

OXFORD IB DIPLOMA PROGRAMME



MATHEMATICS:

APPLICATIONS AND INTERPRETATION

STANDARD LEVEL
COURSE COMPANION



ENHANCED ONLINE

Jennifer Chang Wathall
Suzanne Doering
Jane Forrest
David Harris
Nadia Stoyanova Kennedy
Paula Waldman

OXFORD

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Paul Belcher
Jennifer Chang Wathall
Suzanne Doering
Phil Duxbury
Panayiotis Economopoulos
Jane Forrest

Peter Gray
Tony Halsey
David Harris
Lorraine Heinrichs
Ed Kemp
Paul La Rondie

Palmira Mariz Seiler
Michael Ortman
Nuriye Sirinoglu Singh
Nadia Stoyanova Kennedy
Paula Waldman

OXFORD

UNIVERSITY PRESS

Great Clarendon Street, Oxford, OX2 6DP, United Kingdom

Oxford University Press is a department of the University of Oxford. It furthers the University's objective of excellence in research, scholarship, and education by publishing worldwide. Oxford is a registered trade mark of Oxford University Press in the UK and in certain other countries

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First published in 2019

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British Library Cataloguing in Publication Data

Data available

978-0-19-842699-8

1 3 5 7 9 10 8 6 4 2

Paper used in the production of this book is a natural, recyclable product made from wood grown in sustainable forests. The manufacturing process conforms to the environmental regulations of the country of origin.

Printed in Italy by L.E.G.O. S.p.A. Lavis (TN)

Acknowledgements

The author would like to thank the following authors for contributions to digital resources:

Alexander Bradley	David Harris
Ingrid Delange	Georgios Ioannadis
Ben Donaldson	Vilda Markeviciute
Tom Edinburgh	Martin Noon
Jim Fensom	Nadia Stoyanova Kennedy
Jane Forrest	Paula Waldman

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Course Companion definition

The IB Diploma Programme Course Companions are designed to support students throughout their two-year Diploma Programme. They will help students gain an understanding of what is expected from their subject studies while presenting content in a way that illustrates the purpose and aims of the IB. They reflect the philosophy and approach of the IB and encourage a deep understanding of each subject by making connections to wider issues and providing opportunities for critical thinking.

The books mirror the IB philosophy of viewing the curriculum in terms of a whole-course approach and include support for international mindedness, the IB learner profile and the IB Diploma Programme core requirements, theory of knowledge, the extended essay and creativity, activity, service [CAS].

IB mission statement

The International Baccalaureate aims to develop inquiring, knowledgeable and caring young people who help to create a better and more peaceful world through intercultural understanding and respect.

To this end the IB works with schools, governments and international organisations to develop challenging programmes of international education and rigorous assessment.

These programmes encourage students across the world to become active, compassionate, and lifelong learners who understand that other people, with their differences, can also be right.

The IB learner profile

The aim of all IB programmes is to develop internationally minded people who, recognising their common humanity and shared guardianship of the planet, help to create a better and more peaceful world. IB learners strive to be:

Inquirers They develop their natural curiosity. They acquire the skills necessary to conduct inquiry and research and show independence in learning. They actively enjoy learning and this love of learning will be sustained throughout their lives.

Knowledgeable They explore concepts, ideas, and issues that have local and global significance. In so

doing, they acquire in-depth knowledge and develop understanding across a broad and balanced range of disciplines.

Thinkers They exercise initiative in applying thinking skills critically and creatively to recognise and approach complex problems, and make reasoned, ethical decisions.

Communicators They understand and express ideas and information confidently and creatively in more than one language and in a variety of modes of communication. They work effectively and willingly in collaboration with others.

Principled They act with integrity and honesty, with a strong sense of fairness, justice, and respect for the dignity of the individual, groups, and communities. They take responsibility for their own actions and the consequences that accompany them.

Open-minded They understand and appreciate their own cultures and personal histories, and are open to the perspectives, values, and traditions of other individuals and communities. They are accustomed to seeking and evaluating a range of points of view, and are willing to grow from the experience.

Caring They show empathy, compassion, and respect towards the needs and feelings of others. They have a personal commitment to service, and act to make a positive difference to the lives of others and to the environment.

Risk-takers They approach unfamiliar situations and uncertainty with courage and forethought, and have the independence of spirit to explore new roles, ideas, and strategies. They are brave and articulate in defending their beliefs.

Balanced They understand the importance of intellectual, physical, and emotional balance to achieve personal well-being for themselves and others.

Reflective They give thoughtful consideration to their own learning and experience. They are able to assess and understand their strengths and limitations in order to support their learning and professional development.

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Digital contents



Digital content overview

Click on this icon here to see a list of all the digital resources in your enhanced online course book. To learn more about the different digital resource types included in each of the chapters and how to get the most out of your enhanced online course book, go to page ix.



Syllabus coverage

This book covers all the content of the Mathematics: applications and interpretation SL course. Click on this icon here for a document showing you the syllabus statements covered in each chapter.



Practice exam papers

Click on this icon here for an additional set of practice exam papers.



Worked solutions

Click on this icon here for worked solutions for all the question in the book

Introduction

The new IB diploma mathematics courses have been designed to support the evolution in mathematics pedagogy and encourage teachers to develop students' conceptual understanding using the content and skills of mathematics, in order to promote deep learning. The new syllabus provides suggestions of conceptual understandings for teachers to use when designing unit plans and overall, the goal is to foster more depth, as opposed to breadth, of understanding of mathematics.

What is teaching for conceptual understanding in mathematics?

Traditional mathematics learning has often focused on rote memorization of facts and algorithms, with little attention paid to understanding the underlying concepts in mathematics. As a consequence, many learners have not been exposed to the beauty and creativity of mathematics which, inherently, is a network of interconnected conceptual relationships.

Teaching for conceptual understanding is a framework for learning mathematics that frames the factual content and skills; lower order thinking, with disciplinary and non-disciplinary concepts and statements of conceptual understanding promoting higher order thinking. Concepts represent powerful, organizing ideas that are not locked in a particular place, time or situation. In this model, the development of intellect is achieved by creating a synergy between the factual, lower levels of thinking and the conceptual higher levels of thinking. Facts and skills are used as a foundation to build deep conceptual understanding through inquiry.

The IB Approaches to Teaching and Learning [ATLs] include teaching focused on conceptual understanding and using inquiry-based approaches. These books provide a structured inquiry-based approach in which learners can develop an understanding of the purpose of what they are learning by asking the questions: why or how? Due to this sense of purpose, which is always situated within a context, research shows that learners are more motivated and supported to construct their

own conceptual understandings and develop higher levels of thinking as they relate facts, skills and topics.

The DP mathematics courses identify twelve possible fundamental concepts which relate to the five mathematical topic areas, and that teachers can use to develop connections across the mathematics and wider curriculum:

Approximation	Modelling	Representation
Change	Patterns	Space
Equivalence	Quantity	Systems
Generalization	Relationships	Validity

Each chapter explores two of these concepts, which are reflected in the chapter titles and also listed at the start of the chapter.

The DP syllabus states the essential understandings for each topic, and suggests some concept specific conceptual understandings relevant to the topic content. For this series of books, we have identified important topical understandings that link to these and underpin the syllabus, and created investigations that enable students to develop this understanding. These investigations, which are a key element of every chapter, include factual and conceptual questions to prompt students to develop and articulate these topical conceptual understandings for themselves.

A tenet of teaching for conceptual understanding in mathematics is that the teacher does not **tell** the student what the topical understandings are at any stage of the learning process, but provides investigations that guide students to discover these for themselves. The teacher notes on the ebook provide additional support for teachers new to this approach.

A concept-based mathematics framework gives students opportunities to think more deeply and critically, and develop skills necessary for the 21st century and future success.

Jennifer Chang Wathall

Investigation 1

A Measuring a potato

- 1 Make a list of all the physical properties of a potato. Which of these properties can you measure? How could you measure them? Are there any properties that you cannot measure? How do we determine what we can measure?
- 2 **Factual** What does it mean to measure a property of an object? How do we measure?
- 3 **Factual** Which properties of an object can we measure?
- 4 **Conceptual** Why do we use measurements and how do we use measuring to define properties of an object?

In every chapter, investigations provide inquiry activities and factual and conceptual questions that enable students to construct and communicate their own conceptual understanding in their own words. The key to concept-based teaching and learning, the investigations allow students to develop a deep conceptual understanding. Each investigation has full supporting teacher notes on the enhanced online course book.

Gives students the opportunity to reflect on what they have learned and deepen their understanding.

Reflect Why might extrapolation be risky?

Developing inquiry skills

Apply what you have learned in this section to represent the first opening problem with a tree diagram.

Hence find the probability that a cab is **identified** as yellow.

Apply the formula for conditional probability to find the probability that the cab was yellow **given** that it was identified as yellow.

How does your answer compare to your original subjective judgment?

Every chapter starts with a question that students can begin to think about from the start, and answer more fully as the chapter progresses. The developing inquiry skills boxes prompt them to think of their own inquiry topics and use the mathematics they are learning to investigate them further.

The Towers of Hanoi

The problem

The aim of the Towers of Hanoi problem is to move all the disks from peg A to peg C, following the rules:

1. Move only one disk at a time.
2. A larger disk may not be placed on top of a smaller disk.
3. At all times, never place two disks on the same peg.

For 3 disks, what is the minimum number of moves needed to complete the problem?

Explore the problem

Use an online simulation to explore the Towers of Hanoi problem for 3 disks and 4 disks.

What is the minimum number of moves needed in each case?

Is there a pattern for the number of moves needed as you increase the number of disks? Can you use this pattern to predict the number of moves needed for 5 disks?

Try to find a rule

Assume the minimum number of moves follows an arithmetic sequence. Use the minimum number of moves for 3 and 4 disks to predict the minimum number of moves for 5 disks. Check your prediction using the simulation.

Does the minimum number of moves follow an arithmetic sequence?

Find a formula

Use the simulation to write down the number of moves for $n = 1$ and $n = 2$.

Organise your results in the table below.

Look for a pattern. Is there a formula for the number of moves for n disks?

Try to find a formula

Return to the problem with 3 disks. Consider the first stage of a partial solution to the problem. The largest disk is on peg A but not yet moved.

Consider your previous solution. What is the minimum number of moves needed to move the largest disk to peg C?

When the largest disk is on peg C, how many moves would it take to move the two smaller disks from peg A to peg C?

How many moves would it take to move the two smaller disks from peg C to peg A?

Use your answers to these questions to write a formula for the minimum number of moves needed to complete the problem with n disks. This is an example of a recursive formula. What does this mean?

How many moves would it take to move the two smaller disks from peg A to peg C?

What is the formula for the minimum number of moves?

Try another formula

Use the simulation to explore the problem by finding an explicit formula. How does this differ from the recursive formula? How does it differ from the recursive formula? How does it differ from the recursive formula?

How can you use the explicit formula to find the minimum number of moves for n disks?

How can you use the explicit formula to find the minimum number of moves for n disks?

How can you use the explicit formula to find the minimum number of moves for n disks?

The modelling and investigation activities are open-ended activities that use mathematics in a range of engaging contexts and to develop students' mathematical toolkit and build the skills they need for the IA. They appear at the end of each chapter.

International-mindedness

How do you use the Babylonian method of multiplication?

Try 36×14

TOK

Is it possible to know things about which we can have no experience, such as infinity?

TOK and International-mindedness are integrated into all the chapters.

The chapters in this book have been written to provide logical progression through the content, but you may prefer to use them in a different order, to match your own scheme of work. The Mathematics: applications and interpretation Standard and Higher Level books follow a similar chapter order, to make teaching easier when you have SL and HL students in the same class. Moreover, where possible, SL and HL chapters start with the same inquiry questions, contain similar investigations and share some questions in the chapter reviews and mixed reviews – just as the HL exams will include some of the same questions as the SL paper.

How to use your enhanced online course book

Throughout the book you will find the following icons. By clicking on these in your enhanced online course book you can access the associated activity or document.



Prior learning

Clicking on the icon next to the “Before you start” section in each chapter takes you to one or more worksheets containing short explanations, examples and practice exercises on topics that you should know before starting, or links to other chapters in the book to revise the prior learning you need.



Additional exercises

The icon by the last exercise at the end of each section of a chapter takes you to additional exercises for more practice, with questions at the same difficulty levels as those in the book.



Animated worked examples

This icon leads you to an animated worked example, explaining how the solution is derived step-by-step, while also pointing out common errors and how to avoid them.

Click on the icon on the page to launch the animation. The animated worked example will appear in a second screen.

Example 14

The distance, d metres, that a rock falls varies directly with the square of the time taken, t seconds.

- a If the rock falls 6 metres in 2 seconds, write an equation for d in terms of t .
- b Find the distance the rock has fallen after 5 seconds.



9.3 Example 14

Example 14

The distance, d metres, that a rock falls varies directly with the square of the time taken, t seconds.

• When three people are available the wall takes one hour to build. Find the time it takes to build the wall when four people are available to work on it.

• To ensure that the wall is built on time, how many people must be available to work on it?

