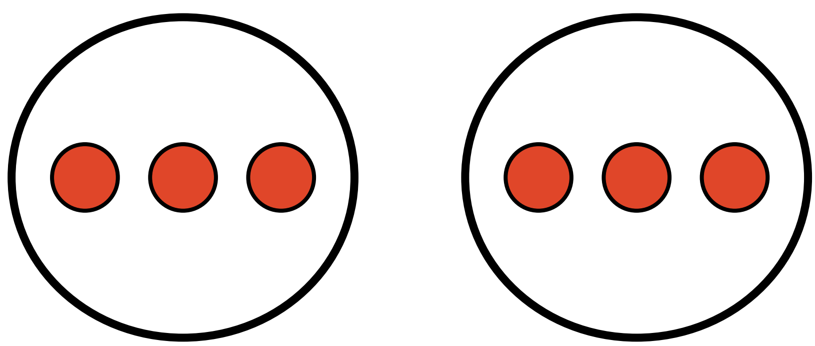
1

Equal Groups, Authentic, Concrete Objects



2

Equal Groups, Concrete Tools

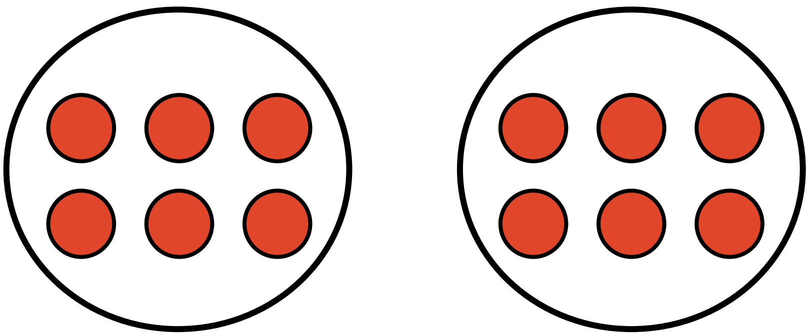


Why is it important to introduce multiplicative concepts with authentic, concrete material?

2 groups of 6 things

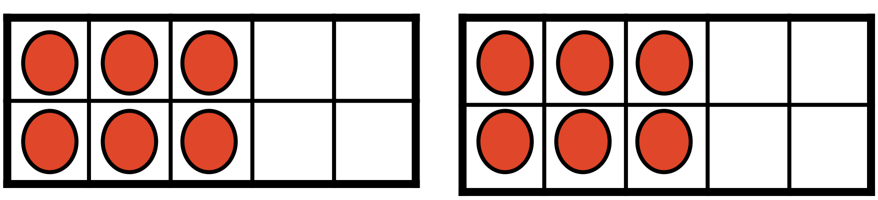
3

Equal Groups, Concrete Tools



4

Equal Groups, Concrete Tools, Ten Frames



What is the difference between #3 and #4?
How does this difference support/enhance understanding?

Ponder This:

Battista notes in his book, <u>Cognitive-Based Assessment and Teaching of Multiplication and Division</u>, that, "…to be effective, mathematics teaching must be carefully guided and support students' construction of personally meaningful mathematical ideas." (Baroody and Ginsburg, 1990; Battista, 1999, 2001, Branford, Brown, and Cocking, 1999; De Corte, Greer, and Verschaffel, 1996; Greeno, Collins and Resnick, 1996; Hiebert and Carpenter, 1992; Lester, 1994, National Research Council, 1989; Prawat, 1999, Romberg, 1992; Schoenfeld, 1994; Steffe and Kieren, 1994; von Glasersfeld, 1995).

What the research says:

Sherin and Fuson state,, "…by the time of formal instruction, students already possess the fundamental conceptual capabilities required for conceptualizing multiplication. Indeed, it has been documented that, as early as kindergarten, children can solve simple multiplication problems." (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993).

2 groups of 5 and 1

Have-a-Go:
3 x 6

Tools:
real-life objects, e.g., keys, beans, candy, packs of gum, pairs of shoes, nickels and pennies, $5 and $1

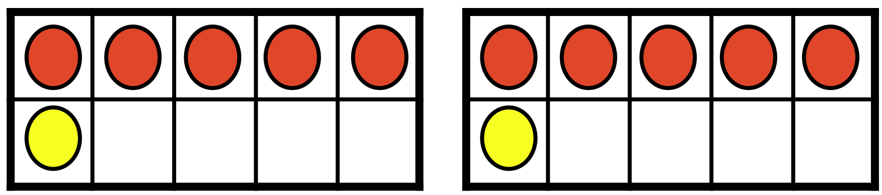
two-colored counters, unit cubes, Unifix cubes, pattern blocks, etc.

Models:
equal groups, set model, arrays, ten frame

Strategies:
count all, subitize, doubles facts, repeated addition, skip counting

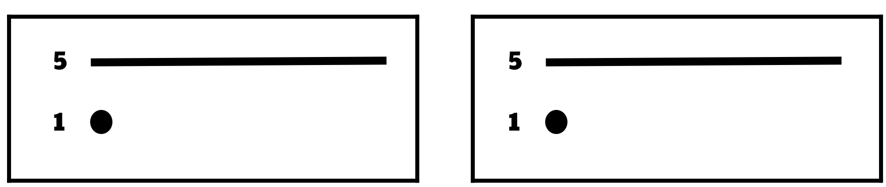
5

Equal Groups, Concrete Tools, Ten Frames, Showing 5 and some more (decomposing)



6

Equal Groups, Pictorial, Ten Frame Structure, Showing 5 and some more (decomposing)

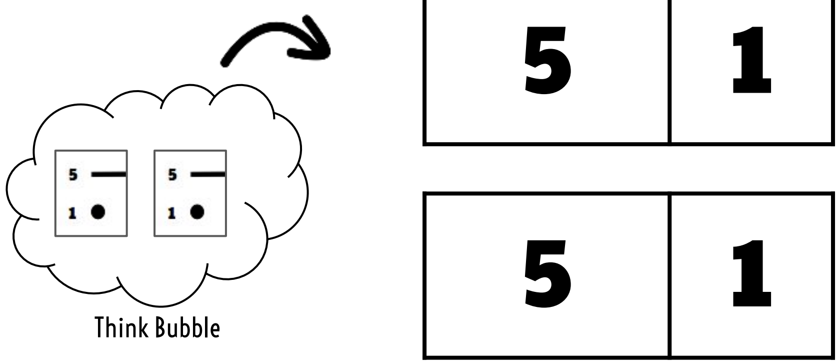


Why might a teacher introduce the idea of using pictorials to represent numerical amounts greater than one early in the development of concepts of multiplication?

6 is equivalent to 5 and 1 more

7

Equal Groups, Decompose (multiplicand), Bar Model



Think Bubble

8

Verbal Expression, Decompose (the second factor using addition)

2 groups of 5 and 1

(or 2 groups of 6)

Six is also equivalent to 3 and 3 more, what verbal expressions can you create?

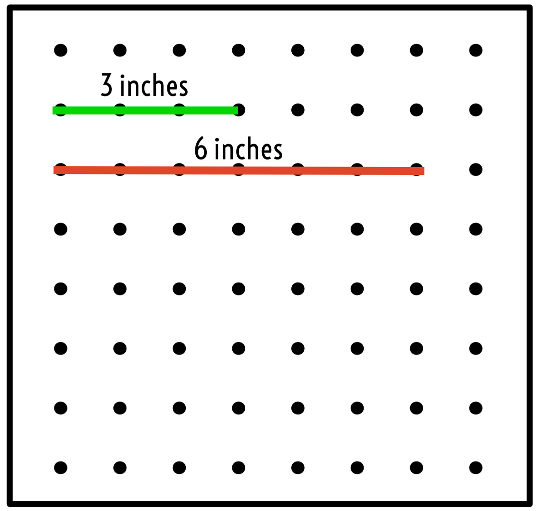
Ponder This:
"...the use of visual representations in mathematical argumentation fell into disfavor in the nineteenth century when it proved misleading in several cases. As a consequence, mathematicians generally moved away from visual arguments and instead developed a strong preference for a symbolic model of reasoning that has held sway in mathematics to the present time." (Stylianou, Silver 2004)

What the research says:
"The decade of the 1980s was an important watershed...The importance of visual processing and external manifestations of this cognition in mathematics was increasingly recognized." (Presmeg 2006)

a 3 inch rubber band stretched to twice its length (2 x 3)

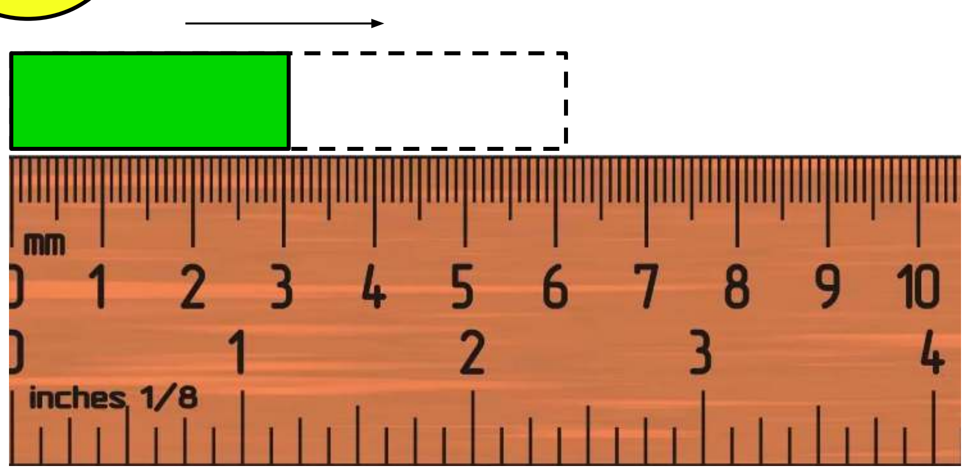
9

Scaling on a Geoboard



10

Scaling on a Ruler



Contextualize:
The concept of multiplication as scaling is often not addressed until 4th or 5th grade. However, it is important that students begin to think of the multifaceted nature of multiplication from early on.

In the example, shown in #9, we are not just making 2 groups of 3, we are actually stretching, scaling up the 3 by a factor of 2 to product a new length.

When placed in this context and acted out, learners can begin to see multiplication as more than repeated addition and differently from equal groupings.

Have-a-Go:
2 x 4

Tools:
geoboards, rubber bands, clay, rulers, yarn, tape

Models:
number line

Strategies:
doubling, tripling, doubles facts, counting on

John is preparing for a party. He buys 5 packs of the variety soda. Each pack has 6 cans of soda.

5 groups of 6 things

5 times (2 + 2 + 2)

5 x 6

5 groups of 6 things

Expressions:
This expression involves factors that are either subitizable and/or are in amounts that students skip count by rote, but the products are not as quickly seen at one glance.

Tools:
real-life objects, e.g., keys, beans, candy, packs of gum, pairs of shoes

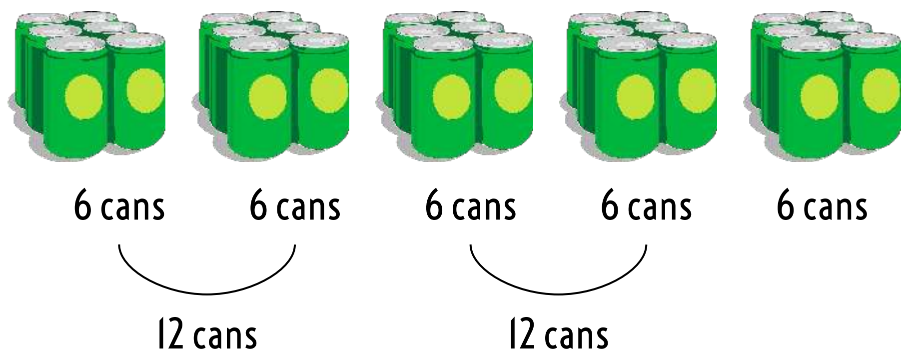
two-colored counters, unit cubes, Unifix cubes, pattern blocks, etc.

Models:
equal groups, set model, arrays , number line diagrams

Strategies:
count all, subitize, doubles facts, repeated addition, skip counting, decomposing groups, partitioning amounts within groups, Distributive Property (informally), Associative Property (informally)

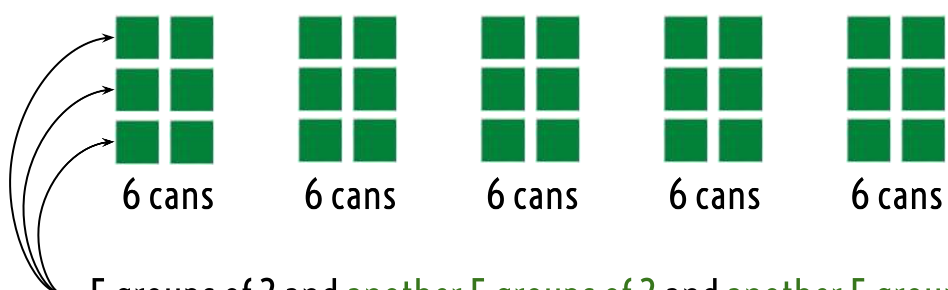
1

Equal Groups, Authentic, Concrete Objects, Composing into larger Group



2

Equal Groups, Arrays, Concrete Objects



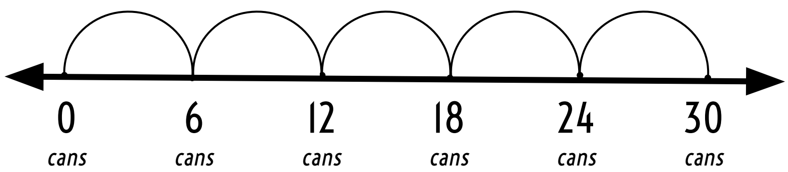
5 groups of 2 and another 5 groups of 2 and another 5 groups of 2

Why is it important for students to begin to "see" regrouping possibilities early on?

2 groups of 12 cans and 1 group of 6 cans

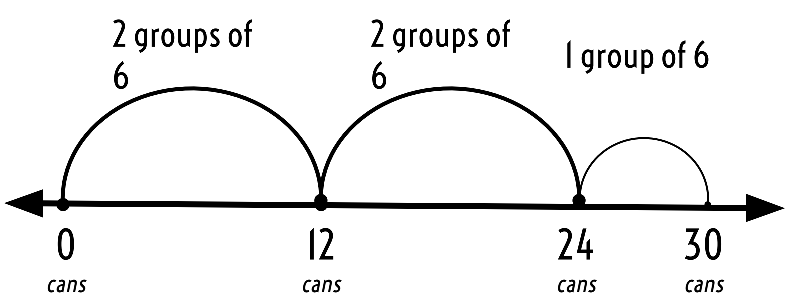
3

Number Line Diagram, Skip Counting



4

Number Line Diagram, Composing, Doubling, Distributive and Associative Property Multiplication (informally)



What are some other ways to show 5 x 6 on the number line?

Ponder This:
Sherin and Fuson (2005) argue, "…during the period in which multiplication is the focus of explicit classroom attention, changes in strategy use are primarily driven by the learning of ***number-specific computational*** resources."

"Stated simply, students acquire a great deal of knowledge about specific numbers—such as 4, 12, and 32—and this knowledge allows the use of new strategies or the use of old strategies in new contexts".