• If $d = 6$ and $t = 2$, write an equation for d in terms of t .

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• If $d = 6$ and $t = 2$, write an equation for d in terms of t .

Click here for a transcript of the audio track.

Tips

Remember that it is a constant. Check to make sure your answer is reasonable.

Things to remember and extra tips will appear here.



Graphical display calculator support

Supporting you to make the most of your TI-Nspire CX, TI-84+ C Silver Edition or Casio fx-CG50 graphical display calculator (GDC), this icon takes you to step-by-step instructions for using technology to solve specific examples in the book.

Click on the icon for the menu and then select your GDC model.

Example 9

Find the equation of the following graph. There are two methods you can use to do this on your GDC.



TI-Nspire CX

TI-84+

Casio fx-CG50



Data sets

Access a spreadsheet containing a data set relevant to the text associated with this icon.



Teacher notes

This icon appears at the beginning of each chapter and opens a set of comprehensive teaching notes for the investigations, reflection questions, TOK items, and the modelling and investigation activities in the chapter.

Assessment opportunities

This Mathematics: applications and interpretation enhanced online course book is designed to prepare you for your assessments by giving you a wide range of practice. In addition to the activities you will find in this book, further practice and support are available on the enhanced online course book.



End of chapter tests and mixed review exercises

This icon appears twice in each chapter: first, next to the “Chapter summary” section and then next to the “Chapter review” heading.

Click here for an end-of-chapter summative assessment test, designed to be completed in one hour.

Chapter summary



Chapter review

Click here for a mixed review exercise



Click here for the mixed review, a summative assessment consisting of exercises and exam-style questions, testing the topics you have covered so far.

Each chapter in the printed book ends with a “Chapter review”, a summative assessment of the facts and skills learned in the chapter, including problem-solving and exam-style questions.

Exam-style questions

Exam-style questions

21 P1: a Find the binomial expansion of $\left(1 - \frac{x}{4}\right)^5$ in ascending powers of x .
(3 marks)

The number of darker bars shows the difficulty of the question (one dark bar = easy; three dark bars = difficult).

[Click here for further exam practice](#)



Plenty of exam practice questions, in Paper 1 (P1) or Paper 2 (P2) style. Each question in this section has a mark scheme in the worked solutions document found on the enhanced online course book, which will help you see how marks are awarded.

Exam practice exercises provide exam style questions for Papers 1 and 2 on topics from all the preceding chapters. Click on the icon for the exam practice found at the end of chapters 4, 9 and 13 in this book.

Answers and worked solutions

Answers to the book questions

Concise answer to all the questions in this book can be found on page 608.



Worked solutions

Worked solutions for **all** questions in the book can be accessed by clicking the icon found on the Contents page or the first page of the Answers section.

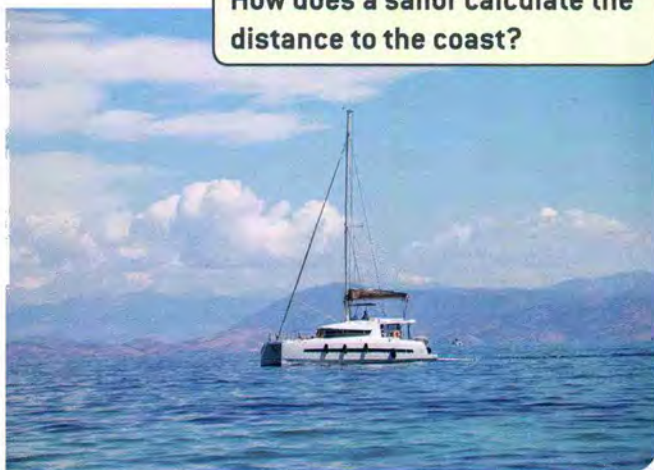
Answers and worked solutions for the digital resources

Answers, worked solutions and mark schemes (where applicable) for the additional exercises, end-of-chapter tests and mixed reviews are included with the questions themselves.

1 Measuring space: accuracy and 2D geometry

Almost everything we do requires some understanding of our surroundings and the distance between objects. But how do we go about measuring the space around us?

How does a sailor calculate the distance to the coast?



How can you calculate the distance travelled by a satellite in orbit?



How do scientists measure the depths of lunar craters by measuring the length of the shadow cast by the edge of the crater?



Concepts

- Quantity
- Space



Microconcepts

- Numbers
- Algebraic expressions
- Measurement
- Units of measure
- Approximation
- Estimation
- Upper/lower bound
- Error/percentage error
- Trigonometric ratios
- Angles of elevation and depression
- Length of arc

How can you measure the distance between two landmarks?



How far can you see? If you stood somewhere higher or lower, how would that affect how much of the Earth you can see?

Think about the following:

- If you stood 10 metres above the ground, what would be the distance between you and the farthest object that you can see?
- One World Trade Center is the tallest building in New York City. If you stand on the Observatory floor, 382 m above the ground, how far can you see?
- How can you make a diagram to represent the distance to the farthest object that you can see? How do you think the distance you can see will change if you move the observation point higher?



Developing inquiry skills

Write down any similar inquiry questions that might be useful if you were asked to find how far you could see from a local landmark, or the top of the tallest building, in your city or country.

What type of questions would you need to ask to decide on the height of a control tower from which you could see the whole of an airfield? Write down any similar situations in which you could investigate how far you can see from a given point, and what to change so you could see more (or less).

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

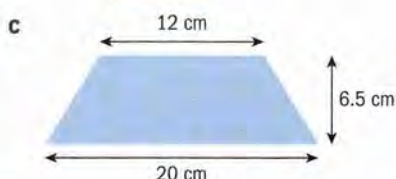
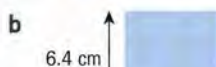
- 1 Find the circumference of a circle with radius 2 cm.
eg $2\pi(2) = 12.6 \text{ cm}$ (12.5663...)
- 2 Find the area of a circle with radius 2 cm.
 $\pi(2)^2 = 12.6 \text{ cm}$ (12.5663...)
- 3 Find the area of:
 - a triangle with side 5 cm and height towards this side 8.2 cm
 $A = \frac{5 \times 8.2}{2} = 20.5 \text{ cm}^2$
 - a square with side 3 cm
 $3^2 = 9 \text{ cm}^2$
 - a trapezoid with bases 10 m and 7 m, and height 4.5 m.
 $\frac{10+7}{2} \times 4.5 = 8.5 \times 4.5 = 38.25 \text{ m}^2$

Skills check

[Click here for help with this skills check](#)



- 1 Find the circumference of a circle with radius $r = 5.3 \text{ cm}$.
- 2 Find the area of a circle with radius 6.5 cm.
- 3 Find the areas of the following shapes.



1.1 Measurements and estimates

Investigation 1

A Measuring a potato

- 1 Make a list of all the physical properties of a potato. Which of these properties can you measure? How could you measure them? Are there any properties that you cannot measure? How do we determine what we can measure?
- 2 **Factual** What does it mean to measure a property of an object? How do we measure?
- 3 **Factual** Which properties of an object can we measure?
- 4 **Conceptual** Why do we use measurements and how do we use measuring to define properties of an object?

B Measuring length

- 5 Estimate the length of the potato.
- 6 Measure the length of the potato. How accurate do you think your measurement is?

C Measuring surface area

Recall that the area that encloses a 3D object is called the surface area.

- 7 Estimate the surface area of the potato and write down your estimate.

Use a piece of aluminium foil to wrap the potato and keep any overlapped areas to a minimum. Once the potato is entirely wrapped without any overlaps, unwrap the foil and place it over grid paper with 1 cm^2 units, trace around it and count the number of units that it covers.

- 8 Record your measurement. How accurate do you think it is?
- 9 Measure your potato again, this time using sheets of grid paper with units of 0.5 cm^2 and 0.25 cm^2 . Again, superimpose the aluminium foil representing the surface area of your potato on each of the grids, trace around it on each sheet of grid paper, and estimate the surface area.
- 10 Compare your three measurements. What can you conclude?
- 11 **Factual** How accurate are your measurements? Could the use of different units affect your measurement?

D Measuring volume

You will measure the volume of a potato, which has an irregular shape, by using a technique that was used by the Ancient Greek mathematician Archimedes, called displacement. The potato is to be submerged in water and you will measure the distance the water level is raised.

- 12 Estimate the volume of the potato.
- 13 What units are you using to measure the volume of water? Can you use this unit to measure the volume of a solid?

Note the height of the water in the beaker before you insert the potato. Slowly and carefully place the potato in the water, and again note the height of the water. Determine the difference in water level.

International-mindedness

Where did numbers come from?



- ➔ 14 Record your measurement for the volume of the potato.
- 15 **Conceptual** If you used a beaker with smaller units, do you think that you would have a different measurement for the volume?
- E Measuring weight**
- 16 Estimate the weight of the potato and write down your estimate.
- 17 Use a balance scale to measure the weight of the potato. Which units will you use?
- 18 **Conceptual** Could the use of different units affect your measurement?
- F Compare results**
- 19 Compare your potato measurements with the measurements of another group. How would you decide which potato is larger? What measures can you use to decide?
- 20 **Conceptual** How do we describe the properties of an object?

Measurements help us compare objects and understand how they relate to each other.

Measuring requires approximating. If a smaller measuring unit is chosen then a more accurate measurement can be obtained.

When you measure, you first select a property of the object that you will measure. Then you choose an appropriate unit of measurement for that property. And finally, you determine the number of units.

Investigation 2

Margaret Hamilton worked for NASA as the lead developer for Apollo flight software. The photo here shows her in 1969, standing next to the books of navigation software code that she and her team produced for the Apollo mission that first sent humans to the Moon.

- 1 Estimate the height of the books of code stacked together, as shown in the image. What assumptions are you making?
- 2 Estimate the number of pages of code for the Apollo mission. How would you go about making this estimate? What assumptions are you making?
- 3 **Factual** What is an estimate? What is estimation? How would you go about estimating? How can comparing measures help you estimate?
- 4 **Conceptual** Why are estimations useful?



Estimation (or estimating) is finding an approximation as close as possible to the value of a measurement by sensible guessing. Often the estimate is used to check whether a calculation makes sense, or to avoid complicated calculations.

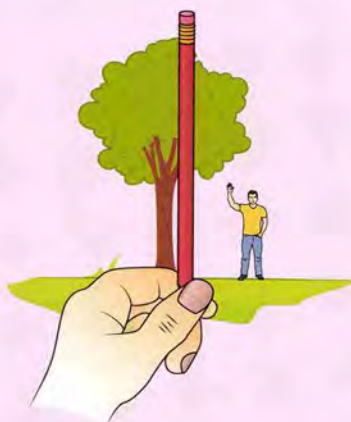
Estimation is often done by comparing the attribute that is measured to another one, or by sampling.

Did you know?

The idea of comparing and estimating goes way back. Some of the early methods of measurement are still in use today, and they require very little equipment!

The logger method

Loggers often estimate tree heights by using simple objects, such as a pencil. An assistant stands at the base of the tree, and the logger moves a distance away from the tree and holds the pencil at arm's length, so that it matches the height of the assistant. The logger can then estimate the height of the tree in "pencil lengths" and multiply the estimate by the assistant's height.



The Native American method

Native Americans had a very unusual way of estimating the height of a tree. They would bend over and look through their legs!



This method is based on a simple reason: for a fit adult, the angle that is formed as they look through their legs is approximately 45 degrees. Can you explain how this method works?

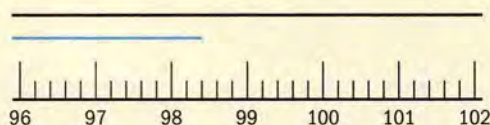


1.2 Recording measurements, significant digits and rounding

Investigation 3

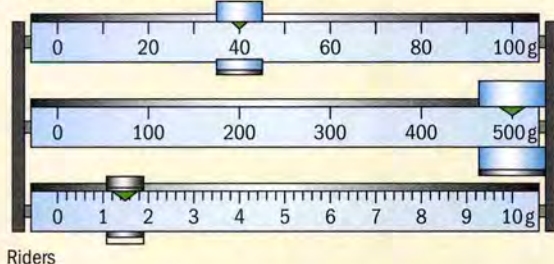
When using measuring instruments, we are able to determine only a certain number of digits accurately. In science, when measuring, the **significant figures** in a number are considered only those figures (digits) that are definitely known, plus one estimated figure (digit). This is summarized as “all of the digits that are known for certain, plus one that is a best estimate”.

- 1 Read the temperature in degrees Fahrenheit from this scale.



What is the best reading of the temperature that you can do? How many digits are significant in your reading of this temperature?

- 2 A pack of coffee is placed on a triple-beam balance scale and weighed. The image below shows its weight, in grams.



Find the weight of the pack of coffee by carefully determining the reading of each of the three beam scales and adding these readings. How many digits are significant in your reading of this weight?