What the research says:
"As students learn, they develop an increasingly rich network of knowledge about specific numbers. In essence, their number-specific resources merge and, because of this merging, it does not make sense to speak of students using one strategy or another." Sherin and Fuson (2005)

5 rows with 6 in each

Have-a-Go:
4 x 6

Tools:
grid paper

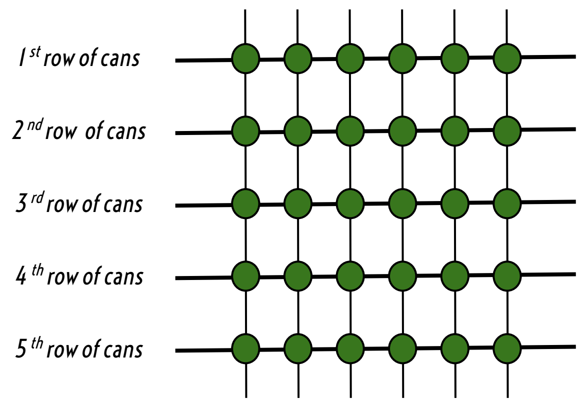
two-colored counters, unit cubes

Models:
equal groups, set model, arrays , crosshatch model, ratio table

Strategies:
count all, subitize, doubles facts, repeated addition, skip counting, decomposing groups, partitioning amounts within groups, partial products from multiplication expressions

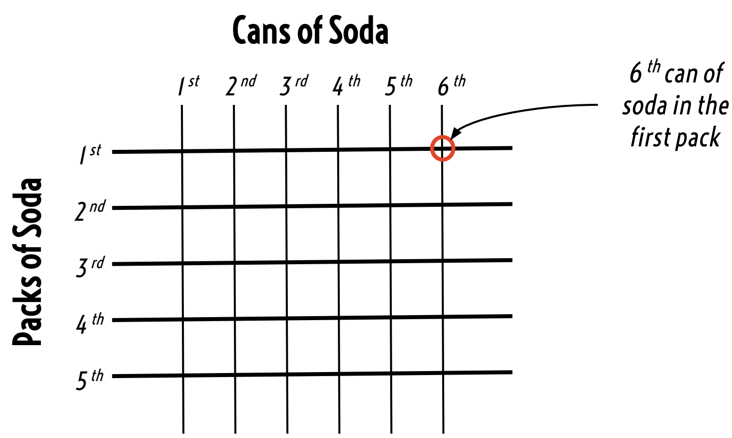
5

Equal Groups, Crosshatch Model, Array Model



6

Crosshatch Model

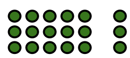
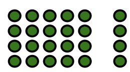
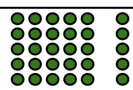


If the scenario about the soda cans included the line, "Each pack had 2 lime sodas, 2 orange sodas and 2 grape sodas," how might the representation in this section look different?

6 is equivalent to 5 and 1 more

7

Ratio Table, Pictorial with 5 structure

# of Packs	Visual	Total # of Cans
1		6
2		12
3		18
4		24
5		30

8

Ratio Table, Repeated Addition

# of Packs	Repeated Addition	Total # of Cans
1	6	6
2	6 + 6	12
3	6 + 6 + 6	18
2 + 3 = 5	6 + 6 + 6 + 6 + 6	12 + 18 = 30

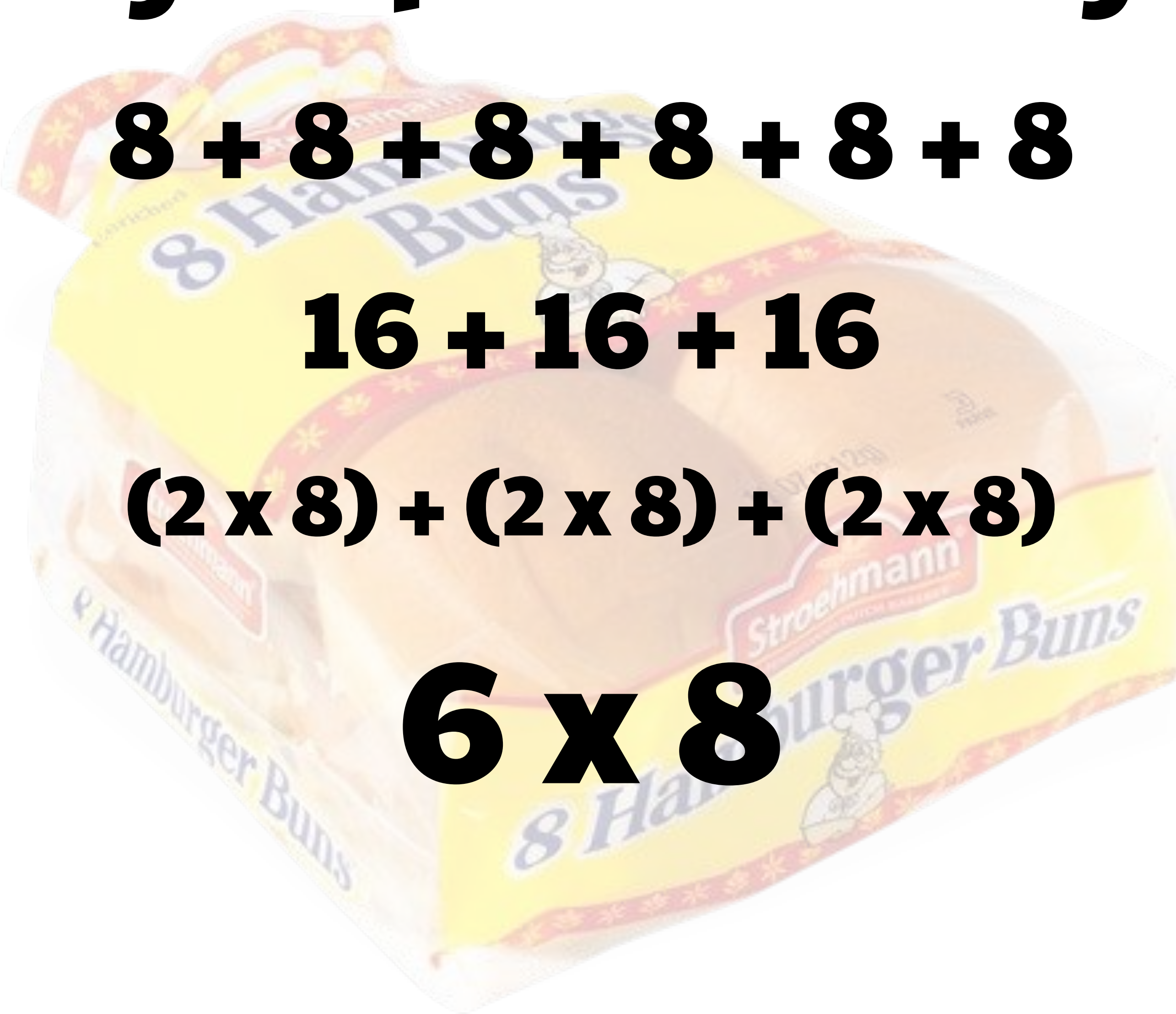
How can we adjust the visual to connect the "Visual" column in #7 to the "Repeated Addition" column in #8?

Ponder This:
Beginning in Grade 3, "...the push for students to look at a number and realize it is so many units of another number is much greater than it was in earlier grades. For example, before, students looked at 18 and thought 10 + 8; even though that is still clearly the case, starting at this level, we want students to look at 18 and think nine 2s OR two 9s OR three 6s OR six 3s. The ability to think multiplicatively is the foundation for future success in math." (Small, 2015).

What the research says:
"Number sense, critically important to students' mathematical development, is inhibited by over-emphasis on the memorization of math facts in classrooms and homes. The more we emphasize memorization to students the less willing they become to think about numbers and their relations and to use and develop number sense." (Boaler, 2009)

Krystle and Nelson bought 6 packs of hamburger buns. Each pack has 8 hamburger buns. They now want to figure out how many hamburgers they should buy so that they have at least one hamburger for every hamburger bun.

6 groups of 8 things



8 + 8 + 8 + 8 + 8 + 8

16 + 16 + 16

(2 x 8) + (2 x 8) + (2 x 8)

6 x 8

6 groups of 8 things or 3 groups of 8 things twice

Expressions:
This expression is one of a group of multiplication facts that research calls the 'harder' facts. They generally span from 6 x 6 to 8 x 8 (Van de Walle, 2004). The expression used in these examples is one that a significant majority of students get incorrect on timed exams.

Tools:
real-life objects, e.g., keys, beans, candy, packs of gum, pairs of shoes

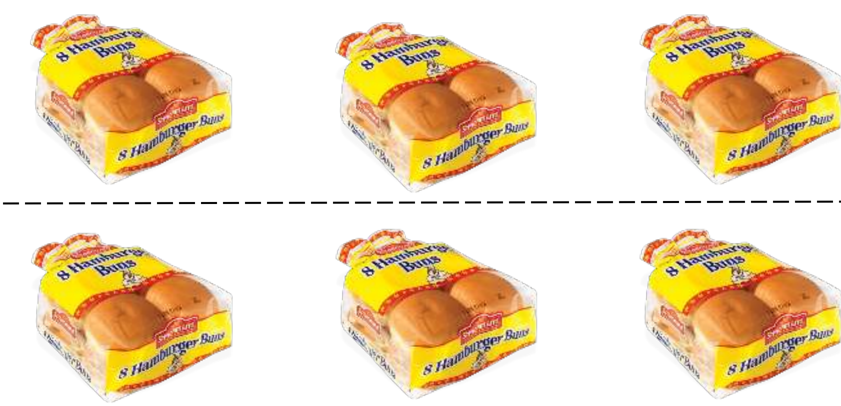
two-colored counters, unit cubes, Unifix cubes, pattern blocks, etc.

Models:
equal groups, set model, arrays, open arrays, ten frame

Strategies:
count all, count on, subitize, doubles facts, skip counting, repeated addition, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication

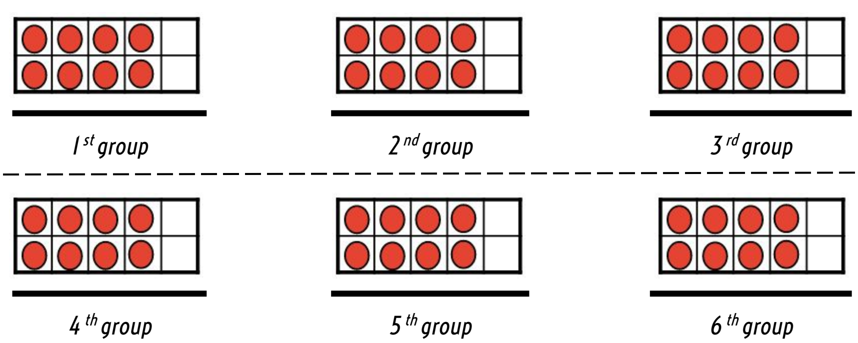
1

Equal Groups, Authentic, Concrete Objects, Partitioning, Arrays



2

Equal Groups, Concrete Tools, Arrays, Ten Frame

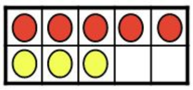


Using the model in #2, what other expressions would be equivalent to 6 x 8? How do you know?

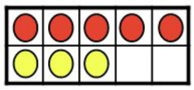
6 groups of 5 and 6 groups of 3

3

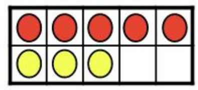
Equal Groups, Concrete Tools, Ten Frame Showing 5 and some more



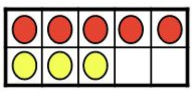
1st group



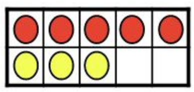
2nd group



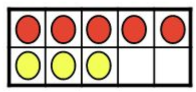
3rd group



4th group



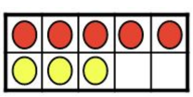
5th group



6th group

4

Equal Groups, Concrete Tools, Ten Frame, Decompose the multiplicand



1st group

5 and 3

2nd group

5 and 3

3rd group

5 and 3

4th group

5 and 3

5th group

5 and 3

6th group

In #4 what other decisions could a teacher make if a student was not ready to go from the concrete in the ten frame to the numerical representation of 5 and 3?

Ponder This:
"Mathematics facts are important but the memorization of math facts through times table repetition, practice and timed testing is unnecessary and damaging." (Boaler, 2015)

What the research says:
"Adopt this simple rule and stick with it: Do not subject any student to fact drills unless the student has developed an efficient strategy for the facts included in the drill."

"Drill can strengthen strategies with which students feel comfortable - ones they "own" - and will help to make these strategies increasingly automatic."

"Drill prior to development of effective methods is simply a waste of precious instructional time." (Van de Walle, 2004).

5 + 2 + 1 repeated six times

Have-a-Go:
9 x 6

Tools:
real-life objects, e.g., keys, beans, candy, packs of gum, pairs of shoes

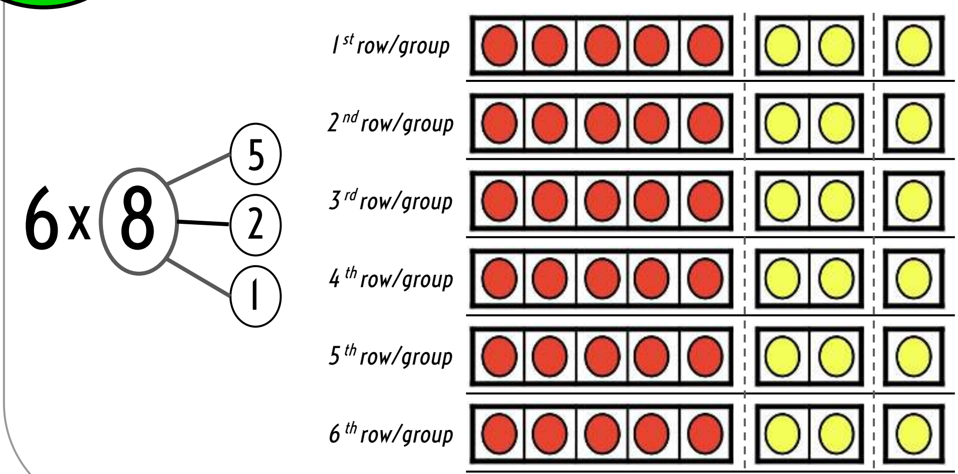
two-colored counters, unit cubes, Unifix cubes, pattern blocks, etc.

Models:
equal groups, set model, arrays, open arrays, ten frame, pictorial base-tens

Strategies:
count all, count on, subitize, doubles facts, skip counting, repeated addition, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication

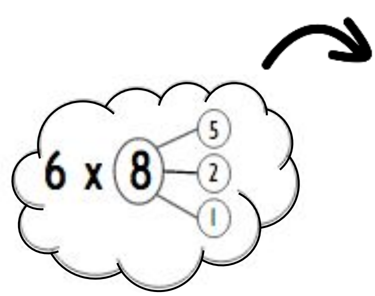
5

Number Bond, Decompose the multiplicand, Array/Equal Groups



6

Number Bond, Decomposing, Skip Counting, Equal Groups



Think Bubble

1st row/group 5 + 2 + 1

2nd row/group 5 + 2 + 1

3rd row/group 5 + 2 + 1

4th row/group 5 + 2 + 1

5th row/group 5 + 2 + 1

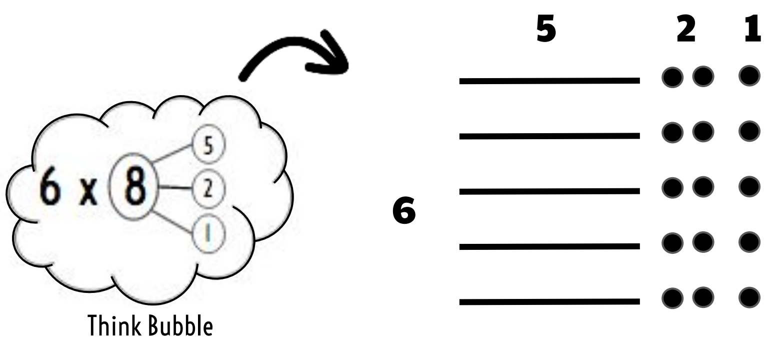
6th row/group 5 + 2 + 1

How might we encourage students to explore and play with these configurations? What might they discover?

6 groups of 5 + 2 + 1

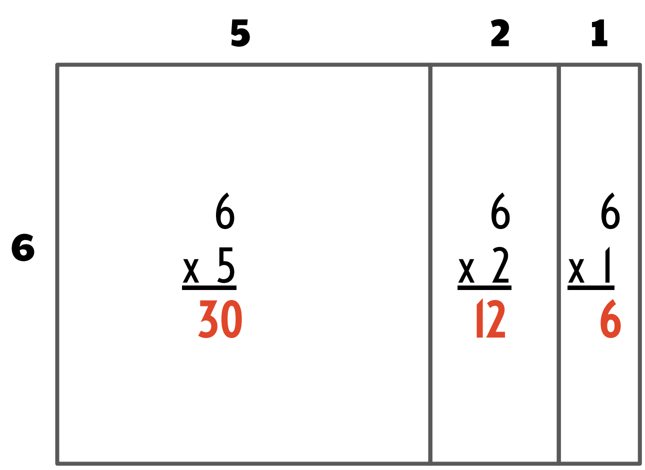
7

Equal Groups, Pictorials, Decompose the multiplicand, Pictorial, Array



8

Area/Open Array, partial products



How are the representations related?
How is the word problem modeled?

Ponder This:
"Mathematics is about reasoning and patterns and making sense of things. Mathematics is about problem solving." (Van de Walle, 2004)

What the research says:
"Importantly the study also found that those who learned through strategies achieved 'superior performance' over those who memorized, they solved problems at the same speed, and showed better transfer to new problems."

"The brain researchers concluded that automaticity should be reached through understanding of numerical relations, achieved through thinking about number strategies." (Delazer et al, 2005)

Ms. Soler bought 4 boxes of a dozen Krispy Kreme doughnuts. She also bought 3 boxes of a dozen Dunkin' Donuts. She planned to keep a dozen donuts for her family, but use the rest in a donut-eating contest.

(12 + 12 + 12 + 12) + (12 + 12 + 12)

Given the situation, what question does this equation answer?

(12 + 12 + 12 + 12) + (12 + 12 + 12) - 12

Given the situation, what question does this equation answer?

(4 x 12) + (3 x 12)

7 x 12

4 groups of 12 things and 3 groups of 12 things

Expressions:
The 12 factor in this expression begin to get students thinking about and using place value understandings.

Tools:
real-life objects, e.g., keys, beans, candy, packs of gum, pairs of shoes

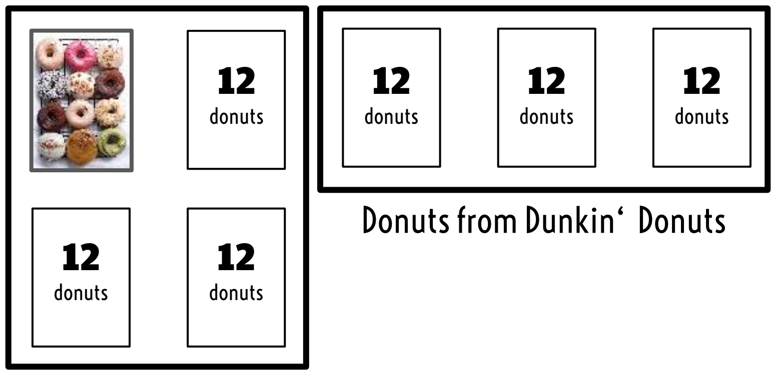
two-colored counters, unit cubes, Unifix cubes, pattern blocks, etc.

Models:
equal groups, set model, arrays, open arrays, ten frame, base-ten blocks, pictorial base-tens

Strategies:
~~count all~~, count on, subitize, doubles facts, skip counting, repeated addition, patterns with 10, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication

1

Equal groups, Ten Frame Structure, Array model, Partial use of Authentic, Concrete Materials

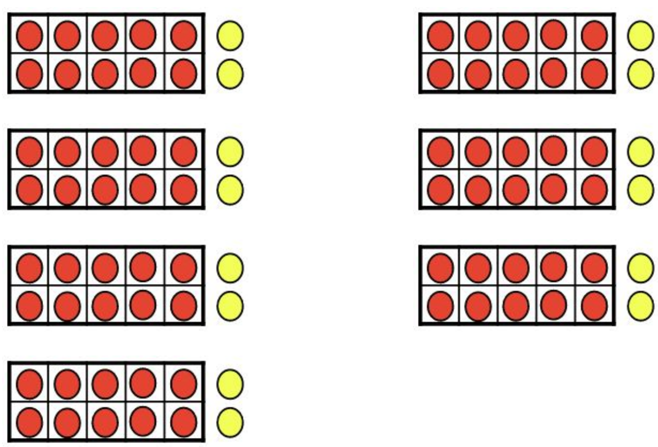


Donuts from Dunkin' Donuts

Doughnuts from Krispy Kreme

2

Equal Groups, Concrete Tools, Ten Frames, Array

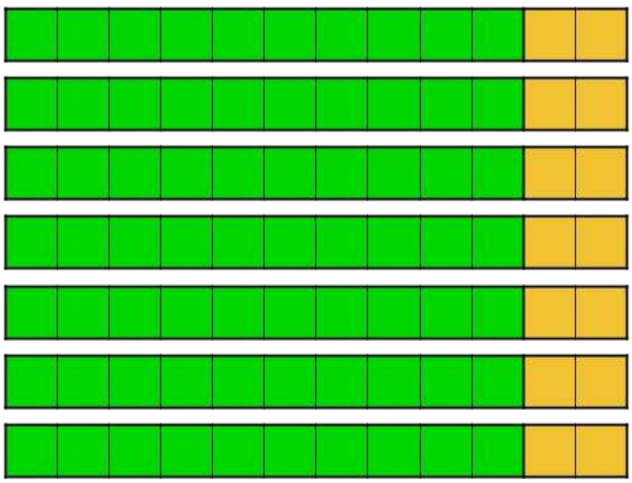


If using the tools and models from #2, would it be beneficial to rearrange the ten frames into strips of 10 and 2 more before progressing to #3? If so, why?

7 x 10 plus 7 x 2

3

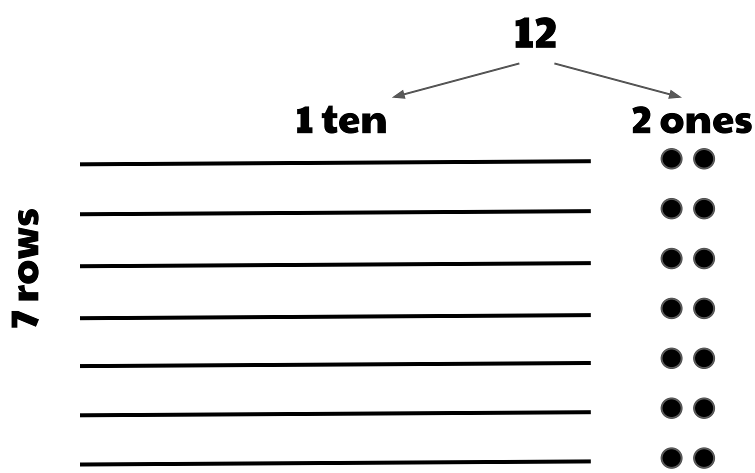
Equal Groups, Base-Ten Blocks



7 tens and 14 ones

4

Decompose the Multiplicand, Array/Equal Groups, Pictorial Base-Tens



How does stating partial products in unit form help prepare students for regrouping based upon place value understandings?

Ponder This:

"Psychologists have long known that people more easily learn a body of knowledge by focusing on its structure (i.e., underlying patterns and relationships) than by memorizing individual facts by rote."

"Furthermore, psychologists have long known that well-connected factual knowledge is easier to retain in memory and to transfer to learning other new but related facts than are isolated facts."

"As with any worthwhile knowledge, meaningful memorization of the basic combinations entails discovering patterns or relationships." (Baroody, 2006)

What the research says:

"Physical representations serve as tools for mathematical communication, thought, and calculation, allowing personal mathematical ideas to be externalized, shared, and preserved." (National Research Council, 2001)

How does the use of concrete tools support an understanding of the Properties of Operations?

Have-a-Go:
8 x 12

Tools:
money ($10, $1), grid paper

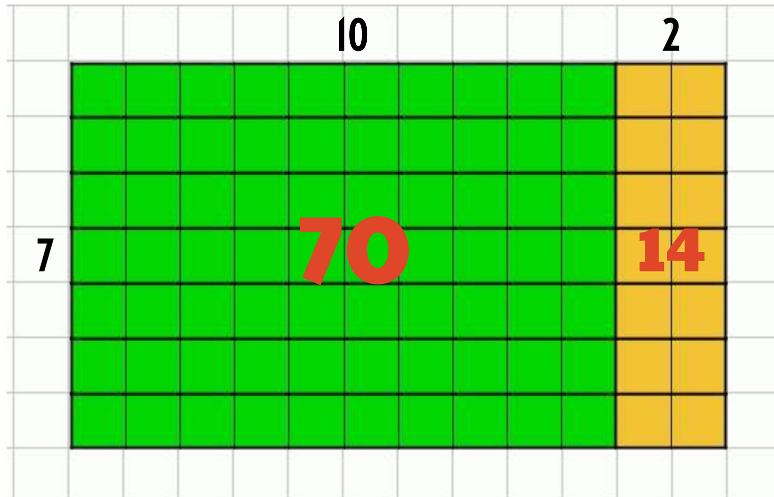
square tiles, unit cubes, Unifix or Snap cubes, etc.

Models:
equal groups, arrays, open arrays, base-ten blocks, pictorial base-tens

Strategies:
~~count all~~, count on, subitize, skip counting, repeated addition, patterns with 10, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication

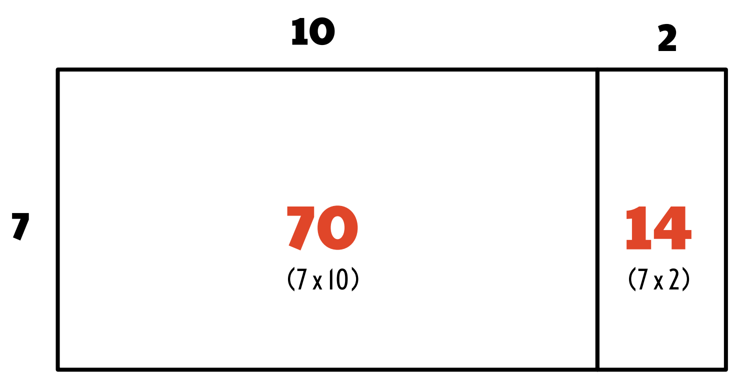
5

Array/Area Model, Partial Products, Decomposing by Place Value, Grid Paper



6

Open Array, Partial Products, Application of Distributive Property of Multiplication

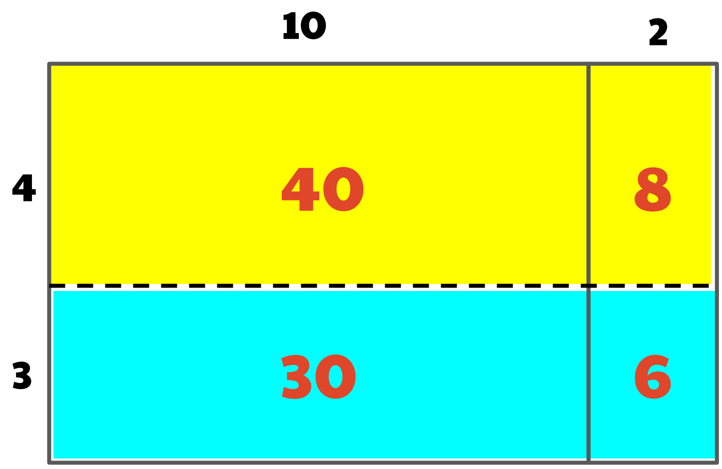


What function does the grid paper play in #5?
How might you use grid paper with #6?

4 groups of 10 and 2
3 groups of 10 and 2

7

Open Array, Decomposing by Place Value, Partial Products



8

Distributive Property of Multiplication over Addition

4 x 12 and 3 x 12 = 7 x 12

(4 x 10) + (4 x 2) and (3 x10) + (3 x 2)

40 + 8 + 30 + 6 = 40 + 30 + 8 + 6 = 84

What are some other ways that 7 x 12 could be modeled using the Distributive Property of Multiplication?

Ponder This:
"The Distributive Law is used frequently in algebraic manipulation; however, we rarely address it with real numbers, because we can simply perform the operation within the brackets rather than distribute. Consider then, how strange the property must seem to students who see it for the first time algebraically rather than with numbers. If students are already familiar with this property before using it in algebra, then it will make sense conceptually." (Southall, 2017)

What the research says:
"Calculating with number sense means that one should look at the numbers first and then decide on a strategy that is fitting-and effective. Developing number sense takes time; algorithms taught to early work against the development of good number sense. Children who learn to think, rather than apply the same procedures by rote regardless of the numbers, will be empowered." (Fosnot and Dolk, 2001)

How does working on a number line support seeing and using patterns?

Have-a-Go:
15 x 6

Tools:
ruler, yardstick/meter stick

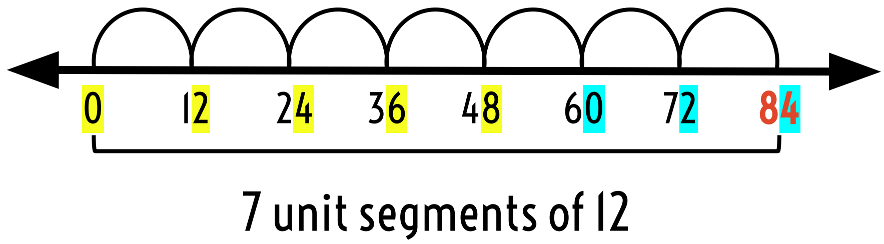
square tiles, unit cubes, Cuisenaire Rods

Models:
equal groups, number line diagram

Strategies:
count all (included, but most students might not need this strategy), count on, subitize, skip counting, repeated addition, patterns with multiples of 10, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication compensation

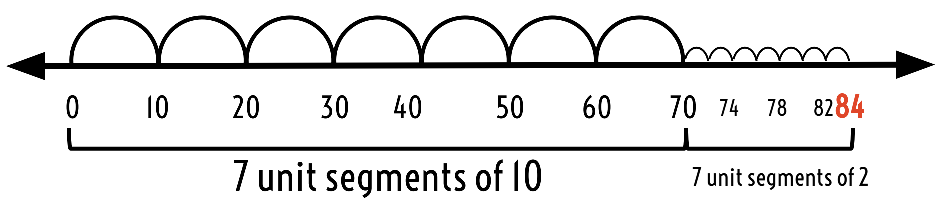
9

Number Line Diagram, Skip Counting, Pattern Recognition



10

Number Line Diagram, Skip Counting by Multiples of 10 and 2s



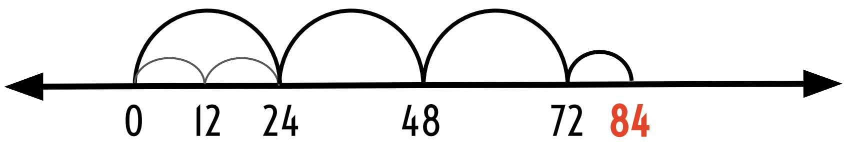
$$7 \times 12 = 7 \times (10 + 2) = (7 \times 10) + (7 \times 2)$$

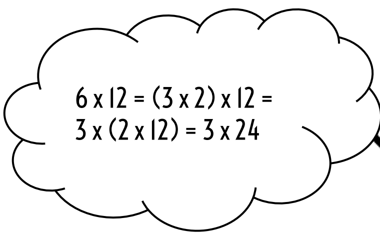
How might we use tools such as rulers, meter sticks, etc., as concrete number lines? Could we use unit cubes, square tiles or Cuisenaire Rods to represent unit segments?