- 3 **Factual** What is the smallest unit to which the weight of the pack of coffee can be read on this scale?
- 4 A laboratory technician compares two samples that were measured as 95.270 grams and 23.63 grams. What is the number of significant figures for each measurement? Is 95.270 grams the same as 95.27 grams? If not, how are the two measurements different?
- 5 **Conceptual** What do the significant figures tell you about the values read from the instrument? What do the significant figures in a measurement tell you about the accuracy of the measuring instrument?
- 6 **Conceptual** How do the reading of the measuring instrument and the measuring units limit the accuracy of the measurement?

TOK

What might be the ethical implications of rounding numbers?

Decimal places

You may recall that in order to avoid long strings of digits it is often useful to give an answer to a number of decimal places (dp). For example, when giving a number to 2 decimal places, your answer would have exactly two digits after the decimal point. You round the final digit up if the digit after it is 5 or above, and you round the final digit down if the digit after it is below 5.

Significant figures

Measuring instruments have limitations. No instrument is advanced enough to determine an unlimited number of digits. For example, a scale can measure the mass of an object only up until a certain decimal place. Measuring instruments are able to determine only a certain number of digits precisely.

The digits that can be determined accurately or with some degree of reliability are called significant figures (sf). Thus, a scale that could register mass only up to hundredths of a gram until 99.99 g could only measure up to 4 digits with accuracy (4 significant figures).

Example 1

For each of the following, determine the number of significant figures.

21.35, 1.25, 305, 1009, 0.00300, 0.002

21.35 has 4 sf and 1.25 has 3 sf.	Non-zero digits are always significant.
305 has 3 sf and 1009 has 4 sf.	Any zeros between two significant digits are significant.
In 0.00300 only the last two zeros are significant and the other zeros are not. It has 3 sf.	A final zero or trailing zeros in the decimal part only are significant.
0.002 has only 1 sf, and all zeros to the left of 2 are not sf.	

Rounding rules for significant figures

The rules for rounding to a given number of significant figures are similar to the ones for rounding to the nearest 10, 100, 1000, etc. or to a given number of decimal places.

EXAM HINT

In exams, give your answers as exact or accurate to 3 sf, unless otherwise specified in the problem.



Example 2

Round the following numbers to the required number of significant figures:

- a** 0.1235 to 2 sf **b** 0.2965 to 2 sf
c 415.25 to 3 sf **d** 3050 to 2 sf

a $0.\underline{12}35 = 0.12$ (2 sf)

b $0.\underline{29}65 = 0.30$ (2 sf)

c $\underline{415}.25 = 415$ (3 sf)

d $\underline{30}50 = 3100$ (2 sf)

Underline the 2 significant figures. The next digit is less than 5, so delete it and the digits to the right.

The next digit is greater than 5 so round up. Write the 0 after the 3, to give 2 sf.

Do not write 415.0, as you only need to give 3 sf.

Write the zeros to keep the place value.

Rounding rule for n significant figures

If the $(n + 1)$ th figure is less than 5, keep the n th figure as it is and remove all figures following it.

If the $(n + 1)$ th figure is 5 or higher, add 1 to the n th figure and remove all figures following it.

In either case, all figures after the n th one should be removed if they are to the right of the decimal point and replaced by zeros if they are to the left of the decimal point.

Exercise 1A

- Round the following measurements to 3 significant figures.
 - 9.478 m
 - 5.322 g
 - 1.8055 cm
 - 6.999 in
 - 4578 km
 - 13 178 kg
- Round the numbers in question 1 parts **a** to **d** to 2 dp.
- Determine the number of significant figures in the following measurements:
 - 0.102 m
 - 1.002 dm
 - 105 kg
 - 0.001020 km
 - 1 000 000 μg
- Find the value of the expression $\frac{12.35 + 21.14 + 1.075}{\sqrt{3.5} - 1}$ and give your answer to 3 significant figures.

Example 3

A circle has radius 12.4 cm. Calculate:

- the circumference of the circle
- the area of the circle.

Write down your answers correct to 1 dp.



Continued on next page



a $C = 2 \times \pi \times 12.4$
 $= 77.9 \text{ cm (1 dp)}$

Use the formula for circumference of a circle,
 $C = 2\pi r$.

The answer should be given correct to 1 dp,
 so you have to round to the nearest tenth.

b $A = \pi(12.4)^2$
 $= 483.1 \text{ cm}^2 \text{ (1 dp)}$

Use the formula for area of a circle, $A = \pi r^2$.
 The answer should be given correct to 1 dp,
 so you have to round to the nearest tenth.

Investigation 4

The numbers of visitors to the 10 most popular national parks in the United States in 2016 are shown in the table.

10 Most Visited National Parks (2016)	
Park	Recreational Visits
1. Great Smoky Mountains NP	11 312 786
2. Grand Canyon NP	5 969 811
3. Yosemite NP	5 028 868
4. Rocky Mountain NP	4 517 585
5. Zion NP	4 295 127
6. Yellowstone NP	4 257 177
7. Olympic NP	3 390 221
8. Acadia NP	3 303 393
9. Grand Teton NP	3 270 076
10. Glacier NP	2 946 681

- Which park had the most visitors? How accurate are these figures likely to be?
- Round the numbers of visitors given in the table to the nearest 10 000.
- Are there parks with an equal number of visitors, when given correct to 10 000? If so, which are they?
- Round the number of visitors, given in the table, to the nearest 100 000.
- Are there parks with an equal number of visitors, when given correct to 100 000? If so, which are they?
- Round the numbers of visitors given in the table to the nearest 1 000 000.
- Are there parks with an equal number of visitors, when given correct to 1 000 000? If so, which are they?
- Determine how many times the number of visitors of the most visited park is bigger than the number of visitors of the least visited park. Which parks are they?
- Determine how many times the number of visitors of the most visited park is bigger than the number of visitors of the second most visited park.





- 10 **Factual** Determine how many times the number of visitors of the most visited park is bigger than the number of visitors of the third most visited park.
- 11 **Conceptual** How did rounding help you compare the numbers of park visitors?
- 12 **Conceptual** What are the limitations of a measurement reading in terms of accuracy?
- 13 **Conceptual** How is rounding useful?

Exercise 1B



- Round each of the following numbers as stated:
 - 8888 to 3 sf
 - 3.749 to 3 sf
 - 27 318 to 1 sf
 - 0.00637 to 2 sf
 - $\sqrt{62}$ to 1 dp
- Round the numbers in the table to the given accuracy.
- Round the following amounts to the given accuracy:
 - 502.13 EUR to the nearest EUR
 - 1002.50 USD to the nearest thousand USD
 - 12 BGN to the nearest 10 BGN
 - 1351.368 JPY to the nearest 100 JPY
- A circle has radius 33 cm. Calculate the circumference of the circle. Write down your answer correct to 3 sf.
- The area of a circle is 20 cm^2 correct to 2 sf. Calculate the radius of the circle correct to 2 sf.
- Estimate the volume of a cube with side 4.82 cm. Write down your answer correct to 2 sf.

	Number	Round to the nearest ten	Round to the nearest hundred	Round to the nearest thousand
a	2815			
b	75391			
c	316479			
d	932			
e	8253			

1.3 Measurements: exact or approximate?

Accuracy

The accuracy of a measurement often depends on the measuring units used. The smaller the measuring unit used, the greater the accuracy. If I use a balance scale that measures only to the nearest gram to weigh my silver earrings, I will get 11 g. But if I use an electronic scale that measures to the nearest hundredth of a gram, then I get 11.23 g.

TOK

To what extent do instinct and reason create knowledge?

Do different geometries (Euclidean and non-Euclidean) refer to or describe different worlds? Is a triangle always made up of straight lines? Is the angle sum of a triangle always 180° ?

The accuracy would also depend on the precision of the measuring instrument. If I measured the weight of my earrings three times, the electronic scale might produce three different results: 11.23 g, 11.30 g and 11.28 g. Usually, the average of the available measures is considered to be the best measurement, but it is certainly not exact.

Each measuring device (metric ruler, thermometer, theodolite, protractor, etc.) has a different **degree of accuracy**, which can be determined by finding the smallest division on the instrument. Measuring the dimensions of a rug with half a centimetre accuracy could be acceptable, but a surgical incision with such precision most likely will not be good enough!

A value is **accurate** if it is close to the **exact value** of the quantity being measured. However, in most cases it is not possible to obtain the exact value of a measurement. For example, when measuring your weight, you can get a more accurate measurement if you use a balance scale that measures to a greater number of decimal places.

Investigation 5

The **yard** and the **foot** are units of length in both the British Imperial and US customary systems of measurement. Metal yard and foot sticks were the original physical standards from which other units of length were derived.

In the 19th and 20th centuries, differences in the prototype yards and feet were detected through improved technology, and as a result, in 1959, the lengths of a yard and a foot were defined in terms of the metre.

In an experiment conducted in class, several groups of students worked on measuring a standard yardstick and a foot-long string.

- 1 Group 1 used a ruler with centimetre and millimetre units and took two measurements: one of a yardstick and one of a foot-long string. Albená, the group note taker, rounded off the two measurements to the nearest centimetre and recorded the results for the yard length as 92 cm and for the foot length as 29 cm. Write down the possible values for the unrounded results that the group obtained. Give all possible unrounded values for each measure as intervals in the form $a \leq x < b$.
- 2 Group 2 used a Vernier caliper, which is able to measure lengths to tenths of a millimetre. They also took measurements of a yardstick and of a piece of string a foot long. Velina, the group note taker, rounded off the two measurements to the nearest millimetre and recorded the results as 91.5 cm and 31.5 cm. Write down the possible values for the unrounded results that the group obtained. Give all possible unrounded values for each measure as intervals in the form $a \leq x < b$.



International-mindedness

SI units

In 1960, the International System of Units, abbreviated SI from the French, “système International”, was adopted as a practical system of units for international use and includes metres for distance, kilograms for mass and seconds for time.



- 3 **Conceptual** Can you explain why the intervals in parts 1 and 2 **include** the endpoint a but **exclude** the endpoint b ?
- 4 **Conceptual** Based on your conclusions in parts 1 and 2, make a conjecture about the interval in which the exact value should lie. How big is this interval in relation to the unit used? How would you determine the left and the right ends of the interval?

The left and the right ends of an interval in which an exact value of a measurement lies are respectively called the **lower bound** and the **upper bound**.

The lower bound and the upper bound are half a unit below and above a rounded value of a measurement. Thus the upper bound is calculated as the rounded measurement $+ 0.5$ unit, and the lower bound is found as the rounded measurement $- 0.5$ unit.

Did you know?

The exact values of one yard and one foot are defined by an international agreement in 1959. A yard was defined as 0.9144 metres exactly, and a foot was defined as 0.3048 metres exactly.

Example 4

- a Jane's weight is 68 kg to the nearest kg. Determine the upper and lower bounds of her weight.
- b Rushdha's height is measured as 155 cm to the nearest cm. Write the interval within which her exact height lies.

a The upper and lower bounds are 68.5 and 67.5, respectively.

b $154.5 \leq h < 155.5$

The range of possible values for Jane's weight is 68 ± 0.5 kg.

The range of possible values for Rushdha's height is 155 ± 0.5 cm.

Example 5

Majid ran 100 metres in 11.3 seconds. His time is measured to the nearest tenth of a second. Determine the upper and lower bounds of Majid's running time.

Lower bound = $11.3 - 0.05 = 11.25$

Upper bound = $11.3 + 0.05 = 11.35$

Lower bound is $11.3 - 0.5$ unit, and upper bound is $11.3 + 0.5$ unit.

Majid's time is given to the nearest tenth of a second. A unit is a tenth of a second or 0.1 sec, and 0.5 unit is 0.05 sec.



Example 6

A rectangular garden plot was measured as 172 m by 64 m. Determine the lower and upper bounds of its perimeter.

$$171.5 \leq L < 172.5$$

$$63.5 \leq W < 64.5$$

Then the lower bound of the perimeter is $2 \times (171.5 + 63.5) = 470$, and the upper bound of the perimeter is $2 \times (172.5 + 64.5) = 474$.

$$\text{Thus } 470 \leq P < 474.$$

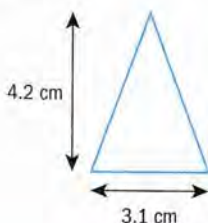
For the lower bound of the perimeter, use the shortest possible lengths of the sides: the measured values $- 0.5$ m; for the upper bound, use the longest possible lengths: the measured values $+ 0.5$ m.

Exercise 1C

- Find the range of possible values for the following measurements, which were rounded to the nearest mm, tenth of m, and hundredth of kg respectively:
 a 24 mm b 3.2 m c 1.75 kg
- A triangle has a base length of 3.1 cm and corresponding height 4.2 cm, correct to 1 decimal place. Calculate the upper and lower bounds for the area of the triangle as accurately as possible.
- With 72 million bicycles, correct to the nearest million, Japan is at the top of the list of countries with most bicycles per person. On average, Japanese people travel about 2 km by bicycle, correct to the nearest km, each day. Calculate the upper bound for the total distance travelled by all the bicycles in Japan.
- To determine whether a business is making enough profit, the following formula is used:

$$P = \frac{S - C}{S}$$

where P is relative profit, S is sales and C is costs. If a company has \$340 000 worth of sales and \$230 000 of costs, correct to 2 significant figures, calculate the maximum and minimum relative profit to the appropriate accuracy.



Since measurements are approximate there is always error in the measurement results. A measurement error is the difference between the exact value (V_E) and the approximate value (V_A), ie

$$\text{Measurement error} = V_A - V_E$$

Investigation 6

Tomi and Massimo measured the length of a yardstick and the length of a foot-long string and obtained 92.44 cm for the length of a yard and 31.48 cm for the length of a foot.



TOK

Do the names that we give things impact how we understand them?



- 1** Given that the exact value of 1 yard is 91.44 cm and of 1 foot is 30.48 cm, find the measurement error in the results obtained by Tomi and Massimo.

Tomi thinks that the two measurements were equally inaccurate. Massimo thinks that one of the two measurements is more accurate than the other.

- 2** Who do you agree with: Tomi or Massimo? Why? Explain.

Massimo decides to find the magnitude of his measurement error as a percentage of the measured length.

- 3** Write down the error in measuring the length of 1 yard as a fraction of the exact length of 1 yard. Give your answer as a percentage.
- 4** Write down the error in measuring the length of 1 foot as a fraction of the exact length of 1 foot. Give your answer as a percentage.
- 5** **Conceptual** In what ways could expressing measurement errors as a percentage of the exact value be helpful?
- 6** **Conceptual** How could measurement errors be compared?

The percentage error formula calculates the error as a percentage of the measured quantity. For example, a weight measured as 102 kg when the exact value is 100 kg gives a measurement error of 2 kg and percentage error of $\frac{2}{100} = 2\%$. A weight measured as 27 kg when the exact value is 25 kg gives the same measurement error of 2 kg but a percentage error of $\frac{2}{25} = 8\%$.

Percentage error = $\left| \frac{V_A - V_E}{V_E} \right| \times 100\%$, where V_A is the approximate value and V_E is the exact value.

Example 7

The fraction $\frac{22}{7}$ is often used as an approximation of π .

- a** How close (to how many decimal places) does $\frac{22}{7}$ approximate π ?
- b** Find the percentage error of this approximation, giving your answer to 2 dp.

a $\frac{22}{7} - \pi = 0.001264\dots$

Thus $\frac{22}{7}$ approximates π to 2 dp.

Measurement error = $V_A - V_E$, where V_A is the approximate value and V_E is the exact value.



Continued on next page



$$\begin{aligned} \text{b Percentage error} &= \left| \frac{\frac{22}{7} - \pi}{\pi} \right| \times 100\% \\ &= 0.04\% \end{aligned}$$

$$\text{Percentage error} = \left| \frac{V_A - V_E}{V_E} \right| \times 100\%$$

Be careful to take the absolute value of the fraction as the percentage error is always a positive number!

In multistep calculations, you must be careful not to round figures until you have your final answer. Premature rounding of initial or intermediate results may lead to an incorrect answer.

TOK

How does the perception of the language being used distort our understanding?

Example 8

Calculate the density of a cube of sugar weighing 2.45 grams, where the side of the cube is 1.2 cm, correct to 1 dp.

$$\text{Volume} = (1.2)^3 = 1.728 \text{ cm}^3$$

$$\begin{aligned} \text{Density} &= \frac{2.45}{1.728} = 1.41782 \dots \text{ g/cm}^3 \\ &= 1.4 \text{ g/cm}^3 \text{ (correct to 1 dp)} \end{aligned}$$

If you first rounded the mass to 1 dp, then calculated the volume to 1 dp and then divided, you would obtain:

$$\begin{aligned} \frac{2.5}{1.7} &= 1.47058 \dots \text{ g/cm}^3, \\ &= 1.5 \text{ g/cm}^3 \text{ (correct to 1 dp)} \end{aligned}$$

This gives a percentage error of

$$\left| \frac{1.5 - 1.41782 \dots}{1.41782 \dots} \right| \times 100\% = 5.80\%$$

Make sure to avoid premature rounding!

Rounding of intermediate results during multistep calculations reduces the accuracy of the final answers. Thus, make sure to round your final answer only.

Exercise 10

- Find the percentage error in using 3.14 instead of π .
- In 1856, Andrew Waugh announced Mount Everest to be 8840 m high, after several years of calculations based on observations made by the Great Trigonometric Survey. More recent surveys confirm the height as 8848 m. Calculate the percentage error made in the earlier survey.
- Eratosthenes estimated the circumference of the Earth as 250 000 stadia (the length of an athletic stadium). If we assume he used the most common length of stadia of his time, his estimate of the circumference of the Earth would be 46 620 km. Currently, the accepted average circumference of the Earth is 40 030.2 km. Find the percentage error of Eratosthenes' estimate of the circumference of the Earth.



- 4 The temperature today in Chicago is 50°F . Instead of using the standard conversion formula $^{\circ}\text{C} = \frac{5}{9} \times (^{\circ}\text{F} - 32)$, Tommaso uses his grandmother's rule, which is easier but gives an approximate value: "Subtract 32 from the value in $^{\circ}\text{F}$ and multiply the result by 0.5."
- a Calculate the actual and an approximate temperature in $^{\circ}\text{C}$, using the standard formula and Tommaso's grandmother's rule.
- b Calculate the percentage error of the approximate temperature value in $^{\circ}\text{C}$.

1.4 Speaking scientifically

Investigation 7

- 1 a Complete the table, following the pattern:

Number							
Written as powers of 10	10^3	10^2	10^1	10^0		10^{-2}	10^{-3}
Written as decimals	1000						
Written as fractions	$\frac{1000}{1}$				$\frac{1}{10}$		

- b When you move from left to right, from one column to the next, which operation would you use?
- c How would you write 10^{-4} as a decimal and as a fraction?
- d Write 10^{-n} as a fraction.
- 2 Complete the table, following the pattern:

Number							
Written as powers of 2	2^3	2^2	2^1	2^0		2^{-2}	2^{-3}
Written as decimals	8						
Written as fractions	$\frac{8}{1}$		$\frac{2}{1}$		$\frac{1}{2}$		

- a Find 2^0 .
- b Find 10^0 .
- c Find x^0 .
- d How are algebraic expressions similar to and different from a numerical expression?

Numerical expressions consist only of numbers and symbols of operations (addition, subtraction, multiplication, division and exponentiation), whereas **algebraic expressions** contain numbers, variables and symbols of operations.

Exponentiation is a mathematical operation, written as a^n , where a is called the base and n the exponent or power.

If the exponent, n , is a **positive integer**, it determines **how many times** the base, a , is multiplied by itself. For example, 8^2 means 8×8 .

If the exponent, n , is a **negative integer** it determines how many times to **divide 1** by the base, a . For example, 8^{-2} means $\frac{1}{8^2}$ or $\frac{1}{8 \times 8}$.

The base, a , is the factor in the expression $a \times a \times a \times \dots \times a$ if the exponent, n , is positive and is the factor in the expression $\frac{1}{a \times a \times a \times \dots \times a}$ if the exponent, n , is negative.

International-mindedness

Indian mathematician Brahmagupta is credited with the first writings that included zero and negative numbers in the 7th century.

Investigation 8

- 1 a Complete the first and the third columns of the table. The middle column you can choose to finish or not.

Expression	Expanded expression	Written as power of 10
$10^2 \times 10^3$	$(10 \times 10) \times (10 \times 10 \times 10)$	10^5
$10^4 \times 10^5$		
	$(10 \times 10 \times 10 \times 10) \times (10 \times 10 \times 10 \times 10 \times 10)$	
		10^8
$10^5 \times 10^6$		
$10^1 \times 10^{10}$		
$(10^2)^3$	$(10 \times 10) \times (10 \times 10) \times (10 \times 10)$	

- b Follow the pattern from the table and rewrite $10^m \times 10^n$ as a single power of 10.
- c Rewrite $10^2 \times 10^0$ as a single power of 10.
- d What can you conclude for the value of 10^0 ? Why?
- e Follow the pattern and rewrite $x^m \times x^n$ as a single power.
- f Use powers and multiplication to write three expressions whose value is 10^{11} .
- g Rewrite $(10^2)^3$ as a single power.
- h Follow the pattern and rewrite $(10^m)^n$ as a single power.
- i Follow the pattern from part h, and rewrite $(x^m)^n$ as a single power.
- j Write the expanded expression for $(2 \times 3)^5$. Then rewrite each term of the expanded expression as a product of two single powers.
- k Follow the pattern in part j, and rewrite $(x \times y)^n$ as a product of two single powers.





- 2 a** Complete the first and the third columns of the table. The middle column you can choose to finish or not.

Expression	Expanded expression	Written as power of 10
$\frac{10^3}{10^2}$	$\frac{10 \times 10 \times 10}{10 \times 10}$	10^1
$\frac{10^5}{10^3}$		
	$\frac{10 \times 10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10}$	
		10^4
$\frac{10^6}{10^6}$		
$\frac{10^3}{10^0}$		

- b** Follow the pattern from the table and rewrite $\frac{10^m}{10^n}$ as a single power of 10.
- c** Rewrite $\frac{10^2}{10^0}$ as a single power of 10. What would that be?
- d** Follow the pattern and rewrite $\frac{x^m}{x^n}$ as a single power.
- e** Use powers and division to write three expressions whose value is 10^5 .
- f** Write the expanded expression of $\left(\frac{2}{3}\right)^5$. Rewrite the expanded expression as a quotient of two powers.
- g** Write the expanded expression of $\left(\frac{x}{y}\right)^n$. Rewrite the expanded expression as a quotient of two powers.

Laws of exponents

Law	Example
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = \frac{1}{x}$	$4^{-1} = \frac{1}{4}$
$x^m x^n = x^{m+n}$	$x^2 x^3 = x^{2+3} = x^5$

Continued on next page

International-mindedness

Archimedes discovered and proved the law of exponents,
 $10^x \times 10^y = 10^{x+y}$.

Law	Example
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{x^6}{x^2} = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2 \times 3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3 y^3$
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$
$x^{-n} = \frac{1}{x^n}$	$x^{-3} = \frac{1}{x^3}$

International-mindedness

Abu Kamil Shuja was called al-Hasib al-Misri, meaning “the calculator from Egypt”, and was one of the first mathematicians to introduce symbols for indices in the 9th century.

Example 9

Use the laws of exponents to express each of the following in terms of powers of a single number:

a $\frac{2}{2^3} + (2^2)^3$ b $\frac{4^3 \times 4^{-7}}{4^2}$

a $2^{-2} + 2^6 = 64.25$	Use your GDC, or use $\frac{x^m}{x^n} = x^{m-n}$ for $\frac{2}{2^3}$ and $(x^m)^n = x^{mn}$ for $(2^2)^3$.
b $\frac{4^{-4}}{4^{-2}} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16} = 0.0625$	Use your GDC, or use $x^m x^n = x^{m+n}$ to simplify the numerator and then $\frac{x^m}{x^n} = x^{m-n}$ to simplify the quotient.

Example 10

Write the following expressions using a single power of x :

a $(x^2)^3$ b $\left(\frac{1}{x^2}\right)^4$ c $(x^3)^{-1}$ d $\frac{x^3 \times x^{-1}}{x^5}$

a x^6	Use $(x^m)^n = x^{mn}$.
b $\left(\frac{1}{x^2}\right)^4 = (x^{-2})^4 = x^{-8}$	Use $x^{-n} = \frac{1}{x^n}$ to write $\frac{1}{x^2}$ as x^{-2} , and then apply $(x^m)^n = x^{mn}$.
c $(x^3)^{-1} = \frac{1}{x^3} = x^{-3}$	Use $(x^m)^n = x^{mn}$, or use $x^{-n} = \frac{1}{x^n}$ twice.
d $\frac{x^2}{x^5} = x^{-3}$	Use $\frac{x^m}{x^n} = x^{m-n}$.



Exercise 1E

- 1 Calculate the following numerical expressions, giving your answer as a single power of an integer.

a $2^3 \times 2^3$

b $5^2 \times 5^1$

c $\frac{6^7}{6^5}$

d $\frac{4^2}{5^{-2}}$

e $8^6 \times 8^{-3}$

f $(3^2)^4$

g $\frac{3^{-4}}{3^2 \times 3^9}$

h $\frac{2^7 \times 2^{-4}}{2^3}$

i $5^4 \times 3^4$

j $\frac{20^3}{4^3}$

- 2 Simplify the following algebraic expressions:

a $x^{-2} \times (x^2)^3 \times (x^2 \times x^6)$

b $\frac{x^0 \times (x^2)^3}{x^2 \times x^{-3}}$

Standard form

In standard form (also known as scientific notation), numbers are written in the form $a \times 10^n$, where a is called the **coefficient** or **mantissa**, with $1 \leq a < 10$, and n is an integer.

Scientific notation is certainly economical; a number such as *googol*, which in decimal notation is written as 1 followed by 100 zeros, is written simply as 10^{100} in scientific notation.