How are the number line diagrams similar? How are they different?

11

Number Line Diagram, Associative Property of Multiplication





Think Bubble

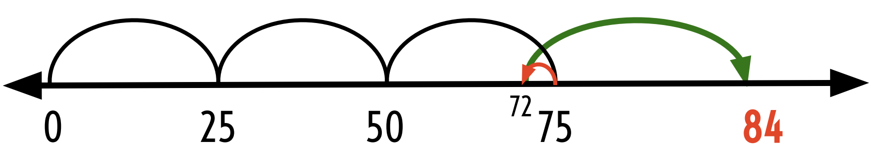
2 x 12 = 1 x 24

6 x 12 = 3 x 24

7 x 12 = (3 x 24) + (1 x 12) or (3 x 24) + 12

12

Number Line Diagram, Associative Property of Multiplication, Compensation



7 x 12 = (3 x 24) + (1 x 12), consequently…

7 x 12 = **(3 x 25) - (3 x 1) + (1 x 12)** = 75 - 3 + 12 =

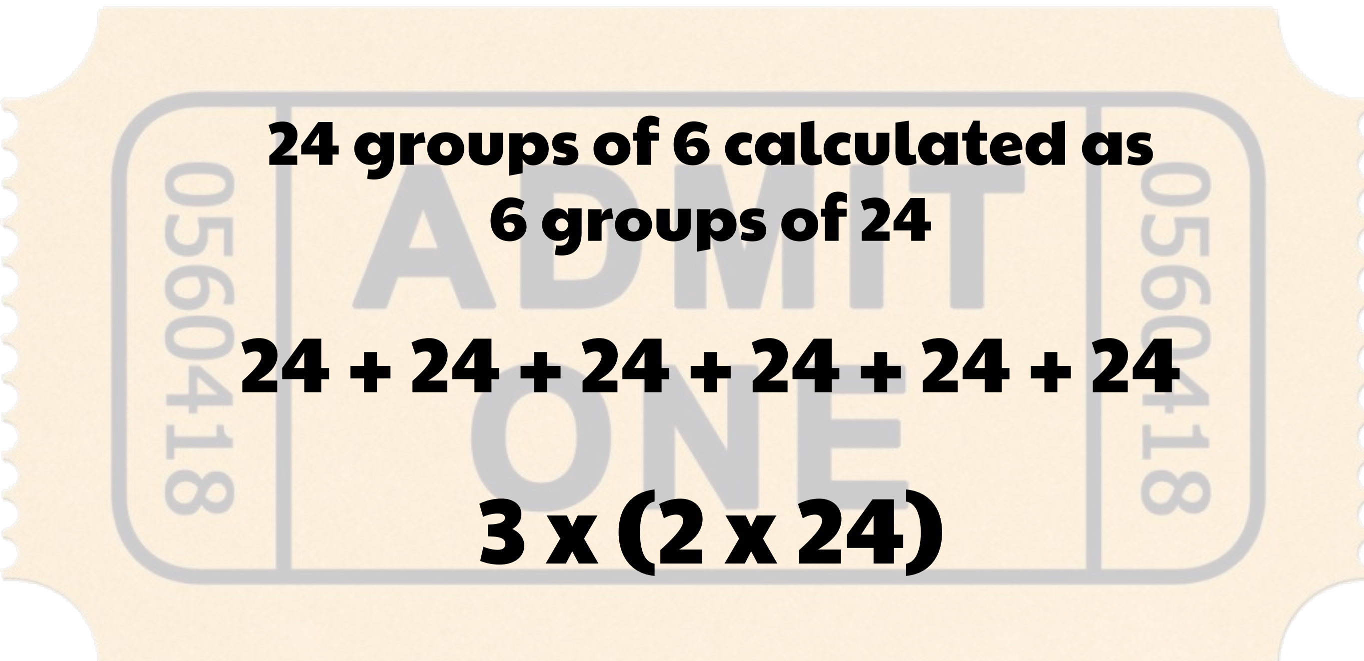
72 + 12 = 72 + 10 + 2 = 82 + 2 = 84

How is the Associative Property of Multiplication applied in #11?

Ponder This:
"The potential of the number line does not stop at providing a simple way to picture all rational numbers geometrically. It also lets you form geometric models for the operations of arithmetic." (National Research Council, 2001)

What the research says:
"Jumps in levels are made internally by students, not by teachers or curriculum. This does not mean that students must progress through the levels with no help. Teaching helps students by providing them with the right kinds of encouragement, support and challenges – having students work on problems that stretch, but do not overwhelm, their reasoning, asking good questions, having them discuss their ideas with other students, and sometimes showing them ideas that they don't invent themselves." (Battista, 2012)

It was 'Bargain Tuesday' at Bow Tie Cinemas. The tired attendant counted 24 people still waiting in line to pay $6 to see Black Panther. She used the Commutative Property of Multiplication to figure out how much money she was going to collect from the 24 people still waiting to pay.



24 groups of 6 calculated as
6 groups of 24

24 + 24 + 24 + 24 + 24 + 24

3 x (2 x 24)

6 x 25 - 6

6 x 24

24 x 6 has the same product as 6 x 24

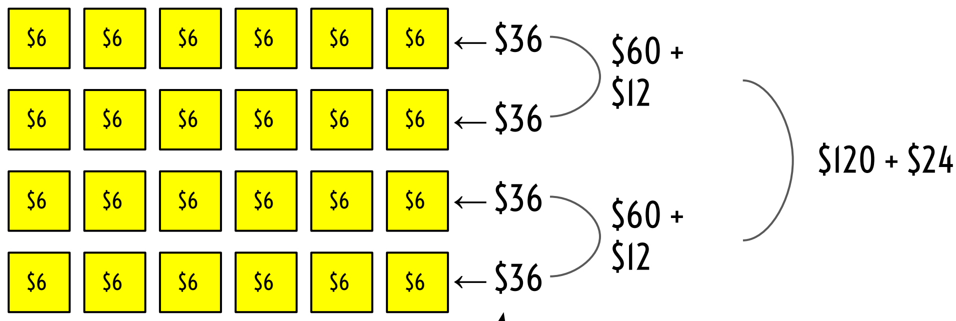
Expressions:
This expression lends itself to compensation and many ways to decompose.

Tools:
square tiles, unit cubes, Cuisenaire Rods, money, number bead string, base-ten blocks

Models:
equal groups, arrays

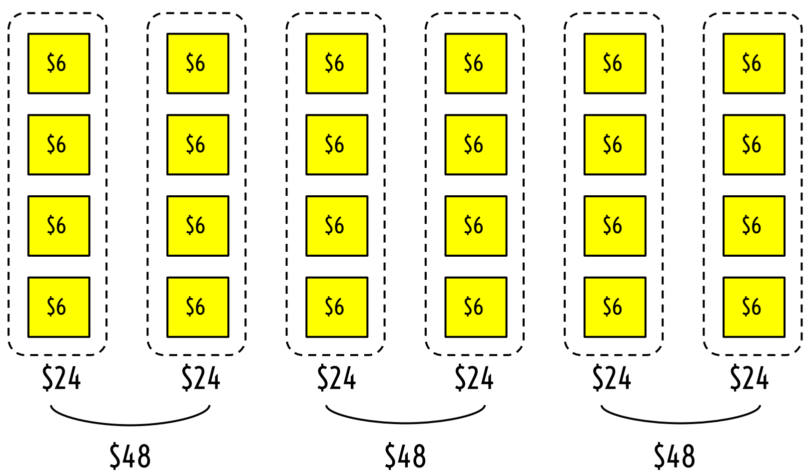
Strategies:
subitize, skip counting, repeated addition, patterns with multiples a factor, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication, Commutative Property of Multiplication, compensation, composing into larger groups, equivalent forms, decomposing by place value

1 Concrete Objects, Partial Products, Array, Equal Groups



24 people each paying $6
24 x 6 = 4 x 36

2 Equal Groups, Array, Partitioning, Composing into larger Groups



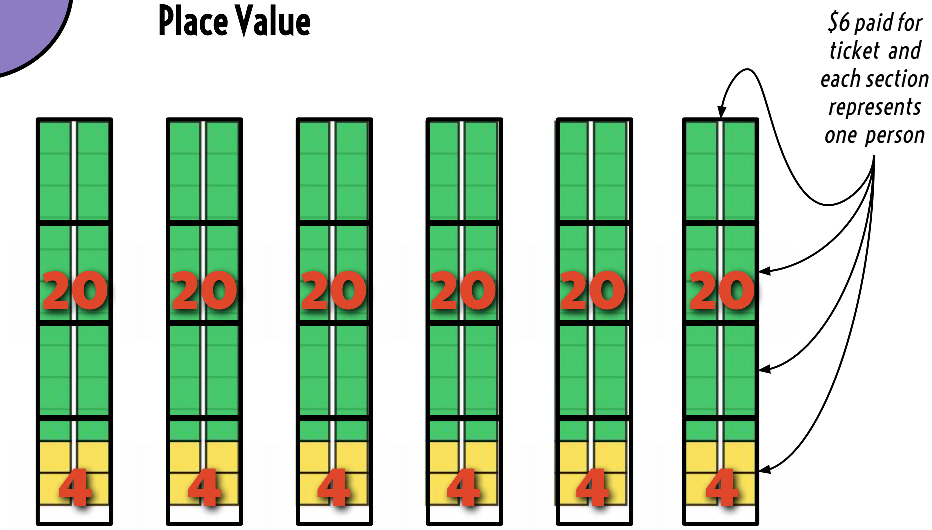
6 groups of 24 is equivalent to 3 groups of 48

In #2 why might someone compute 3 x 50, then take away 6?

24 x 6 is also (6 x 4) x 6 or 6 x (4 x 6) or (12 x 2) x 6 or 12 x (2 x 6)

3

Base-Ten Blocks, Equal Groups, Decomposing by Place Value



4

Repeated Addition, Decomposing by Place Value, Associative Property of Multiplication, Multiply by Multiples of 10

24 x 6 = 6 x 24

(24) + (24) + (24) + (24) + (24) + (24)

(20 + 4) + (20 + 4) +(20 + 4) + (20 + 4) + (20 + 4) + (20 + 4)

(20 + 20 + 20 + 20 + 20 + 20) + (4 + 4 + 4 + 4 + 4 + 4)

(6 x 20) + (6 x 4) = 120 + 24 = 144

In #3, given the story context, what do the red numbers represent?

Ponder This:
"Proficiency with numbers in the elementary and middle grades implies that students can not only appreciate these different notations for a number but also can translate freely from one to another. It also means that they see connections among numbers and operations in the different number systems." (National Research Council, 2001)

What the research says:
"Applying the Distributive Property reinforces our students' understanding of the multiplication process and helps them make sense of the factors as well as the reasonableness of the product ." (Tellish, O'Connell, SanGiovanni, 2016)

How does visualizing our thinking support deeper understanding?

Have-a-Go:
6 x 49

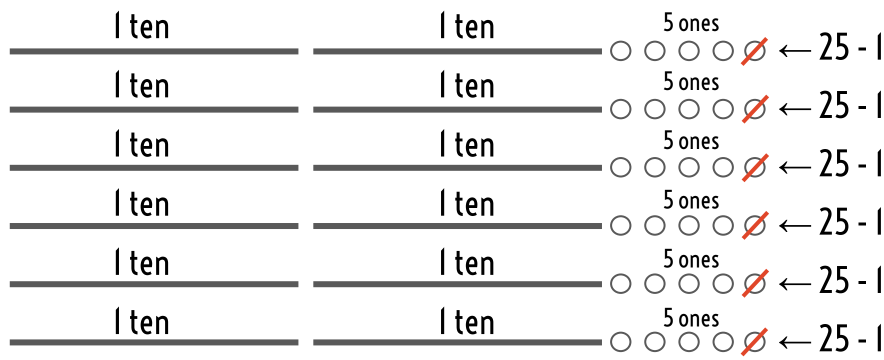
Tools:
square tiles, unit cubes, Cuisenaire Rods, money, number bead string, base-ten blocks, place value discs, place value chart

Models:
equal groups, arrays, pictorial base-tens

Strategies:
subitize, skip counting, repeated addition, patterns with multiples 10, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication, Commutative Property of Multiplication, compensation, composing into larger groups, equivalent forms, decomposing by place value, doubling, halving, partitioning

5

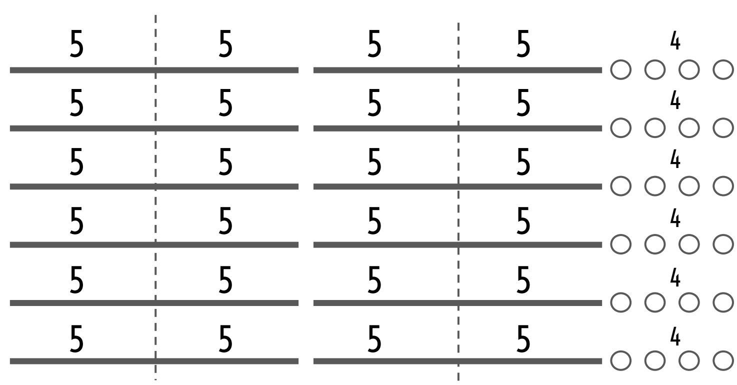
Pictorial Base Tens, Compensation, Decomposing by Place Value



$6 \times (10 + 10 + 5) - (6 \times 1) = (6 \times 25) - (6 \times 1) = 150 - 6$

6

Pictorial Base-Tens, Halving, Decomposing by Place Value



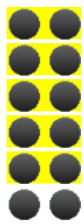
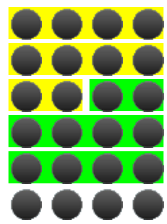
$6 \times 24 = 6 \times (4 \times 5) + (6 \times 4)$

Connect the models to the equations.

How can place value discs in a place value chart reinforce student understanding of regrouping, unitizing and equivalence?