With scientific notation, **Avogadro's constant**, which is the number of particles (atoms or molecules) contained in 1 mole of a substance, is written as $6.022140857 \times 10^{23}$. If not for scientific notation it would take 24 digits to write!

You may have noticed that your graphic display calculator gives any results with lots of digits in scientific notation. However, instead of writing the results in the form $a \times 10^n$, it gives them in the form aEn , where n is an integer. For example, 3×10^5 will be given as 3E5, and 3.1×10^{-3} as 3.1E-3.

To convert from decimal to scientific notation on your GDC, change the Mode from Normal to Scientific. Thus, all of your number entries will be immediately converted to scientific calculator notation. To convert to correct scientific notation, you will need to replace E with 10, eg 5E2 should be written as 5×10^2 .

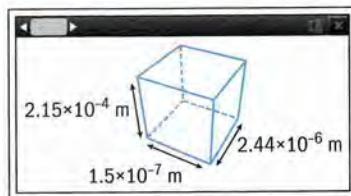
EXAM HINT

Note that in an exam you cannot write 3E5, as this is calculator notation and not the correct scientific notation. The correct notation is 3×10^5 .

Example 11

Find the volume of a computer chip (a cuboid) that is 2.44×10^{-6} m wide, 1.5×10^{-7} m long and 2.15×10^{-4} m high.

The volume of a cuboid = length $\times$ width $\times$ height



Make sure to give your answer in standard form.

Continued on next page

The volume is $7.869 \times 10^{-17} \text{ m}^3$.

To check your answer is sensible, multiply the rounded values $2 \times 2 \times 2 = 8$, and check the powers of 10 come to 10^{-17} .

Exercise 1F

- Find the measurements below in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer:
 - the density of air at 27°C and 1 atm pressure, 0.00161 g/cm^3
 - the radius of a calcium atom, $0.000\,000\,000\,197 \text{ m}$
 - one light-year, $9\,460\,000\,000\,000 \text{ km}$
 - the mass of a neutron, $0.000\,000\,000\,000\,000\,000\,000\,001\,675 \text{ g}$.
- Write down the following numbers found on a calculator display in scientific notation:
 - $1.2\text{E}-1$
 - $5.04\text{E}7$
 - $4.005\text{E}-5$
 - $1\text{E}-3$
- The image of a speck of dust viewed through an electron microscope is 1.2×10^2 millimetres wide. The image is 5×10^2 times as large as the actual size. Determine the width, in millimetres, of the actual speck of dust.
- Convert the following from decimal to scientific notation:
age of the Earth = $4\,600\,000\,000$ years.
 - Convert the following from scientific to decimal notation: 5×10^3 .
- One millilitre has about 15 drops. One drop of water has 1.67×10^{21} H_2O (water) molecules. Estimate the number of molecules in 1 litre of water. Write down your answer in standard form.
- Scientists announced the discovery of a potential "Planet Nine" in our solar system in 2016.
The so-called "Planet Nine" is about 5000 times the mass of Pluto. Pluto's mass is $0.01303 \times 10^{24} \text{ kg}$.
 - Calculate the mass of Planet Nine. Write down your answer in standard form.

- The Earth's mass is $5.97 \times 10^{24} \text{ kg}$. Find how many times Planet Nine is bigger or smaller compared to the Earth. Write down your answer correct to the nearest digit.

TOK

What do we mean when we say that one number is larger than another number?

- The table below shows the population of different regions in 1985 and in 2005.

Place	Population in 1985	Population in 2005
Entire world	4.9×10^9	6.4×10^9
China	1.1×10^9	1.3×10^9
India	7.6×10^8	1.1×10^9
United States	2.4×10^8	3.0×10^8
Bulgaria	8.9×10^6	7.2×10^6

- Determine the ratio of the population of the entire world to that of China in 2005, giving your answer to the nearest integer.
- Find the increase in the world's population between 1985 and 2005.
- Calculate the percentage change in the population of each of the four countries, giving your answers accurate to 3 sf. List the countries according to their percentage change from highest to lowest.
- State whether or not you agree that it is always the case that the country with a bigger percentage change also has a bigger population increase between 1985 and 2005. Justify your answer.



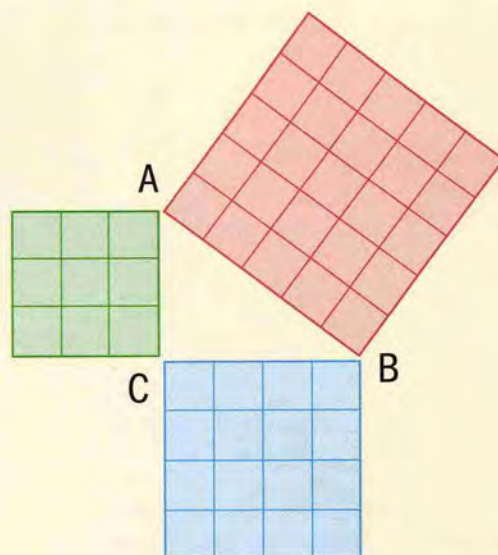
1.5 Trigonometry of right-angled triangles and indirect measurements

Investigation 9

1 Puzzle One:

What do you notice about the three squares built on the sides of the right-angled triangle $\triangle ABC$, where angle $\hat{C} = 90^\circ$?

Can you state the relationship between the sides $[AB]$, $[BC]$ and $[AC]$ of the triangle? If you cannot copy the shapes exactly, then use scissors to cut out copies of the red, blue and green squares, and see whether you can fit the green and blue squares exactly over the red one. You may cut up the blue and green squares along the internal lines to create smaller squares or rectangles.



2 Puzzle Two:

There are 11 puzzle pieces: 8 right-angled triangles and 3 squares of different sizes.

Cut out copies of the puzzle pieces and arrange them in the two frames.

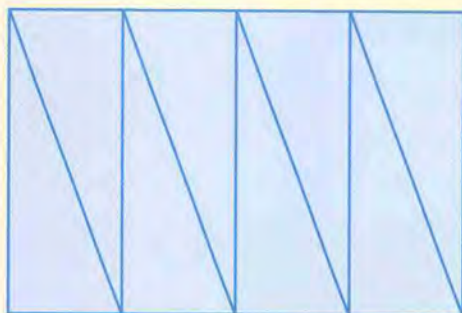
Frames:



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Pieces:



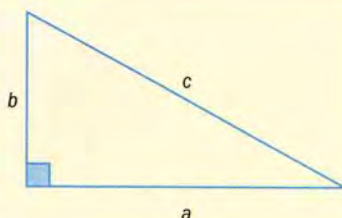
Is there more than one way to arrange the puzzle pieces into the two frames? What is the relationship between the areas of the frames?

What is the relationship between the areas of the three puzzle squares? What relationship between the sides of the right-angled triangles is revealed by the two puzzles?

- 3 **Conceptual** What is the relationship between the sides of any right-angled triangle?

The Pythagorean theorem

For every right-angled triangle $c^2 = a^2 + b^2$, where c is the hypotenuse, and a and b are the other two sides. The hypotenuse is the longest side and is opposite the right angle.



Did you know?

The Pythagorean theorem is named after the Greek mathematician Pythagoras, who lived in the 6th century BC and is credited with the first recorded proof of the theorem.

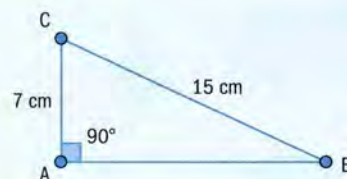
The theorem was known to some cultures well before Pythagoras; the ancient Egyptians knew about the 3-4-5 right-angled triangle (a triangle with sides 3, 4 and 5 units), and they knew how to measure a right angle by stretching a rope with equally spaced knots. The Egyptian surveyors (called "rope stretchers") used knotted cords to make measurements for building and restoring the boundaries of fields after the flooding of the River Nile.



If you know the lengths of two of the sides of a right-angled triangle you can always determine the length of the third side.

Example 12

Calculate the length of [AB] in the right-angled triangle $\triangle ABC$, where $CB = 15$ cm and $AC = 7$ cm.



Using the Pythagorean theorem:

$$AB^2 + 7^2 = 15^2$$

$$AB = \sqrt{15^2 - 7^2}$$

$$= 13.3 \text{ cm (13.2666...)}$$

Rearrange to find AB.

Remember to give the final answer to 3 sf unless a different accuracy is specified in the question.

The converse of the Pythagorean theorem states: If for a triangle $c^2 = a^2 + b^2$, where a , b and c are its sides, then it must be a right-angled triangle.

The converse of the Pythagorean theorem is also useful, as if you know the three sides of a triangle you can determine whether the triangle has a right angle or not.

Example 13

Determine whether or not a triangle with sides 6 cm, 8 cm and 9 cm is a right-angled triangle.

$$\text{Is } 9^2 = 6^2 + 8^2 \text{ true?}$$

Since $81 \neq 100$ we conclude that this triangle does not have a right angle.

To determine whether a triangle with sides 6 cm, 8 cm and 9 cm has a right angle, use the converse of the Pythagorean theorem. Check whether $9^2 = 6^2 + 8^2$.

Example 14

Sam's vegetable garden is a triangle. His plan shows that the plot is a right-angled triangle. Sam measures the sides of his vegetable garden as 17 m, 144 m and 145 m. Show that the plot is indeed a right-angled triangle.

$$\text{Is } 145^2 = 144^2 + 17^2 \text{ true?}$$

Since $21\,025 = 20\,736 + 289$, Sam can conclude that his plot is a right-angled triangle.

To determine whether his plot is a right-angled triangle with sides 17 m, 144 m and 145 m, use the converse of the Pythagorean theorem. Check whether $145^2 = 144^2 + 17^2$.

Did you know?

The existence of irrational numbers is attributed to a Pythagorean, Hippasus [5th century BC]. He proved that $\sqrt{2}$, the length of the hypotenuse of a right-angled triangle with sides 1 and 1, is an irrational number.

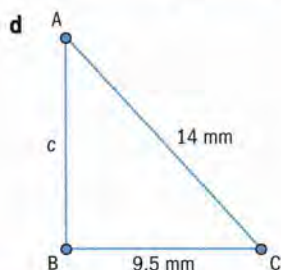
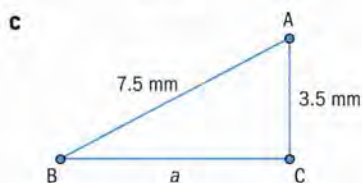
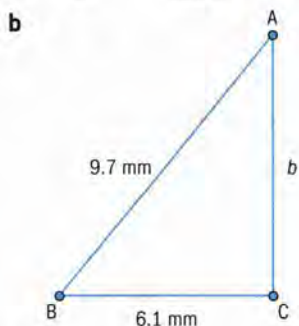
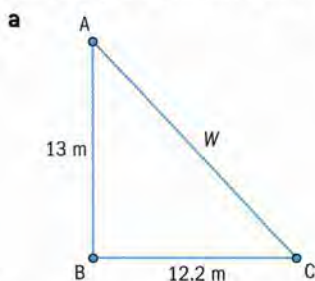
The **irrational numbers** are numbers that are not rational. And **rational numbers** are numbers that can be written as a quotient $\frac{p}{q}$ of two integers p and q , where $q \neq 0$. The legend claims that Hippasus was exiled for his discovery of the irrational numbers.

TOK

Is it ethical that Pythagoras gave his name to a theorem that may not have been his own creation?

Exercise 1G

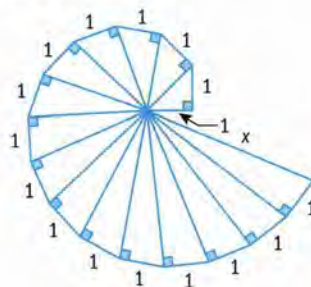
- 1 Find the length of the unknown side in each of the following right-angled triangles. Round your answers to 1 dp.



- 2 Determine whether or not the triangles with the following sides are right-angled triangles:

a 9 cm, 40 cm, 41 cm b 10 m, 24 m, 26 m
c 10, 10, $\sqrt{200}$ d 11.2, 7.5, 8.3

- 3 The spiral in the figure is made by starting with a right-angled triangle with both legs of length 1 unit.



- a Find the hypotenuse of the first triangle. Then, the second right-angled triangle is built with one leg measuring 1 unit and the other leg being the hypotenuse of the first triangle.
- b Find the hypotenuse of this second triangle. A third right-angled triangle is built on the second triangle's hypotenuse, again with the other leg measuring 1 unit.
- c Find the hypotenuse of the third triangle. The process is continued in the same fashion and the hypotenuse of the final triangle is denoted by x .
- d Find the length of x .
- 4 A right-angled triangle has a hypotenuse of 8.2 cm and another side length of 4.3 cm. Draw a diagram. Calculate the area of the triangle. Give your answer correct to 1 dp.



Investigation 10

The right-angled triangles $\triangle ABC$ and $\triangle ADE$ shown in the diagram below are such that $[CB] \perp [AB]$, $[ED] \perp [AD]$ and $\hat{BAC} = 30^\circ$. The lengths of the sides are $AC = 15.0$ cm, $CB = 7.5$ cm, $AB = 13.0$ cm, $AE = 10.0$ cm, $ED = 5.0$ cm and $AD = 8.7$ cm, given to 1 dp.