7

Place Value Chart, Place Value Discs, Regrouping

Tens	Ones
	
12 tens (10 tens and 2 tens)	**24 ones** (10 ones, 10 ones and 2 ones)

8

Place Value Chart, Verbal Expression, Unit Form

Tens	Ones
6 groups of 2 tens	6 groups of 4 ones
12 tens	**24 ones**

12 tens 24 ones is equivalent to
1 hundred 2 tens 2 tens 4 ones
or 1 hundred 4 tens 4 ones

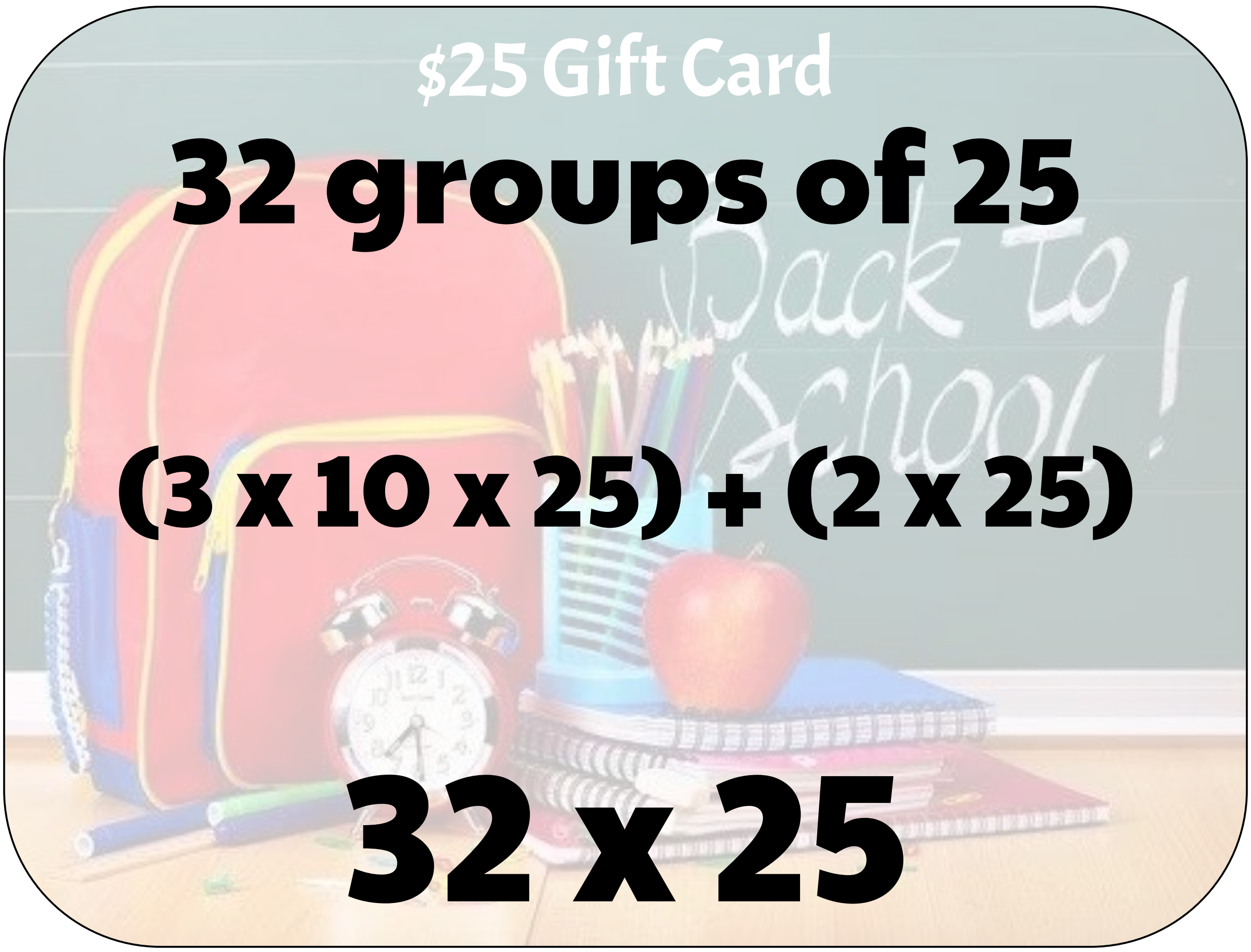
Instead of place value discs, how might you use tally marks within the place value chart to show multiplication?

Ponder This:
"Mathematical ideas are enhanced through multiple representations." (National Research Council, 2001).

What the research says:
Baroody states in, "Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them," that, "Recent research supports the view that the basic number-combination knowledge of mental-arithmetic experts is not merely a collection of isolated or discrete facts but rather a web of richly inter-connected ideas. For example, evidence indicates not only that an understanding of commutativity enables children to learn all basic multiplication combinations by practicing only half of them but also that this conceptual knowledge may also enable a person's memory to store both combinations as a single representation."

"This view is supported by the observation that the calculation prowess of arithmetic savants does not stem from a rich store of isolated facts but from a rich number sense." (Heavey 2003)

Mr. Jones was awarded a grant from the Kellogg Corporation. As a result, each of his 32 students received a $25 gift card to buy school supplies at Target or Walmart.



Expressions:
This expression can help to deepen students' understanding about the Properties of Operations.

Tools:
square tiles, unit cubes, Cuisenaire Rods, money, number bead string, base-ten blocks, place value discs, place value chart

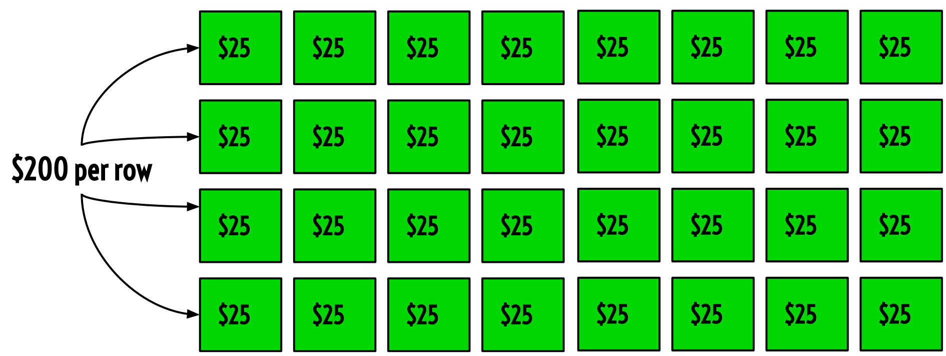
Models:
equal groups, arrays, open array, ratio table, base-ten blocks, pictorial base-tens

Strategies:
subitize, skip counting, repeated addition, patterns with multiples 10, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication, Commutative Property of Multiplication, compensation, composing into larger groups, decomposing by place value, doubling, halving, partitioning, division

32 groups of $25

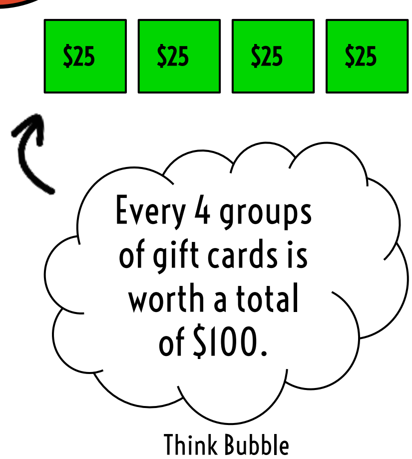
1

Array Model, Partial Products, Concrete Objects, Partial Products



2

Ratio Table, Composing into larger Groups, Doubling



# of Gift Cards	Total Worth
4	$100
8	$200
16	$400
32	$800

Explain the multiplicative relationship in the ratio table?
Can you note cross-connections to other domains?

Using the examples below, fill-in the blank: 8 x (_ x 50). Explain your thinking.

3

Composing into larger Groups, Quotative Division, Partial Products

32 divided to create 4 equal shares per group is 8 groups with 4 equal shares in each group.

Every 4 groups of gift cards is worth a total of $100. I wonder how many groups of 4 are in 32?

Think Bubble

4

Associative Property of Multiplication, Inverse Property of Multiplication

Example A:

32 x 25 =

(8 x 4) x 25 =

8 x (4 x 25) =

8 x 100 = 800

Example B:

32 x 25 =

(8 x 4) x 25 =

8 x (4 x 25) =

(32 ÷ 4) x (4 x 25) =

8 x 100 = 800

Which example in #4 better matches the model in #3? Why?

Ponder This:

In 1989, researcher Magdalene Lambert wrote this about students: “They need to be treated like sense-makers rather than like rememberers and forgetters.”

What the research says:

Baroody states in, “Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them,” that, “Recent research supports the view that the basic number-combination knowledge of mental-arithmetic experts is not merely a collection of isolated or discrete facts but rather a web of richly inter-connected ideas. For example, evidence indicates not only that an understanding of commutativity enables children to learn all basic multiplication combinations by practicing only half of them, but also that this conceptual knowledge may also enable a person’s memory to store both combinations as a single representation.”

“This view is supported by the observation that the calculation prowess of arithmetic savants does not stem from a rich store of isolated facts but from a rich number sense.” (Heavey 2003)

32 groups of (20 + 5)

Have-a-Go:
32 x 35

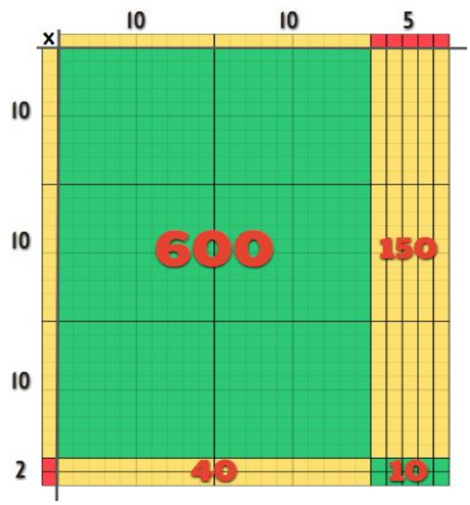
Tools:
square tiles, unit cubes, Cuisenaire Rods, money, number bead string, base-ten blocks, place value discs, place value chart

Models:
equal groups, arrays, open array, ratio table, base-ten blocks, pictorial base-tens

Strategies:
subitize, skip counting, repeated addition, patterns with multiples 10, decomposing, partial products, Associative Property of Multiplication, Distributive Property of Multiplication, Commutative Property of Multiplication, compensation, composing into larger groups, decomposing by place value, doubling, halving, partitioning, division

5

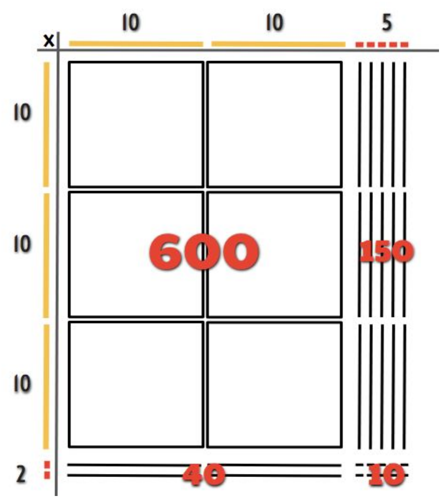
Base-Ten Blocks, Array/Area Model, Partial Products, Application of Distributive Property



600
150
40
+ 10
800

6

Pictorial Base-Ten, Area Model, Partial Products, Application of Distributive Property



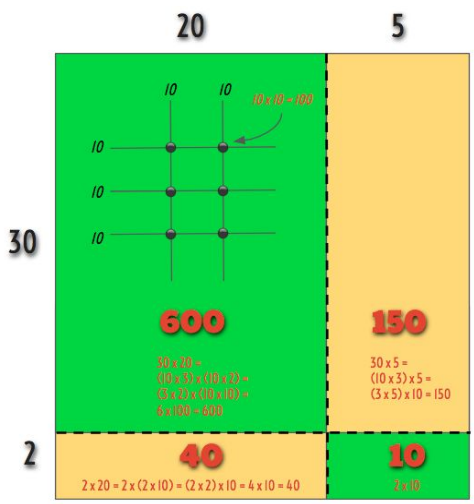
(600 + 150) + (40 + 10) =
750 + 50 = 800

Using concrete base-ten blocks, as modeled in #1, has its limitations. For example, what would happen if you used base-ten blocks to model 99 x 99 or 101 x 24?

Find where (30 x 10) + (30 x 10) is modeled in #7. How else might you look at that section?

7

Open Array, Crosshatch (Line Multiplication) Multiplication Model, Partial Products



(600 + 40) + (150 + 10) =

640 + 160 =

600 + 40 + 100 + 60 =

600 + 100 + (40 + 60) =

700 + 100 = 800

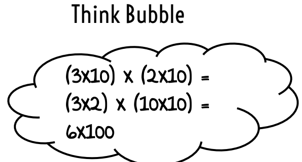
8

Partial Products, Decomposing by Place Value, Distributive Property of Multiplication

32 x 25 =

(30 + 2) x (20 + 5) =

Think Bubble



(30 x 20) + (30 x 5) + (2 x 20) + (2 x 5) =

600 + 150 + 40 + 10 = 800

What are some other ways to model/structure the multiplication of partial products from #8?

Ponder This:
"Knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably." (Ball, Hill and Bass, 2005)

What the research says:
"Goldin and Shteingold's (2001) work on representations suggests that representational systems are important to the learning of mathematics because of the inherent structure contained within each representation. This structure can shape or constrain learning. Furthermore, different representations emphasize different aspects of a concept and so the development of an understanding of a particular concept comes from having a range of representations and being able to move both within and between them." (Harries and Barmby, 2007).

SECTION II:

Tools, Models and Strategies Extended

EQUAL GROUPS

NUMBER LINE DIAGRAM

TEN FRAME

ARRAY/AREA MODEL

CROSSHATCH METHOD/MATRIX MULTIPLICATION

RATIO TABLE

Does creating equal groups always mean making circles?

Featured Model:
Equal Groups

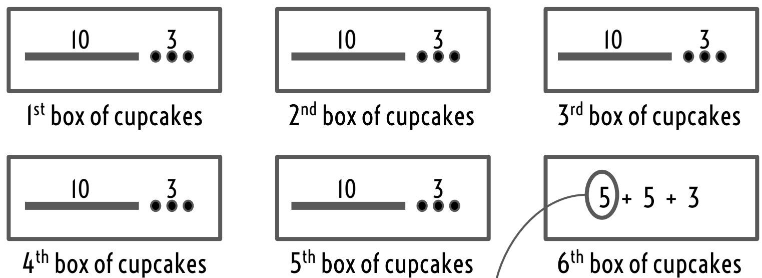
Context:
Ms. Hinds had 6 boxes with a baker's dozen cupcakes in each box. She gave 5 cupcakes to her neighbor and used the rest for her visitors.

Can you think of questions given the context?

Standards:
3.OA.1

3rd

Equal Groups, Pictorial Base-Tens, Decomposing, Partial products



5 x 10 = 50
6 x 3 = 18
50 + 18 + 5 = 68 + 5 =
68 + 2 + 3 = 73 cupcakes

cupcakes given to neighbor

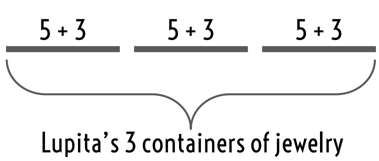
Context:
Lupita had 3 containers that each had 8 pieces of jewelry. Her friend Danai had twice as many containers of jewelry. Danai had the same amount of jewelry in each container.

Can you think of questions given the context?

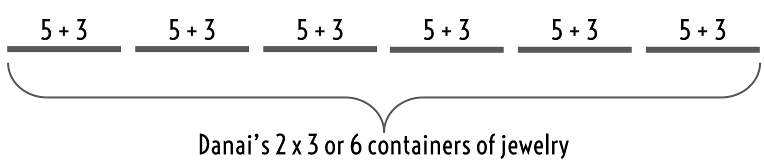
Standards:
4.OA.1

4th

Equal Groups, Decomposing, Doubling, Distributive Property of Multiplication



3 x 8 or 3 times 5 + 3
(3 x 5) + (3 x 3) = 15 + 9 =
15 + 10 - 1 = 25 - 1 = 24
Lupita had 24 pieces of jewelry, so Danai had 48 pieces of jewelry.



What Properties of Operations are being applied in this section?

Why is it important to see where the numbers come from in an equation?

5th

Equal Groups, Associative Property of Multiplication, Commutative Property of Multiplication

2 x ¾ miles	2 x ¾ miles	2 x ¾ miles	2 x ¾ miles	2 x ¾ miles
1st Day	2nd Day	3rd Day	4th Day	5th Day

Total # of miles walked to and from work by Ms. Dawkins in 5 days.

$5 \times (2 \times \frac{3}{4}) = (5 \times 2) \times \frac{3}{4} = 10 \times \frac{3}{4} = \frac{3}{4}$ of $10 = 7\frac{1}{2}$ miles in 5 days

Context:
Marina was trying to get more exercise. Each day Marina walked ¾ of a mile to work and then the same distance back home every day for 5 days.

Can you think of questions given the context?

Standards:
5.NF.4

6th

Equal Groups, List, Decomposing

Last Summer

16 one-cup servings of organic soda

8 oz.	8 oz.
8 oz.	8 oz.
8 oz.	8 oz.
8 oz.	8 oz.
8 oz.	8 oz.
8 oz.	8 oz.
8 oz.	8 oz.
8 oz.	8 oz.