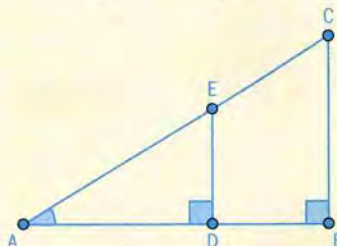


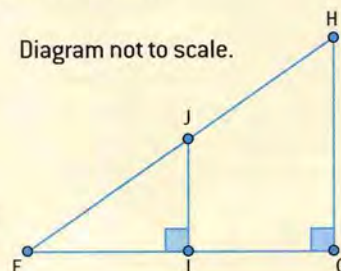
Diagram not to scale.

- Factual** For each side of $\triangle ABC$ and $\triangle ADE$ determine whether it is a hypotenuse, opposite to $\hat{BAC}$ or adjacent to $\hat{BAC}$. On your diagram, label each side of the two triangles with Opp, Adj or Hyp.
- Place each measure of the sides of $\triangle ABC$ and $\triangle ADE$ in the appropriate cell in the table.

	Opp	Adj	Hyp	Opp/Hyp	Adj/Hyp	Opp/Adj
$\triangle ABC$						
$\triangle ADE$						

- Calculate the ratios Opp/Hyp, Adj/Hyp and Opp/Adj for each triangle. Give your answers correct to 1 dp. Place each ratio in the appropriate cell in the table.
- Compare the corresponding ratios for the two triangles. Write down your conclusions.
- The right-angled triangles $\triangle FGH$ and $\triangle FIJ$ shown in the diagram on the right are such that $[HG] \perp [FG]$, $[JI] \perp [FI]$ and $\hat{GFH} = 60^\circ$. The lengths of the sides are $FH = 15.0$ cm, $FG = 7.5$ cm, $FJ = 10.0$ cm and $FI = 5.0$ cm, given to 1 dp.
- Find the lengths of $[HG]$ and $[JI]$.

Diagram not to scale.



- Factual** For each side of $\triangle FGH$ and $\triangle FIJ$, determine whether it is a hypotenuse, opposite to $\hat{GFH}$ or adjacent to $\hat{GFH}$. On your diagram, label each side of the two triangles with Opp, Adj or Hyp.
- Place each measure of the sides of $\triangle FGH$ and $\triangle FIJ$ in the appropriate cell of the table below.
- Calculate the ratios Opp/Hyp, Adj/Hyp and Opp/Adj for each triangle. Give your answers correct to 1 dp. Place each ratio in the appropriate cell in the table.

	Opp	Adj	Hyp	Opp/Hyp	Adj/Hyp	Opp/Adj
$\triangle FGH$						
$\triangle FIJ$						

- Compare the corresponding ratios for $\triangle FGH$ and $\triangle FIJ$. Compare these ratios with the ratios that you calculated for $\triangle ABC$ and $\triangle ADE$. Is the conclusion different from the one you arrived at for triangles $\triangle FGH$ and $\triangle FIJ$? If so, how?
- Conceptual** What can you say about the ratios of corresponding pairs of sides in similar triangles? How does changing the acute angles in the triangle affect the ratios of the sides?

Trigonometric ratios

Trigonometry (from Greek *trigōnon*, “triangle”, and *metron*, “to measure”) studies relationships between sides and angles of triangles. Astronomers in the 3rd century noted for the first time what you saw

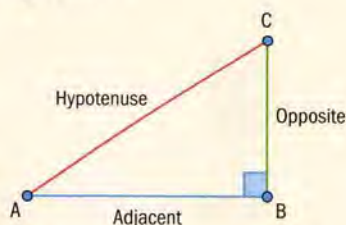
in your investigation, that the ratios $\frac{\text{Adjacent}}{\text{Hypotenuse}}$, $\frac{\text{Opposite}}{\text{Hypotenuse}}$ and

$\frac{\text{Opposite}}{\text{Adjacent}}$ are constant for all right-angled triangles with the same acute angles.

TOK

How certain is the shared knowledge of mathematics?

The ratios of the sides of a right-angled triangle are called **trigonometric ratios**. Three common trigonometric ratios are the **sine (sin)**, **cosine (cos)** and **tangent (tan)**. These are defined for acute angle $\hat{A}$ in the right-angled triangle below:



$$\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos \hat{A} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan \hat{A} = \frac{\text{Opposite}}{\text{Adjacent}}$$

In the definitions above, “Opposite” refers to the length of the side opposite angle $\hat{A}$, “Adjacent” refers to the length of the side adjacent to angle $\hat{A}$, and “Hypotenuse” refers to the side opposite the right angle.

Some people use the mnemonic **SOH-CAH-TOA**, pronounced “soh-kuh-toh-uh”, to help them remember the definitions of sine, cosine and tangent. The table below shows the origins of SOH-CAH-TOA.

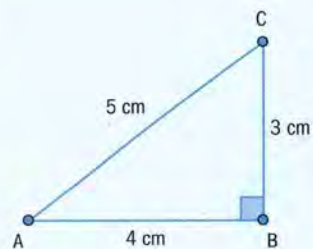
Abbreviation	Verbal description	Definitions
SOH	Sine is Opposite over Hypotenuse	$\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$
CAH	Cosine is Adjacent over Hypotenuse	$\cos \hat{A} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$
TOA	Tangent is Opposite over Adjacent	$\tan \hat{A} = \frac{\text{Opposite}}{\text{Adjacent}}$

Reflect What are trigonometric ratios?



Example 15

Find the size of $\hat{A}$ in triangle $\triangle ABC$, where angle $\hat{A}BC = 90^\circ$.



$$\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin \hat{A} = \frac{3}{5}$$

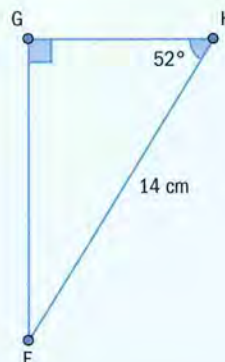
$$\hat{A} = \sin^{-1}\left(\frac{3}{5}\right)$$

$$\hat{A} = 36.9^\circ, \text{ correct to 3 sf.}$$

Use $\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$. Once we know the trigonometric ratio, we can calculate angle $\hat{A}$ by using the $\sin^{-1}$ function on the GDC and finding $\sin^{-1}\left(\frac{3}{5}\right)$. Remember to give your answer to 3 sf unless otherwise specified.

Example 16

Shown here is the right-angled triangle $\triangle FGH$, where $FH = 14$ cm, $\hat{F}HG = 52^\circ$ and angle $\hat{F}GH = 90^\circ$. Find the length of side $[GH]$.



$$\cos \hat{F}HG = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\cos 52^\circ = \frac{GH}{14}$$

$$\begin{aligned} GH &= 14 \cos 52^\circ \\ &= 8.62 \text{ cm (3 sf)} \end{aligned}$$

Since we want to find the length of side $[GH]$ we use the cosine. We use

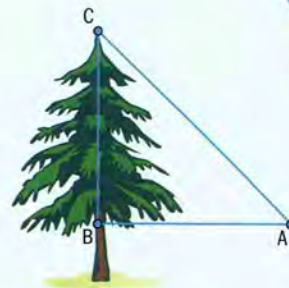
$$\cos \hat{F}HG = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

We substitute the values for $\hat{F}HG$ and the hypotenuse in the cosine ratio and do the calculations.

If at least one side and one non-right angle are known in a right-angled triangle, then all other angles and side lengths can be determined.

Example 17

Emma is standing in front of a big tree. She measures her distance from the tree as $AB = 15$ m. She also measures $\hat{A} = 40^\circ$. Find the length BC . State what other information Emma needs in order to calculate the height of the tree.

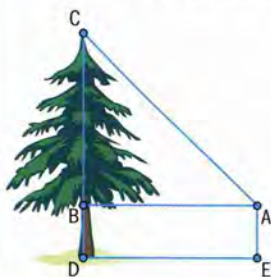


$AB = 15$ m and $\hat{A} = 40^\circ$.

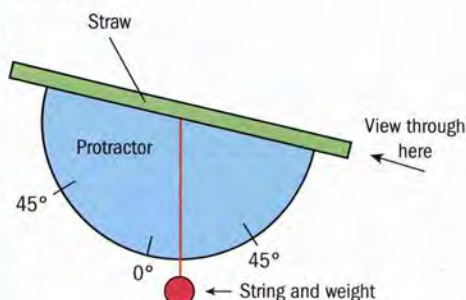
$$\tan 40^\circ = \frac{CB}{15}$$

$$CB = 15 \tan 40^\circ \\ = 12.6 \text{ m (3 sf)}$$

Note that to obtain the height of the tree Emma will have to add to CB her height, which is represented by EA and DB .



$\hat{A}$ can be measured with an instrument called a clinometer, as shown in the image below.



$$\text{Use } \tan \hat{A} = \frac{\text{Opposite}}{\text{Adjacent}}$$

Trigonometric ratios are very helpful in solving practical problems. Distances which are inaccessible can be indirectly measured through use of trigonometry.

Investigation 11

The distance between two landmarks A and B (trees or other) cannot be measured directly as there is a building standing between them. How could you find the distance AB ?

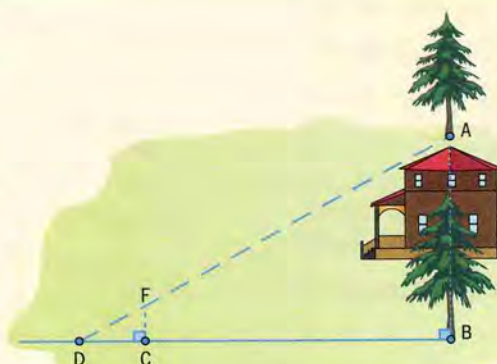
Group work:

- Find two landmarks A and B (trees or other) such that you cannot measure the distance between A and B directly. For example, there may be a building or a tall wall between them and only the top of A is visible from B.





- 2 Mark a line from B perpendicular to [AB] using a piece of string.
- 3 Mark two points C and D on this line. Stand at point D, with your partner standing at point C. Your partner now walks parallel to [AB] (or perpendicular to [BD]) until you decide that your partner is directly between yourself and landmark A. Mark this point F.
- 4 Measure the lengths of [DB], [DC] and [FC].
- 5 Calculate the length of [AB] from $\frac{DB}{DC} = \frac{AB}{FC}$.



Choose two different points for C and D, and repeat steps 3 to 5, this time swapping roles with your partner. Did you get the same result for the distance between A and B when you chose different points for C and D? Think of some sources of error in this experiment that may explain any differences you obtained. How could you check whether the person at C indeed walked perpendicularly to [DB] in step 3?

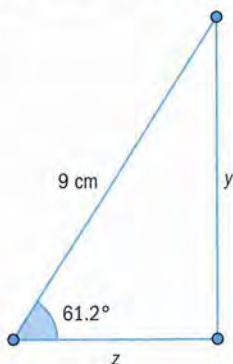
- 6 Now, measure $\hat{ADB}$ and use trigonometric ratios to find the distance AB. Can you find AB by using different trigonometric ratios? If so, calculate AB twice. Did you get the same result for AB? What might be some sources of error in finding AB using trigonometric ratios?
- 7 Compare the values for AB found in parts 5 and 6. Which of the two methods do you think produces a more accurate result for the distance AB? Why? Could you think of other methods to find the distance AB indirectly?
- 8 What kind of real-life questions can be answered by using trigonometric ratios?

Exercise 1H

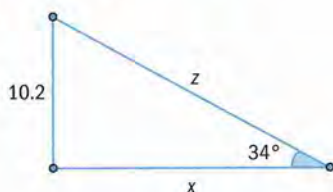


- 1 Determine the length of the unknown sides for each of the right-angled triangles below:

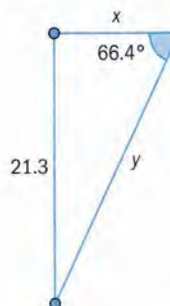
a



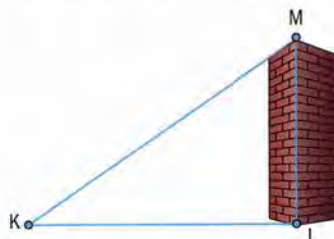
b



c



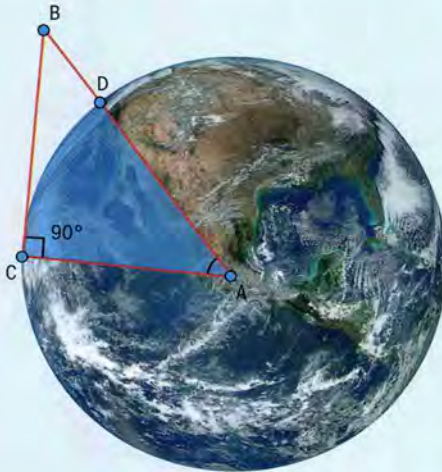
- 2 A ladder [KM] is 8.5 m long. It leans against a wall so that $\hat{LKM} = 30^\circ$ and $\triangle KLM$ is a right-angled triangle.



- a Find the distance KL.
 - b Find how far up the wall the ladder reaches.
 - c The instructions for use of the ladder state that it should not lean against a wall at more than 55° . Find the maximum height up the wall that the ladder can reach.
- 3 A hiker, whose eye is 1.6 metres above ground, stands 50 metres from the base of a vertical cliff. The angle between the line connecting her eye and the top of the cliff and a horizontal line is 58° .
 - a Draw a diagram representing the situation.
 - b Find the height of the cliff.

Developing inquiry skills

Imagine that you are standing at point B, which is the length of [BD] above the ground.



Which line segment in $\triangle ABC$ do you think represents your line of sight to the horizon?

Why do you think $\triangle ABC$ is a right-angled triangle?

Assume that the radius of the Earth is 6370 km.

How can you use what you have learned in this section to calculate the distance to the farthest object that you can see?

- a If $BD = 10$ m, what would be the distance to the farthest object that you can see?
- b If $BD = 382$ m, what would be the distance to the farthest object that you can see?
- c Write an expression that represents the distance to the farthest object you can see, if $BD = h$ km.

What other assumptions are you making?

International-mindedness

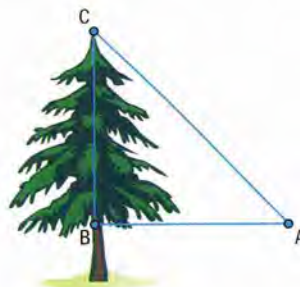
The word “sine” started out as a totally different word and passed through Indian, Arabic and Latin before becoming the word that we use today.



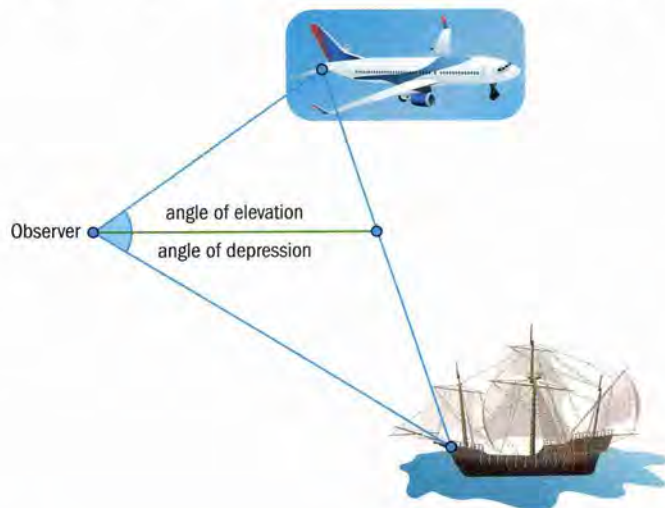
1.6 Angles of elevation and depression

Since ancient times, sailors and surveyors have measured angles to determine measurements that are inaccessible directly, such as the heights of trees, clouds or mountains, and distances to coasts or buildings.

In the problem about measuring the height of a tree from the previous section, angle $B\hat{A}C$ is the angle between the line of sight of the observer to the top of the tree and a horizontal line $[AB]$ from their eye level. This angle is called the **angle of elevation**.



Similarly, when the object is below the horizontal line at eye level, an **angle of depression** is formed, as shown on the diagram below.



The **angle of elevation** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking upwards.

The **angle of depression** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking downwards.

Reflect Why are angles of elevation and depression useful?

Investigation 12

In this investigation you will take measurements of accessible distances and angles, and then use them to calculate inaccessible heights. You will measure angles with a device called a clinometer and then perform the calculations. But first you have to make the clinometer yourself.

1 Making a clinometer

A clinometer is an instrument for measuring angles above or below a horizontal line (angles of elevation or depression). A metal tube allows the user to sight the top (or the base) of an object, and a protractor-like device on the side measures the angle.

To construct a clinometer you will need:

- drinking straw (instead of a metal tube)
- protractor
- string
- washer
- sticky tape
- tape measure.

2 Clinometer assembly

Attach the drinking straw to the top straight edge of the protractor with tape.

Attach the washer to one end of the string, and attach the string to the 0 marking on the top edge of the protractor and let it hang at least 5 cm below the curved edge of the protractor so that it can swing freely.

You are done! Now try it out.

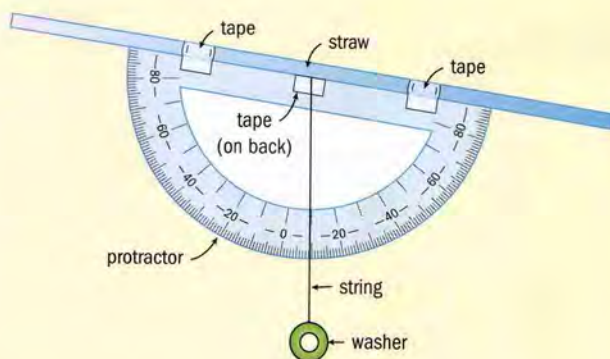
3 Using your clinometer

Start by holding the clinometer so that the straw is parallel to the ground.

Slowly tilt the straw up until you can see the very top of the object that you are measuring through the straw.

Have your partner record the angle measure that the string passes through. Your measure should be between 0° and 90° .

- Choose a tree or other tall object nearby whose height you are going to measure.
- Move away from the object until you can see the top of the object through the straw of your clinometer.
- One person should look through the clinometer at the top of the object you are measuring. The other person reads the angle that the string defines. Record this measurement. Remember that the number will be between 0° and 90° .





- d Next, measure the distance to the object from the point on the ground at which the angle was measured. Record this measurement.

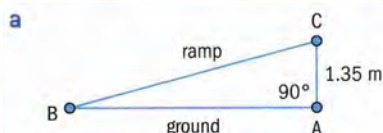
Measurement	Measure of angle of elevation	Measure of distance to the object	Calculated height of the object	Eye-level height	Total height of the object
1					
2					
3					

- e Measure the height from the ground to the eye level of the person measuring the angle.
- f Draw a clear and labelled diagram and write on it the information you have collected.
- g Use the measurements you obtained to do the relevant calculations to find the height.
- h After you have found out how high the object is from eye level, add the person's eye-level height to that number. This should give you the total height of the object.
- i You can do a few sets of measurements and compare your results. What factors do you think could have affected the accuracy of your measurements and final result?
- 4 Suppose you are standing on the third floor of a building and want to find the distance to a building across the street. What information would you need and what measurements would you need to take in order to calculate this distance? Suppose you do have the necessary information and have obtained the measurements. Draw a diagram and explain how you would find the distance. How would you calculate the angle of depression if you know the side measurements?
- 5 **Factual** How are angles of elevation/depression calculated?
- 6 **Conceptual** Why are angles of elevation/depression useful?

Example 18

A ramp, [BC], is to be constructed to give wheelchair access to the door of a house 1.35 m above the ground. Safety regulations require that the angle of the ramp be less than 8° .

- a Draw a diagram and label the vertices and the sides.
- b Find the length of the ramp [BC], where $\hat{A}BC = 8^\circ$.
- c Find the distance AB, giving your answer to 1 dp.



b $\sin \hat{B} = \frac{\text{Opposite}}{\text{Hypotenuse}}$

$$\sin 8^\circ = \frac{1.35}{BC}$$

Use $\sin \hat{B} = \frac{\text{Opposite}}{\text{Hypotenuse}}$ and solve for BC.

Continued on next page



$$BC = \frac{1.35}{\sin 8^\circ}$$

$$BC = 9.70 \text{ m (3 sf)}$$

$$\text{c } \tan \hat{B} = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan 8^\circ = \frac{1.35}{AB}$$

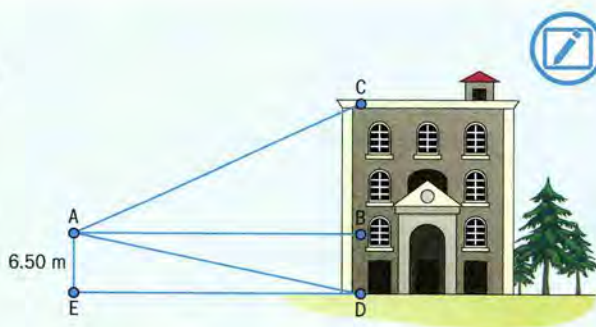
$$AB = \frac{1.35}{\tan 8^\circ}$$

$$\text{so } AB = 9.61 \text{ m (3 sf)}$$

Use $\tan \hat{B} = \frac{\text{Opposite}}{\text{Adjacent}}$ and solve for AB.

Example 19

From his studio window at point A, which is 6.50 m above the ground, Tomchik views the top and the base of a building on the opposite side of the street. The angles of elevation and depression of the top and the base of the building are 55° and 18° respectively. Find the height of the building he is viewing, giving your answer correct to 2 dp.



From $\triangle ABD$,

$$\tan \hat{BAD} = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan 18^\circ = \frac{6.50}{AB}$$

$$AB = 20.00 \text{ m (2 dp)}$$

From $\triangle ABC$,

$$\tan \hat{BAC} = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan 55^\circ = \frac{CB}{20}$$

$$CB = 20 \tan 55^\circ$$

$$CB = 28.56 \text{ m (2 dp)}$$

Then the height of the building is

$$28.56 + 6.50 = 35.06 \text{ m}$$

For $\triangle ABD$, we use $\tan \hat{BAD} = \frac{\text{Opposite}}{\text{Adjacent}}$,
as we know that $BD = AE = 6.50 \text{ m}$ and $\hat{BAD} = 18^\circ$.

For $\triangle ABC$, we use also $\tan \hat{BAC} = \frac{\text{Opposite}}{\text{Adjacent}}$,
as we now know that $AB = 20 \text{ m}$ and $\hat{BAC} = 55^\circ$.

To find the height of the building you need to add the lengths of sides [CB] and [DB].



Exercise 11

- The angle of depression from the top of a cliff to a boat at sea is 17° . The boat is 450 m from the shore.
 - Draw a diagram representing the situation.
 - Find the height of the cliff, giving your answer rounded to the nearest metre.
- Your family wants to buy an awning for a French window that will be long enough to keep out the sun when it is at its highest point in the sky. The height of the French window is 2.80 m. The angle of elevation of the sun from this point is 70° . Find how long the awning should be. Write down your answer correct to 2 dp.
- Scientists measure the depths of lunar craters by measuring the length of the shadow cast by the edge of the crater using photos. In a photograph, the length of the shadow cast by the edge of the Moltke crater is about 606 metres, given to the nearest metre. The sun's angle of elevation is 65° . Find the depth of the crater, giving your answer rounded to the nearest metre.



- The height of a building is 72 m. Find the angle of elevation from a point on ground level that is 55 m away from the base of the building.
- From a boat 160 m out at sea, the angle of elevation to the coast is 18° . The angle of elevation to the top of a lighthouse on the coast is 22° .
 - Draw a diagram representing the situation.
 - Find the height of the lighthouse.

Investigation 13

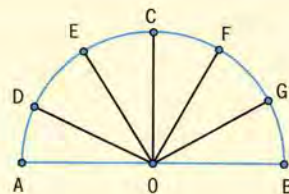
- Complete the following table.

Angle	Section of a circle with this angle	Length of the corresponding arc
360°		$2\pi r$
270°		
180°		
90°		
45°		
1°		
α°		



Continued on next page

- 2 A window, which is a semicircle of radius 12 cm, is divided into six congruent sections, as shown in the diagram.
Find the length of the arc of the entire semicircular window.
- 3 Find the length of the arc of one of the six sections of the window.
- 4 If the window is divided into sections with a central angle of 1° , find the length of the arc of such a window section.
- 5 Find the length of the arc of a window section if its central angle were α° .
- 6 Generate a formula for finding the length of the arc of a circle with central angle α° and radius r , based on your conclusions above.
- 7 **Conceptual** How are the arc length and circumference of a circle related?



Length of arc formula

The length of the arc of a sector of a circle with radius r and central angle

$$\alpha^\circ \text{ is } \alpha \times \frac{2\pi r}{360} \text{ or } \alpha \times \frac{\pi r}{180}.$$

TOK

What does it mean to say that mathematics is an axiomatic system?

Example 20

Find the length of the arc of a circle with radius 2 cm and central angle 120° .

The length of the arc can be found in two ways.

$$\begin{aligned} \text{a } \frac{1}{3} \times 2\pi(2) \\ = 4.19 \text{ cm (3 sf)} \end{aligned}$$

$$\begin{aligned} \text{b } 120 \times \frac{\pi r}{180} &= 120 \times \frac{2\pi}{180} \\ &= 4.19 \text{ cm (3 sf)} \end{aligned}$$

One way would be to use the fact that 120° is $\frac{1}{3}$ of 360° , which means that the length of the arc will be $\frac{1}{3}$ of the circumference of a circle with radius 2 cm, thus $\frac{1}{3} \times 2\pi(2)$ or 4.19 cm (3 sf).