4 groups of 8 oz. is a quart and 4 quarts equals a gallon

16 groups of 8 oz. is 128 oz. or 1 gallon
6 bottles x $2.69 =
$12 + $1.40 - 0.02 = $13.38

This Summer

16 one-cup servings of organic soda twice

8 oz.	8 oz.	8 oz.	8 oz.
8 oz.	8 oz.	8 oz.	8 oz.
8 oz.	8 oz.	8 oz.	8 oz.
8 oz.	8 oz.	8 oz.	8 oz.
8 oz.	8 oz.	8 oz.	8 oz.
8 oz.	8 oz.	8 oz.	8 oz.
8 oz.	8 oz.	8 oz.	8 oz.
8 oz.	8 oz.	8 oz.	8 oz.

2(128) = 256 oz.
2 groups of 1 gallon equals 2 gallons
12 bottles x 25.4 fl. oz. =
(10 x 25.4) + 2(25.4) = 254 + 50.8 = 304.8 fl. oz.

Context:
Last summer, Carmela served 16 one-cup servings of organic soda for the community's "Going Organic" Fair. It was a hit! This summer she plans to serve twice as much as she did last summer. Last year she bought a six-pack of soda. Each bottle cost $2.69 for 25.4 fl. oz. She did not want to waste any soda .

Can you think of questions given the context?

Standards:
6.RP.3

What are some strategies (computational or otherwise), evident in the 6th grade example?

Featured Model:
Number Line Diagram

Context:
Maria walked 8 blocks to school each day. Due to several days off, she only had 10 days of school in September. Maria did not have to walk home, because her father picked her up everyday.

Can you think of questions given the context?

Standards:
3.OA.1
3.OA.9

Context:
Ida and Ruby loved to read. Every season they had a contest to see who could read the most books. During the summer season Ida read 3 times as many books as Ruby.. Ruby read 6 books.

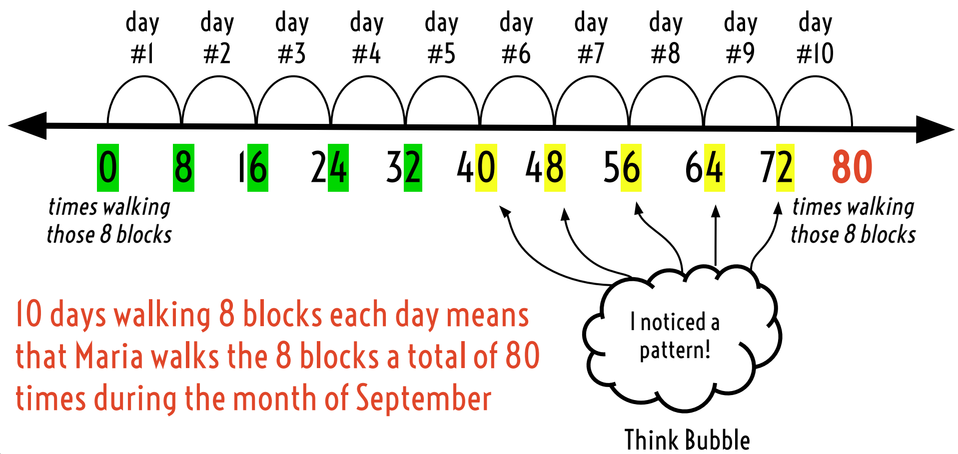
Can you think of a question given the context?

Standards:
4.OA.1

Can you think of other ways to use number line diagrams?

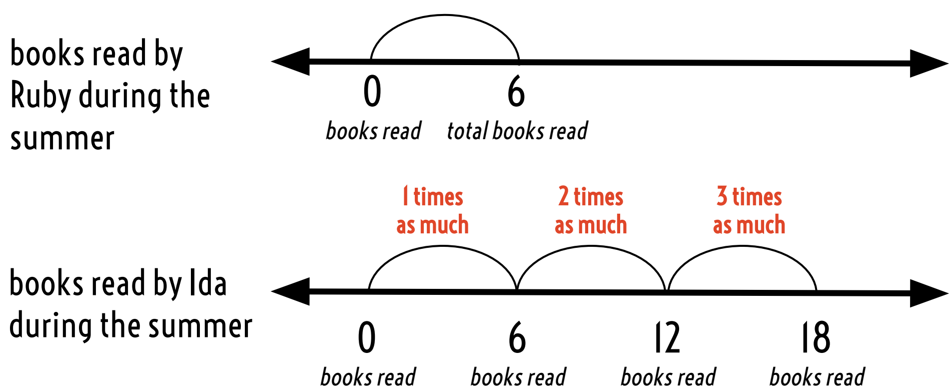
3rd

Number Line Diagram, Skip Counting, Pattern Recognition



4th

Double Number Line Diagram, Skip Counting

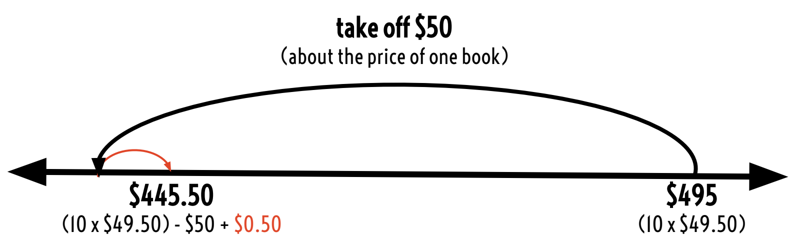


Can you explain the pattern that was noticed in the 3rd grade example?
Mathematically speaking, why is that pattern happening?

How can the number line diagram support flexibility with numbers?

5th

Open Number Line Diagram, Distributive Property of Multiplication, Associative Property of Multiplication



9 copies at $49.50 per copy is equivalent to 10 copies at $49.50, but we have to take out one group of $49.50.

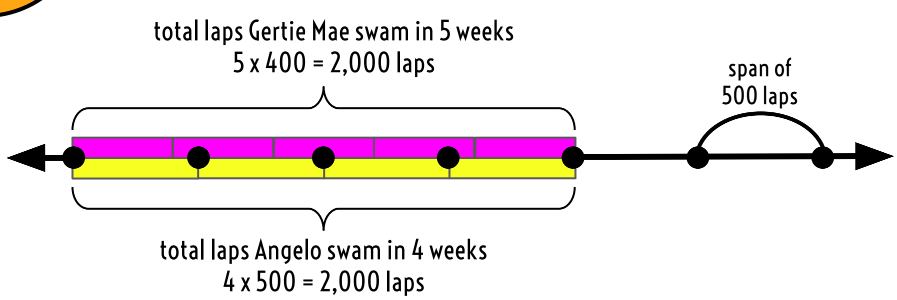
9 x $49.50 = (10 x 49.50) - (1 x $49.50) = $495 - $49.50 =

$495 - ($50 - 0.50) = ($495 - $50) + 0.50 = $445 + $0.50 = $445.50

Mr. Douglass will be under budget and have $450 - $445.50, or $4.50 to spare.

6th

Open Number Line Diagram, Associative Property of Multiplication, Cuisenaire Rods



5 x 400 = 4 x 500 = 5 x (4 x 100) = 4 x (5 x 100) = (4 x 5) x 100 = 20 x 100 = 2,000

- I can see that Angelo swam 100 more laps each week than Gertie Mae.
- I can see that Gertie Mae swam one more week than Angelo, before taking time off.
- I can see that every 2 ½ Gertie Mae swims the same number of laps that Angelo swims in 2 weeks.

Can you explain the following equation in reference to the 5th grade context: 10 x (50 - 0.50) - (50 - 0.50) = ?

Context:
Principal Douglass wanted to purchase 9 copies of the book, *Becoming the Math Teacher You Wish You'd Had*, for his math teachers. With Amazon Prime the book cost $49.50. His budget for this purchase is $450.00.

Can you think of questions given the context?

Standards:
5.NBT.7

Context:
Gertie Mae and Angelo like to swim for fitness and they keep a record of the number of laps they swim per week. Gertie Mae swam 400 laps per week for 5 weeks, then she took a week off to rest. Angelo swam 500 laps per week for 4 weeks before taking a week off to rest.

Can you think of questions given the context?

Standards:
6.RP.3

Featured Model:
Ten Frame

Context:
Dr. Erica Greve was a veterinarian. On Monday she had a very busy day. She gave 7 vaccination shots to each of the 7 cats she saw that day. She also gave 3 vaccination shots to each of the 2 dogs she saw that day.

Can you think of questions given the context?

Standards:
3.OA.3

Context:
Malala was a well known florist in her community. She was known for having the most beautiful floral arrangements. To prepare of an upcoming event, she ordered 14 dozen yellow roses and 11 dozen red roses. Each dozen cost $32.

Can you think of questions given the context?

Standards:
4.NBT.5

Can you think of other ways to use the ten-frame?

3rd

Ten Frame, Partial Products, Decomposing

7	7	7	7	7
7	7	3	3	

7 groups of 7 is equivalent to 10 groups of 7 minus 3 groups of 7.

Think Bubble

7 x 7 = 10 x 7 - 21 = 70 - 20 - 1 = **49**
2 x 3 = **6**
49 + **6** = 49 + 1 + 5 = 50 + 5 = **55**
Dr. Erica Greve gave 55 vaccination shots on Monday.

4th

Ten Frame, Partial Products, Doubling, Halving, Decomposing by Place Value

$32	$32	$32	$32	$32
$32	$32	$32	$32	$32

$32	$32	$32	$32	$32

$32	$32	$32	$32	$32
$32	$32	$32	$32	$32

10 x 32 = 320

2 x 320 = (2 x 300) + (2 x 20) = **640**

Half of 320 or 5 x 32 =

(5 x 30) + (5 x 2) = 150 + 10 = **160**

640 + 160 = 700 + 100 = $800

How would you show any of the situations featured by using a number line diagram?

Do we use decimals and fractions interchangeably? If so, when? If not, why not?

5th

Ten Frame, Pattern, Distributive Property of Multiplication, Associative Property of Multiplication

\$5.75	\$5.75	\$5.75	\$5.75	\$5.75
\$5.75	\$5.75	\$5.75		

$\frac{3}{8}$ of the t-shirts were yellow, $\frac{3}{8}$ of \$46

3 x 5.75 = 15 + (3 x 0.75)

15 + (9 x 0.25)

15 + 2.25 =

\$17.25

8 x 2 = 16

8 x 1 = 8

8 x ½ = 4

8 x ¾ = 6 or 8 x 0.75 = 6

8 x 5 is 40, so each t-shirt had to cost at least \$5. Now I have \$6 dollars to split with 8 t-shirts. 8 times what is 6?

Think Bubble

Context:
Darren spent \$46 on 8 t-shirts for his construction company. Three of the t-shirts were yellow to indicate apprentices.

Can you think of questions given the context?

Standards:
5.NBT.7

6th

Ten Frame, Distributive Property

c + p	c + p	c + p	c + p	c + p
c + p	c + p	c + p	c + p	

nine times c plus p

9(c + p) = 9(8.36 + 15)

9(15 + 5 + 3 + 0.36) = 9(23.36) =

10(23.36) - 23.36 = 233.60 - 23.36 = **\$213.24**

Context:
Kelly loved to bake and was planning on making her specialty, chocolate double fudge cake. During the holiday season she made 9 of these cakes for friends. Kelly figured out that the ingredients for each cake, *c*, costs \$8.36. She charged each person, *p*, \$15 to make a cake, plus the cost of the ingredients.

Can you think of questions given the context?

Standards:
6.EE.2

Can you explain the pattern in the 5th Grade example? Continue the pattern to solve for *n* in the equation $8 \cdot n = 2$.

Does creating equal groups always mean making circles?

Featured Model:
Array/Area Model

Context:
Ms. Hinds had 6 boxes with a baker's dozen cupcakes in each box. She gave 5 cupcakes to her neighbor and used the rest for her visitors.

Can you think of questions given the context?

Standards:
3.OA.1

3rd

Array Model, Base-Ten Blocks, Partial Products, Equal Groups, Decomposing by Place Value

6 x 13 = 6 x 10 and 6 x 3 minus 5 = 60 + 18 minus 5 = 78 - 5 = 73 cupcakes

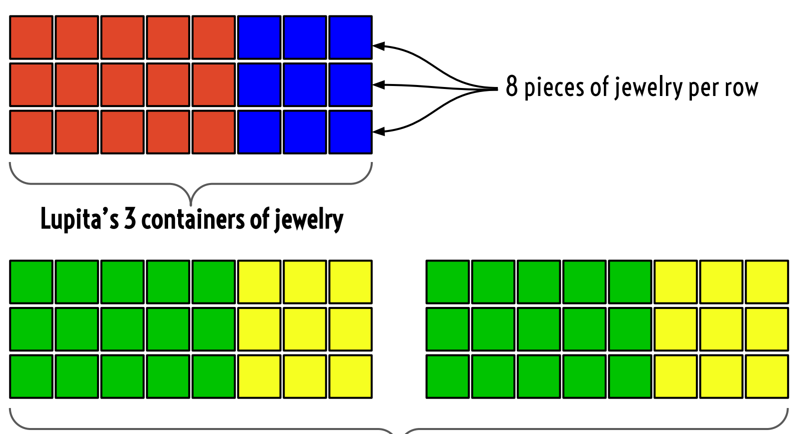
Context:
Lupita had 3 containers that each had 8 pieces of jewelry. Her friend Danai had twice as many containers of jewelry. Danai had the same amount of jewelry in each container.

Can you think of questions given the context?

Standards:
4.OA.1

4th

Array/Area Model, Partial Products, Distributive Property of Multiplication, Square Tiles



Dania's 2 x 3 or 6 containers with 8 pieces of of jewelry in each, which equals 6 x 8 = 2 x (3 x 5 + 3 x 3) = 2 x (15 + 9) = 2 x (24) = 48 pieces of jewelry

How might you rearrange Dania's arrays in the 4th Grade situation to more clearly show (6 x 5) + (6 x 3)?

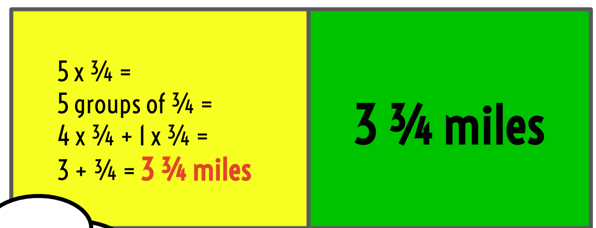
What is the role of mental math when multiplying?

5th

Open Array, Decomposing, Partial Products, Application of Inverse Property of Multiplication

¾ miles ¾ miles

5 days



What whole number can I multiply ¾ by to produce a product equivalent to a whole number?

Think Bubble

(5 x ¾) + (5 x ¾) = 2 x 3 ¾ =

(2 x 3) + (2 x ¾) = 6 + $\frac{6}{4}$ =

6 + 1 ½ = **7 ½ miles for 5 days**

Context:
Marina was trying to get more exercise. Each day Marina walked ¾ of a mile to work and the same distance back home every day for 5 days.

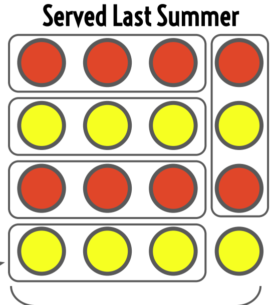
Can you think of questions given the context?

Standards:
5.NF.4

6th

Array, Concrete Objects, Equal Groups

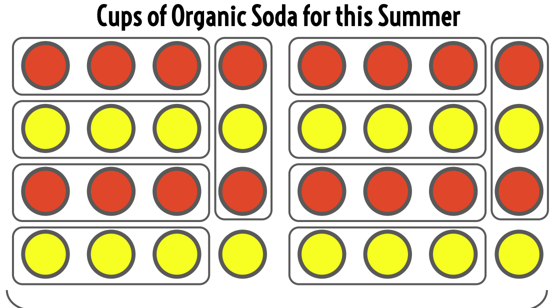
Cups of Organic Soda Served Last Summer



sixteen 8 fl. oz. cups

1 bottle = (3 x 8) + 1.4 = 25.4 fl. oz.
5 x 1.4 fl. oz. left per bottle =
5 + (1 x 0.4) =
5 ones and 20 tenths = 5 + 2 = 7 fl.oz.

Cups of Organic Soda for this Summer



2 (16 cups) means that **you need 11 bottles** as you would need 6 bottled to serve the first 16 cups, but only 5 bottles would be needed for the next 16 cups, as you would only need one oz. from the 6th bottle for first set of 16 cups.

Context:
Last summer, Carmela served 16 one-cup servings of organic soda for the community's "Going Organic" Fair. It was a hit! This summer she plans to serve twice as much as she did last summer. Last year she bought a six-pack of soda. Each bottle cost $2.69 for 25.4 fl. oz. She did not want to waste any soda.

Can you think of questions given the context?

Standards:
6.RP.3

How is the Distributive Property of Multiplication Over Addition being applied in these examples?

Featured Model:
Crosshatch Method

Context:
Maria walked 8 blocks to school each day. Due to several days off, she only had 10 days of school in September. Maria did not have to walk home, because her father picked her up everyday.

Can you think of questions given the context?

Standards:
3.0A.1
3.0A.9

Context:
Ida and Ruby loved to read. Every season they had a contest to see who could read the most books. During the summer season Ida read 3 times as many books as Ruby. Ruby read 6 books.

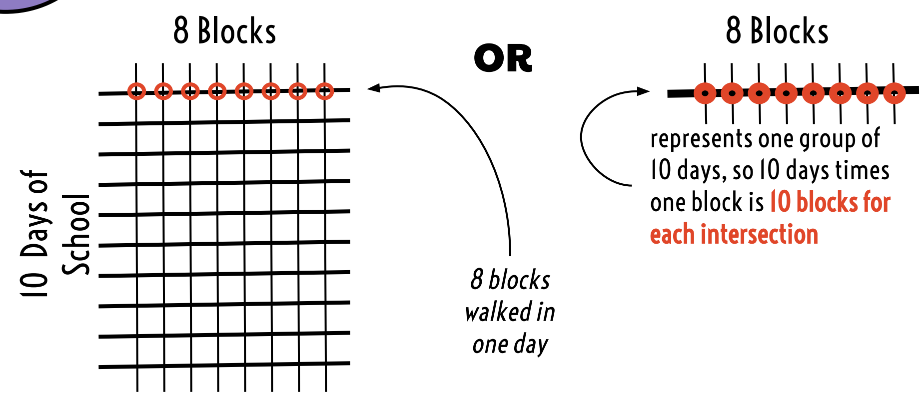
Can you think of a question given the context?

Standards:
4.0A.1

Does creating equal groups always mean making circles?

3rd

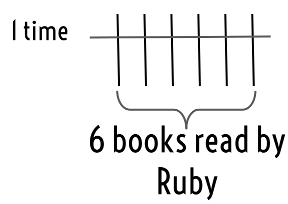
Crosshatch Method (Line AI, Skip Counting

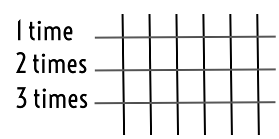


8 blocks repeated 10 times in the month of September is 80 blocks

4th

Crosshatch Method (Line Algorithm)





3 x 6 = 18 books read by Ida

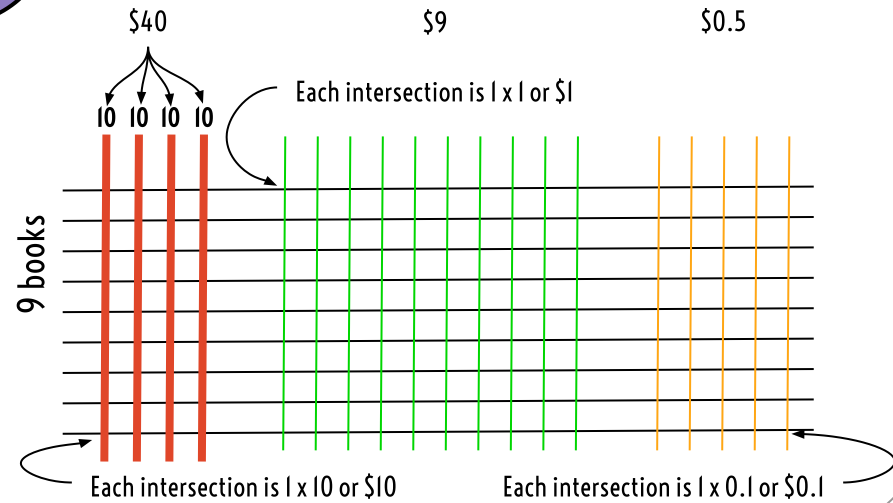
- Ruby and Ida read a total of 6 + 18 or 24 books.
- Ruby read ⅓ of the total books that Ida read.
- Ida read 12 more books than Ruby during the summer season.

What might you do to encourage students to move beyond counting all, and to take place value and multiplicative reasoning into consideration when using this model?

What are the benefits of this model? What are the drawbacks?

5th

Crosshatch Method (Line Algorithm), Partial Products, Distributive Property of Multiplication



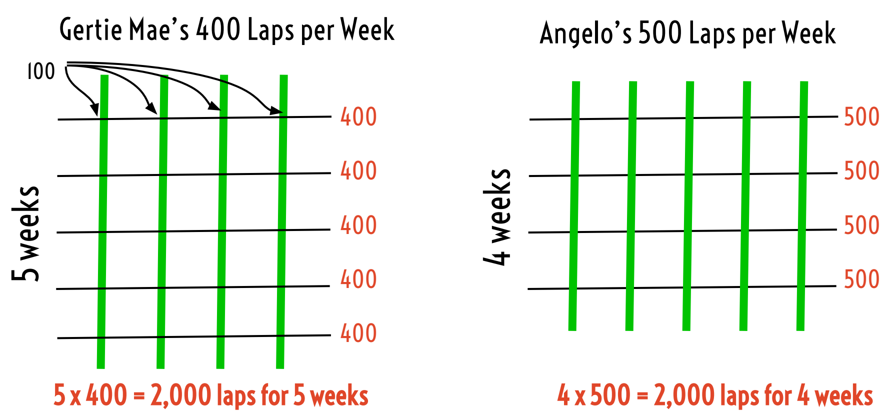
Context:
Principal Douglass wanted to purchase 9 copies of the book, *Becoming the Math Teacher You Wish You'd Had,* for his math teachers. With Amazon Prime the book cost $49.50. His budget for this purchase in $450.00.

Can you think of questions given the context?

Standards:
5.NBT.7

6th

Crosshatch Method (Line Algorithm), Partial Products, Skip Counting



- Gertie Mae swims 80% of the laps Angelo swims each week.
- Angelo swims 25% more laps than Gertie Mae each week.

Context:
Gertie Mae and Angelo like to swim for fitness and they keep a record of the number of laps they swim per week. Gertie Mae swam 400 laps per week for 5 weeks then she took a week off to rest. Angelo swam 500 laps per week for 4 weeks before taking a week off to rest.

Can you think of questions given the context?

Standards:
6.RP.3

How might you use the Crosshatch Method to help students conceptually visualize 30 x 40, so they get 1,200 and not 120?

Featured Model:
Ratio Table

Context:
Dr. Erica Greve was a veterinarian. On Monday she had a very busy day. She gave 7 vaccination shots to each of the 7 cats she saw that day. She also gave 3 vaccination shots to each of the 2 dogs she saw that day.

Can you think of questions given the context?

Standards:
3.OA.3

How do ratio tables build proportional reasoning?

3rd

Ratio Table, Decomposing

# of Cats	Expression	Total # of Shots
1	1 cat x 7 shots for each cat	7
2	2 x 7	21
3	3 x 7	21
4	4 x 7	28
7	3 x 7 and 4 x 7	21 + 28

# of Dogs	Expression	Total # of Shots
1	1 cat x 3 shots for each dog	3
2	2 x 3	6

21 + 28 = 20 + 20 + 9 = 49
49 + 6 = 49 + 1 + 5 =
50 + 5 = 55 shots given

Context:
Malala was a well known florist in her community. She was known for having the most beautiful floral arrangements. To prepare of an upcoming event, she ordered 14 dozen yellow roses and 11 dozen red roses. Each dozen cost $32.

Can you think of questions given the context?

Standards:
4.NBT.5

4th

Ratio Table, Doubling

# of Dozens	Expression for # of Roses by # of Dozens	Total # of Roses	Cost for # of Dozen	Total Cost
1	1 group of 12 roses	12	1 dozen x $32	$32
2	2 x 12	24	2 x 32	$62
4	4 x 12	48	4 x 32	$128
10	10 x 12	120	10 x 32	$320

(14 x 32) + (11 x 32) = (10 x 32) + (4 x 32) + (10 x 32) + (1 x 32) =
2 x (10 x 32) + (5 x 32) = 2 x 320 + (128 + 32) = 640 + 160 = $800

What if the table was horizontal instead of vertical? What might be the benefits? What might be the drawbacks?

What needs to happen in earlier grades to make flexibility with ratio tables more accessible to students in later grades?

5th

Ratio Table, Halving, Associative Property of Multiplication

# of t-shirts	Expression	Total Price
8	1 x 46	\$46
4	½ of 46	\$23
2	¼ of 46 or ½ of 23	10 + 1.50
1	⅛ of 46 or ½ of 11.50	5 + 0.75

$\frac{1}{8} \times 46 = \frac{1}{8} \times (2 \times 23) = (2 \times \frac{1}{8}) \times 23 = \frac{1}{4} \times 23 =$

$\frac{1}{4} \times (2 \times 11.50) = (2 \times \frac{1}{4}) \times 11.50 = \frac{1}{2} \times 11.50 = 5.75$

Context:
Darren spent \$46 on 8 t-shirts for his construction company. Three of the t-shirts were yellow to indicate apprentices.

Can you think of questions given the context?

Standards:
5.NBT.7

6th

Ratio Table, Distributive Property of Multiplication over Addition

# of cakes	Cost (\$)	Profit (P)	Charge to the Person Getting the Cake
1	1 x 8.36	1 x 15	8.36 + 15
10	10 x 8.36	10 x 15	83.60 + 150

$(10 \times 8.36) + (10 \times 15) = 83.60 + 150 =$

$100 + 50 + 80 + 0.6 = 180 + 20 + 30 + 0.6 = 230.60$

$230.60 - (8.36 + 15) = 230.60 - (18 + 2 + 3 + 0.36) = 230.60 - 23.36 =$

213.24

Context:
Kelly loved to bake and was planning on making her specialty, chocolate double fudge cake. During the holiday season she made 9 of these cakes for friends. Kelly figured out that the ingredients for each cake, *c*, costs \$8.36. She charged each person, *p*, \$15 to make a cake, plus the cost of the ingredients.

Can you think of questions given the context?

Standards:
6.EE.2

How does the ability to double or half any number support efficiency, flexibility and the ability to be more strategic with calculations?

APPENDIX:

Useful Resources to Support Implementation of Micro-Progressions

ACTION PLANNING TOOL

LIST OF TOOLS USED IN BOOK

LIST OF MATH MODELS USED IN BOOK

LIST OF COMPUTATIONAL STRATEGIES USED IN BOOK

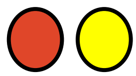
STAGES OF FLUENCY AND STAGE 3 AND 4 MULTIPLICATION FLUENCIES

RECOMMENDED BOOKS AND RECOMMENDED FROM THE WEB

LIST OF TOOLS USED

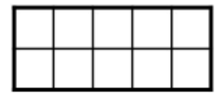
A mathematical tool is a physical thing that learners can access to visualize, structure and organize their thinking. The mathematical tools used in this book are not an exhaustive list of all the tools that could be use when teaching multiplication. The tools listed below are the ones that I chose to use based on my understanding of the intention of the standards and materials that are typically available in most classrooms.

Two Colored Counters



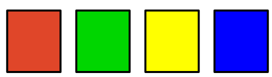
This tool is a discrete, manufactured manipulative. It is used for one-to-one counting, early addition/ subtraction, making arrays, comparing amounts to teach concepts of less than and greater than, as place value discs, addition/subtraction of integers. Similar manipulatives are square tiles, Unifix/Snap Cubes and teddy bear counters. See pgs. 12 - 14, 24, 26

Ten-Frame



This tool has a specific, rectangular structure. It can be purchased or printed from templates found online. It is generally used for structuring and developing concepts of one-to-one counting, addition/subtraction, skip counting by 5s and 10s. This tool can be adapted to be a five-frame, a twenty-frame or even a hundred-frame. See pgs. 13 - 14, 24, 25, 30

Square Tiles



This tool is a discrete, manufactured manipulative. It is generally used for one-to-one counting, early addition/subtraction, making arrays, comparing amounts to teach concepts of less than and greater than, for teaching area and perimeter. Similar manipulatives are two-colored counters, Unifix/Snap Cubes and teddy bear counters. See pgs. 18, 58

Base-ten Blocks

This tool is a proportional, manufactured manipulative. This tool is generally used for teaching place value, decomposing numbers by place value, addition/subtraction, multiplication/division, and decimal concepts. Similar manipulatives are money ($100, $10, $1), Digi-Blocks and the Mortensen Math Blocks. See pgs. 31, 39, 46

Place Value Chart

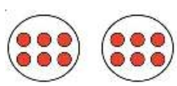
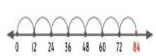
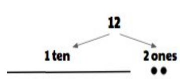
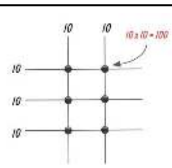
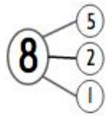
This tool provides structure when applying place value understandings. It can be purchased, printed from templates found online or drawn on chart paper. This tool is generally used for structuring and developing concepts of place value, reading multi-digit whole numbers and decimals, teaching magnitude of a number and addition/subtraction with whole numbers and decimal amounts. This tool can be adapted to show a range of place values. See pg. 41

Grid Paper

This tool is a manufactured product and is used for many purposes outside of the educational setting. Within the educational setting, grid paper is used in many ways, from helping students organize and structure their notes to finding the area or perimeter of geometric figures. When teaching multiplication, the structure of the grid paper allows students to "see" the Properties of Operations, while seeing the product discreetly.
See pg. 32

LIST OF MATH MODELS USED

A math model is a visual representation that shows the mathematical relationship of quantities. Choosing which math models to use is part of the process of becoming more proficient mathematically as different math models may better communicate ideas more precisely than others given a situation. The math models featured in this book are grounded in the CCSS-M and while build upon work in other domains, but do not represent all possible models.

	**Equal Groups:** Used to show equal sets of an amount. While the circle is the common geometric form used when modeling equal groups, the shape can be anything (e.g., rectangles, triangles, line segments) as long as they are all uniform. See pgs. 12, 13, 24, 30
	**Open Array Diagram/Area Model:** Used to show the estimated proportional relationship between the factors (sides of the rectangle) and the product (area of the rectangle). This is sometimes referred to as Can be used to show how factors can be decomposed and easier calculations or to model partial products and the Distributributive Property of Multiplication Over Addition. See pgs. 18, 27, 32, 33
	**Number Line Diagram:** Used to show repeated addition and skip counting. Can be used to show the estimated proportional relationship as equal groups added on. Can be used to model equivalent expressions. Can be used to model the distance from zero. See pgs. 19, 34 - 35
	**Pictorial Base-Tens:** Used a visual, drawn representations of base-ten blocks. A line is used to represent tens and dots to represent ones. A square can be used to represent one-hundred. I would select using another symbol to represent thousands, as drawing a cube takes time and is challenging for students to do precisely. See pgs. 31, 40, 46
	**Crosshatch Model (Line Algorithm):** Used to model the intersection of lines. Each horizontal and vertical line has a weighted value. Unless specified the default value is one. When the lines intersect each point produced is the product of the weighted value of horizontal and vertical lines. Line algorithm See pgs. 20, 60
	**Number Bond:** Used to show how a number can be decomposed in addends that sum to that number. Number Bonds show part-part-whole relationships for addition. See pgs. 26
	**Ratio Table:** Used to show the multiplicative, additive and proportional relationships with scaled quantities. Ratio Tables are sometimes called "Input/Output" Tables. See pgs. 21, 62, 63

LIST OF STRATEGIES USED

A mathematical strategy is an abstraction based upon mathematical structures and the Properties of Operations that we apply/do to figure out (calculate) or show how to arrive at an answer. We often apply mathematical strategies without realizing it. For example, when most people are asked to work out 24 x 6 (which can be represented as 24 groups with 6 things in each group), they often, instead, work out 6 x 24 (which can be represented as 6 groups, with 24 things in each group). This is generally a subconscious application of the Commutative Property of Multiplication.

The strategies listed here do not represent all of the strategies that can be used, but rather the strategies that are consistently used in this book. The strategies used in this book are based on the understandings intended by the CCSS- M.

Decomposing into Addends: Numbers have a hierarchical structure. Every number has numbers 'nested' with it. Decomposing a number is the process of breaking that number into parts (addends) that, when recomposed, would result in a sum of the original number. You decompose a number in order to make it easier to count or calculate. See pgs. 14, 15, 25, 27, 33	**Repeated Addition:** Links the operation of addition to multiplication. Repeated Addition is not the same as multiplication, but it is a strategy that can be used to interpret or create an equivalent form of an amount. Repeated Addition allows us to represent any quantity of equal sets. Repeated Addition supports students in developing the language and concept of how many times the multiplicand is repeated. See pgs. 21, 26
Decomposing by Place Value: In our positional number system digits (or other symbols, if used) have a value based upon their placement. To decompose a number by place value, one has to understand place value enough to know that a digit is going to be worth different amounts based on its position. You decompose a number by place value in order to make it easier to calculate by the powers of 1, 10 or 100. See pgs. 32, 33, 39	**Skip Counting:** Is the skill of rote counting by a set interval or amount. Skip Counting can be used as a strategy when students are not at a Stage 4 Level of Fluency with their multiplication facts. Often the easiest intervals for students to skip count by are: ones, tens, fives and twos, in that order. The strategy of Skip Counting can be extended when working with numbers that are multiples of 10, 100 or 1,000. See pgs. 19, 26, 34
Subitizing: Refers to the ability to 'see' amounts and know how many there are without counting. Subitizing can be developed into an internalized skill and can be encouraged by using configurations that allow for subitizing. See pgs. 12 - 14, 30, 31	**Doubles and Halves:** Refers to the skill of being able to quickly double or half a number. This skill can be used as strategy when one recognizes that an amount can be doubled or halved to make calculations easier. This strategy is often used in conjunction with other strategies. See pgs. 13, 44

Equivalent Forms:
Is about rewriting numbers, expressions or equations in another way to represent the same value. Creating Equivalent Forms is often used when doing mental math. The strategy of Equivalent Forms allows students to create simpler-to-solve calculations. This strategy is often used in conjunction with other strategies. See pgs. 15, 19, 24, 33

Commutative Property of Multiplication:
States that if you have 2 factors, you will get the same product regardless of which factor is the multiplier and which one is the multiplicand. When using the Commutative Property of Multiplication as a strategy, a person is recognizing that one equation might be easier to visualize or solve. See pgs. 38, 39

Associative Property of Addition:
States that the order in which addends are added does not change the sum. When using the Associative Property of Addition as a strategy we can combine addends that are easier to add or decompose addends in order to re-compose them with addends that are easier to add. See pgs. 21, 33, 34

Associative Property of Multiplication:
States that the order in which the factors are multiplied does not change the product. When using the Associative Property of Multiplication as a strategy we can change the order in which we multiply the factors. We can also apply this strategy by decomposing factors into their factors, then multiplying those factors in an order that makes multiplication easier. This strategy is often used with the Doubles and Halves strategy. See pgs. 12, 47

Partial Products:
This strategy is generally used in combination with other strategies such as Decomposing, and often called for when using the Distributive Property of Multiplication. Producing Partial Products is a skill that is needed to apply the Distributive Property of Multiplication, but it does not encompass the entirety of the property. Partial Products is the process of multiplying one factor or part of one factor by another factor or part of another factor. See pgs. 26, 27, 30, 31

Distributive Property of Multiplication Over Addition:
States that when one multiplies a factor, a, by another factor, b, that has been decomposed into an addition expression, $(c + d = b)$, then the partial products produced by multiplying the first factor, a, by each addend of the second factor, $(c + d)$, can be added together to find the entire product of the multiplication expression.
See pgs. 19, 30 - 33

Identity Property of Division:
States that the quotient of any number divided by one is that number. Because multiplication and division are inverse operations, this property also means that any number divided by itself is one. This strategy is often used with the Associative Property of Multiplication.
See pg. 41

Identity Property of Multiplication:
States that the product of any number multiplied by one is that number. This strategy is often used in conjunction with other strategies. See pg. 35

STAGES OF FLUENCY

Stage of Fluency	Definition	Example Student is asked, "What is 34 times 11?"
0	**No Understanding:** Unable to understand the question or concept being asked. Prior knowledge/experience does not support ability to comprehend question or concept.	Student has a blank look on his or her face. Student repeats, "34 times 11."
1	**No Strategy:** Recognizes some aspects of problem being asked, but does not have proficient prior knowledge or understanding to arrive at a solution.	Student has a pondering look and says, "That's multiplication, right? I can't do that yet."
2	**A Strategy, Rote Procedure:** Able to find a solution to the problem, but has only one strategy. The strategy is generally procedural, without conceptual understanding.	Student asks for pencil and paper and begins to write: 34+34+34...eleven times, then adds the ones column, then the tens column. When complete, student says, "Is that right?"
3	**Strategic, Efficient, Flexible:** Able to find a solution to the problem in a few ways, including the use of various math tools, models and/or strategies. Able to explain why and when they would choose each strategy. Can explain conceptually what they are doing using precise mathematical language.	Student is thinking and then, shortly after says, "34 x 11...I don't know my eleven times table that high, but I can do 34 groups of 10, which is 340. Then I would do 34 x 1, which equals 34. I am going to show you in an array, so that I can keep track of my partial products. Now I will add the partial products of 340+34 to get 374 as the product of 34 x 11.
4	**Automaticity, Direct Retrieval:** Able to arrive (verbally) at a correct response within 3 to 5 seconds. If asked, they are able to explain their thinking and strategy(ies) used with precise mathematical language showing a clear understanding of the concepts reflected in the question. Must be distinguished from memorization, without conceptual understanding.	Student says, "Ahhh...that is 374." Teacher asks, "How did you know that so quickly." Student replies, "I know a pattern in the 11 times table. The first digit and the last digit will be the same, because if you break apart the 11 it is a 10+1. If we multiply the 34 x 10 it just moves the digits one place value over to the hundreds. The 34 x 1 is still going to be 34. Now the only place value I have to really think about is the tens. I had 4 tens from the 340 and 3 tens from the 34, so that is 7 tens...and that is how I got my answer."

Based on and adapted from the work of Carpenter, Fennema, Franke, Levi, Empson, 1999, Carpenter & Moser, 1984

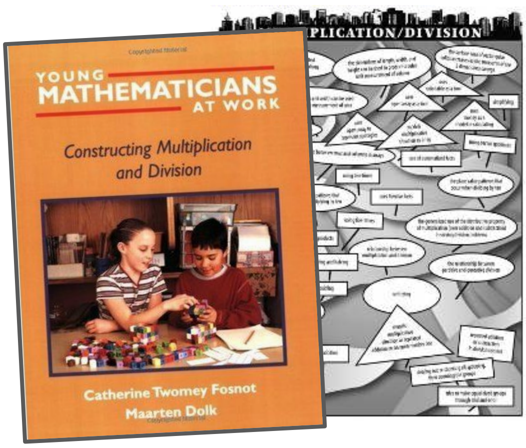
MULTIPLICATION FLUENCIES

Grade of Exposure	A Guide to Stage 3 and 4 Fluencies by the End of the Grade Level of Exposure	
K - 2	Subitizing to 10 Calculating doubles facts Sums to 20 Decomposing numbers Decomposing numbers by place value Applying the Commutative Property of Addition	Skip counting by 1s, 2s, 5s, 10s Using/reading a ten frame Using/reading and doing addition/ subtraction on a number line Organizing counters into rectangular arrays Writing equivalent forms of equations
3	Halving of numbers up to 20 Making equal groups Reading a multiplication expression as, "___ groups of ___ things." Rewriting a multiplication expression in equivalent forms Decomposing factors Skip counting on a number line Recognizing that "x" is an operation symbol to indicate multiplication	Applying the Commutative Property of Multiplication Applying the Associative Property of Multiplication Applying the Distributive Property of Multiplication Making an array and relating it to area Various strategies for working out multiplication facts Writing equivalent forms of multiplication expressions
4	Halving of numbers up to 100 Using various math models to show the application of the Properties of Operations Use various math models to show ___ times as much as ___	Using various math models to show the application of place value Tell 10 and 100 times more of any whole numbers Using various models and strategies used with whole numbers to multiply a whole number by a fraction
5	Applying various Properties of Operations to the multiplication of whole numbers, decimals and fractions	Recognizing that "*" is an operation symbol to indicate multiplication Using various models and strategies used with whole numbers to multiply fraction and decimal amounts
6	Multiplication of multi-digit whole numbers using the standard algorithm Simultaneous use of various Properties of Operations to all multiplication situations involving positive, rational numbers	Recognizing that when you have a factor followed by a set of parenthesis, it indicates the outer factor is the multiplier of the term or expression within the parentheses, e.g., 3(4)=12 Recognizing that "•" is an operation symbol to indicate multiplication, e.g., 3•4=12

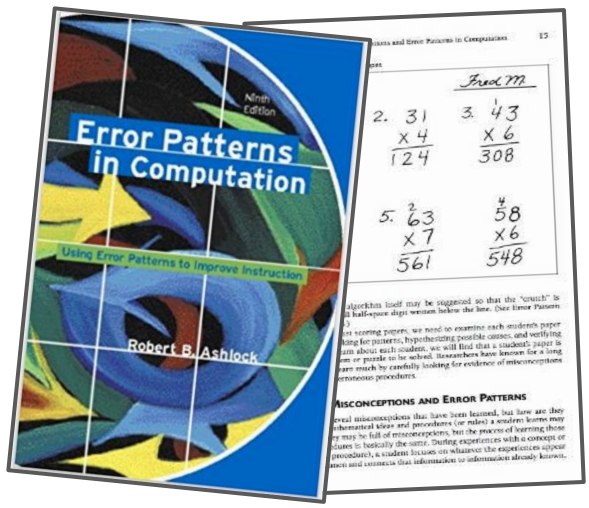
RECOMMENDED BOOKS

The following professional literature is recommended for educators and professional learning communities (PLCs) who want to delve deeper into understanding how students learn and understand multiplication.

Young Mathematicians at Work: Constructing Multiplication and Division
by C. Fosnot and M. Dolk

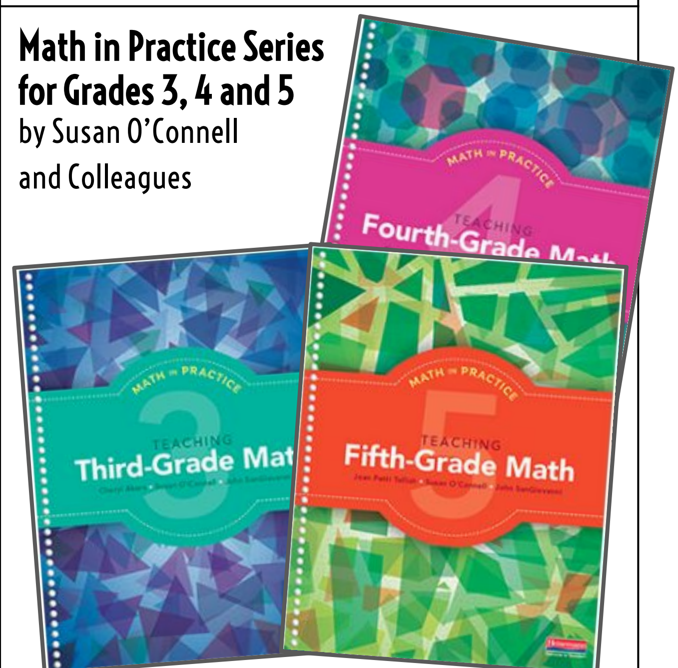


Error Patterns in Computation
by Robert Ashlock



Cognition-Based Assessment & Teaching of Multiplication and Division: Building on Students' Reasoning
by Michael Battista

Math in Practice Series for Grades 3, 4 and 5
by Susan O'Connell
and Colleagues



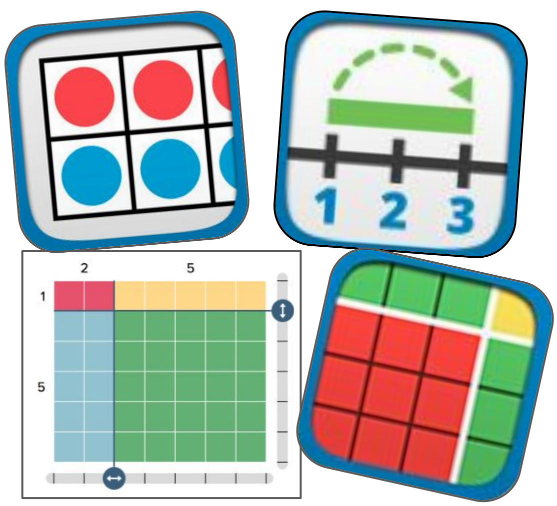
RECOMMENDED FROM THE WEB

The items below are either tools/virtual manipulatives or resources for educators. The tools can be used with students and some have apps that you can be download. The resources can be used to help educators deepen their understanding of the concept of multiplication. These items are FREE!

FREE Math Apps

by Math Learning Center

www.mathlearningcenter.org/resources/apps



GFletchy: Questioning My Metacognition

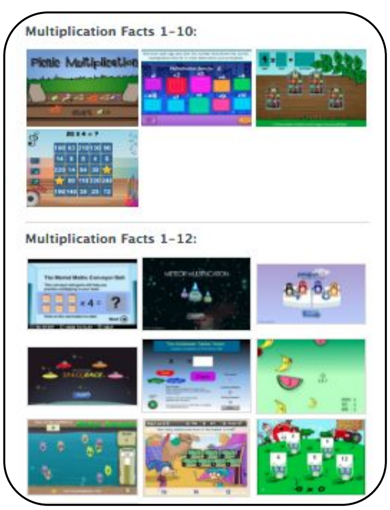
by Graham Fletcher

gfletchy.com/progression-videos

Interactive Sites for Education

By Karen Ogen

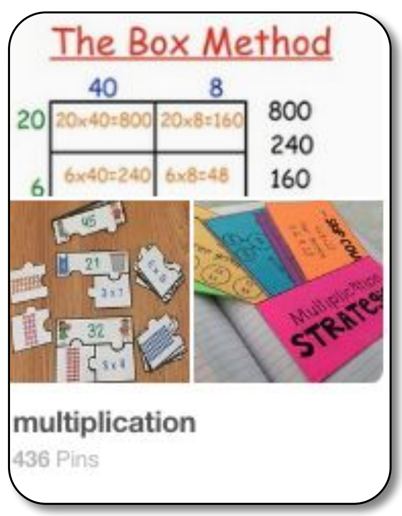
interactivesites.weebly.com/multiplication.html



Dr. Nicki's Pintrest Board

by Dr. Nicki Newton

www.pinterest.com/drnicki7



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