Another way would be to directly use the formula:

$$120 \times \frac{\pi r}{180} \text{ or } 120 \times \frac{2\pi}{180}, \text{ which again results in 4.19 cm (3 sf).}$$

Developing your toolkit

Now do the Modelling and investigation activity on page 46.



Exercise 1J



- 1 Determine the length of each arc with radius r and central angle α given below. Give your answer correct to 2 dp.

- a $r = 5 \text{ cm}$, $\alpha = 70^\circ$
- b $r = 4 \text{ cm}$, $\alpha = 45^\circ$
- c $r = 10.5 \text{ cm}$, $\alpha = 130^\circ$

- 2 A clock is circular in shape with diameter 25 cm. Find the length of the arc between the markings 12 and 5, rounded to the nearest tenth of a centimetre.

- 3 The London Eye is a giant Ferris wheel in London. It is the tallest Ferris wheel in Europe, with a diameter of 120 m. The passenger capsules are attached to the circumference of the wheel, and the wheel rotates at 26 cm per second.

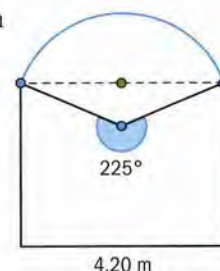


Find:

- a the length that a passenger capsule would travel if the wheel makes a rotation of 200°
- b the time, in minutes, that it would take for a passenger capsule to make a rotation of 200°
- c the time, to the nearest whole minute, that it would take for a passenger capsule to make a full revolution.

- 4 A door with width 4.20 m has an arc as shown in the diagram. Find:

- a the radius of the arc, to the nearest cm
- b the length of the arc, to the nearest cm.



Chapter summary



Measurement

- Measurements help us to compare objects and understand how they relate to each other.
- Measuring requires approximating. If a smaller measuring unit is chosen then a more accurate measurement can be obtained.
- When you measure, you first select a property of the object that you will measure. Then you choose an appropriate unit of measurement for that property. And finally, you determine the number of units.

Estimation

- Estimation (or estimating) is finding an approximation as close as possible to the value of a measurement by sensible guessing. Often the estimate is used to check whether a calculation makes sense or to avoid complicated calculations.
- Estimation is often done by comparing the attribute that we measure to another one or by sampling.

Accuracy

- A value is **accurate** if it is close to the **exact value** of the quantity being measured. However, in most cases it is not possible to obtain the exact value of a measurement. For example, when measuring your weight, you can get a more accurate measurement if you have a balance scale that measures to a greater number of decimal places.



Continued on next page

- The left and the right ends of an interval in which an exact value of a measurement lies are respectively called the **lower bound** and the **upper bound**.
- The lower bound and the upper bound are half a unit below and above a rounded value of a measurement. Thus the upper bound is calculated as the rounded measurement + 0.5 unit and the lower bound is the rounded measurement - 0.5 unit.
- Since measurements are approximate there is always error in the measurement results. A measurement error is the difference between the exact value and the approximate value, ie

$$\text{Measurement error} = V_A - V_E$$

- **Percentage error** = $\left| \frac{V_A - V_E}{V_E} \right| \times 100\%$, where V_A is the approximate value and V_E is the exact value.
- Rounding of intermediate results during multistep calculations reduces the accuracy of the final answers. Thus, make sure to round your final answer only.

Speaking scientifically

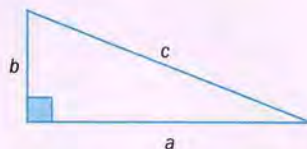
- **Numerical expressions** consist only of numbers and symbols of operations (addition, subtraction, multiplication, division and exponentiation), whereas **algebraic expressions** contain numbers, variables and symbols of operations.
- If the exponent, n , is a **positive integer**, it determines **how many times** the base, a , is multiplied by itself. For example, 8^2 means 8×8 .
- If the exponent, n , is a **negative integer** it determines how many times **to divide** 1 by the base, a . For example, 8^{-2} means $\frac{1}{8^2}$ or $\frac{1}{8 \times 8}$.

Standard form

- In standard form (also known as scientific notation), numbers are written in the form $a \times 10^n$, where a is called the **coefficient** or **mantissa**, with $1 \leq a < 10$, and n is an integer.
- You may have noticed that your graphic display calculator gives any results with lots of digits in scientific notation. However, instead of writing the results in the form $a \times 10^n$, it gives them in the form aEn , where n is an integer. For example, 3×10^5 will be written as 3E5, and 3.1×10^{-3} as 3.1E-3.
- To convert from decimal to scientific notation on your GDC, change the Mode from Normal to Scientific. Thus, all of your number entries will be immediately converted to scientific calculator notation. To convert to correct scientific notation, you will need to replace E with 10, eg 5E2 should be written as 5×10^2 .

Trigonometry of right-angled triangles

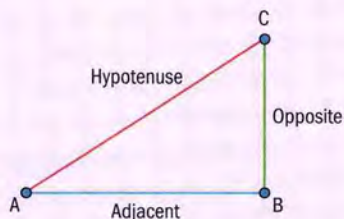
- **Pythagorean theorem:** For every right-angled triangle $c^2 = a^2 + b^2$, where c is the hypotenuse, and a and b are the other two sides. The hypotenuse is the longest side and is opposite the right angle.
- If you know two of the sides of a right-angled triangle you can always determine the third side.
- **The converse of the Pythagorean theorem:** If for a triangle $c^2 = a^2 + b^2$, where a , b and c are its sides, then it must be a right-angled triangle.
- The converse of the Pythagorean theorem is useful, as if you know the three sides of a triangle you can determine whether the triangle has a right angle or not.





Trigonometric ratios

- The ratios of the sides of right-angled triangles are called trigonometric ratios. Three common trigonometric ratios are the sine (sin), cosine (cos) and tangent (tan). These are defined for acute angle $\hat{A}$ in the right-angled triangle below:



$$\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos \hat{A} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan \hat{A} = \frac{\text{Opposite}}{\text{Adjacent}}$$

- If at least one side and one non-right angle are known in a right-angled triangle, then all other angles and side lengths can be determined.
- Trigonometric ratios are very helpful in solving practical problems. Distances that are inaccessible can be indirectly measured through use of trigonometry.

Angles of elevation and depression

- The **angle of elevation** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking upwards.
- The **angle of depression** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking downwards.
- The **length of the arc** of a sector of a circle with radius r and with central angle α° is $\alpha \times \frac{2\pi r}{360}$.

Developing inquiry skills

Does the measure of $\hat{BAC}$ depend on the length of $[BD]$ or how high the observer stands above the ground?

How can you use $\hat{ABC}$ to find the measure of $\hat{BAC}$?

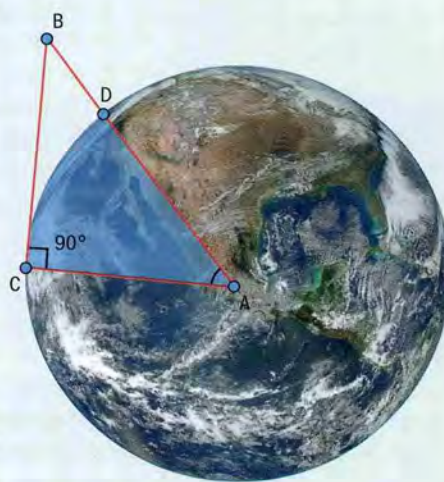
Which is the arc of visibility for an observer at point B?

Does the length of the arc of visibility depend on $\hat{BAC}$?

Assume that the radius of the Earth is 6370 km.

- Calculate the measure of $\hat{BAC}$ if $BD = 10$ m.
- Calculate the measure of $\hat{BAC}$ if $BD = 382$ m.
- Express the measure of $\hat{BAC}$ if $BD = h$ km.
- Calculate the length of the arc of visibility for $BD = 10$ m.
- Calculate the length of the arc of visibility for $BD = 382$ m.
- Calculate the length of the arc of visibility for $BD = h$ km.

What other assumptions are you making?



Chapter review

Click here for a mixed review exercise



- 1 (Group work) Make a list of 10 human activities that are impossible to perform without measuring. Share the list with your peers. Discuss in your small group why measurement matters and how measuring and measurements are used in these activities.

- 2 Find the value of the following algebraic expressions when $a = 1.25$ and $b = 3.365$. Give your answers correct to 3 sf.

a $2a + 2b$

b $\frac{\sqrt{a+b}}{ab}$

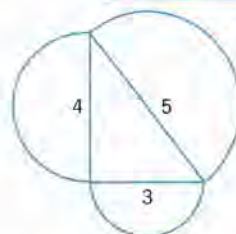
c $\sqrt{a^2 + b^2 - 2ab}$

- 3 Researchers at Ghent University in Belgium have etched a tiny world map (on a scale of 1 to one trillion) onto an optical silicon chip. They reduced the Earth's 40 000 km circumference at the equator down to 40 micrometres, or about half the width of a human hair, to fit it on the chip.

- a Using scientific notation, write down the length of the Earth's circumference as represented on the chip.
- b Find how many times bigger the Earth's circumference is compared to its length on the chip.

- 4 In 2008, the European Union set a carbon emissions target of 130 grams of CO_2 per km for passenger cars. Assuming all cars met this regulation, calculate the upper and lower bounds of the amount of CO_2 in grams that would have been released if people used cars to travel a distance equivalent to the total circumference of the Earth, 40 000 km.

- 5 We used the relationship between the squares built on the sides of a right-angled triangle to discover the Pythagorean theorem. Does the same relationship apply if we draw semicircles on the sides of the right-angled triangle? Find the area of each semicircle shown in the following diagram and draw a conclusion.

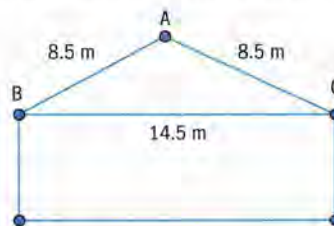


- 6 A rectangular pool measures 7.5 m by 15 m.
- a Find the length of the diagonal, correct to the nearest cm.

Vasil measures the diagonal to be 16 m.

- b Find the percentage error in his measurement. Give your answer to 3 sf.
- 7 Raja is laying tiles on a path that forms the diagonal of a square garden. If Raja is told that the length of the diagonal path is $100\sqrt{2}$ ft, determine the perimeter of the garden, in feet.

- 8 The roof of a house seen from the front is an isosceles triangle $\triangle ABC$ with base 14.5 m and sides of length 8.5 m. Find how high above line [BC] the vertex A is.



- 9 A ramp is placed from a ditch to a main road, which is 0.60 m above the ditch. If the length of the ramp is 3.65 m, determine the distance between the bottom of the ramp and the road.
- 10 A satellite is orbiting the Earth at a height of 800 km. Assume the radius of the Earth to be 6370 km.
- a Draw a diagram representing the situation.
- b Find the distance from the satellite to the Earth's horizon, giving your answer to the nearest km.



- 11** When fire occurs in high-rise buildings, the fire-fighters cannot use the regular stairs or lifts. They can reach higher floors using ladders. If the angle of a ladder to the ground can be at most 50° , determine:

- the length of the ladder that can be extended from the ground and reach a height of 20 m
- the height that a 15 m ladder can reach if extended from a fire truck. The height of the fire truck is 4.15 m.

Calculate your answers to the nearest cm.

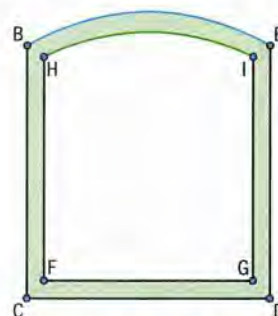
- 12** From the top of a 150 m high-rise building, two cars on the same road below the building are seen at an angle of depression of 45° and 60° on each side of the tower. Find the distance between the two cars, correct to the nearest metre.



- 13** A surveyor wanted to measure the height of a mountain peak. She took a straight road leading to the mountain and measured the angle of elevation from point A on the ground to the top of the mountain D as 33° . She drove 0.5 km further towards the mountain and measured the angle of elevation again as 40° .

- Make a labelled diagram that represents the situation.
- Find the height of the mountain to the nearest metre.

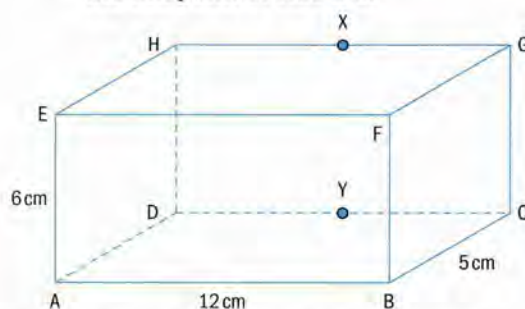
- 14** A customer brings a special window frame, shown in the following diagram, to a glass shop. The wood frame has inside dimensions $FG = 80$ cm and $HF = 136$ cm. Calculate the length of the arc $\widehat{HI}$, if the centre of the circle of the arc is the midpoint of FG .



Exam-style questions

- 15 P1:** The diagram below shows a cuboid of dimensions 5 cm, 6 cm and 12 cm.

X is the midpoint of side HG, and Y is the midpoint of side DC.

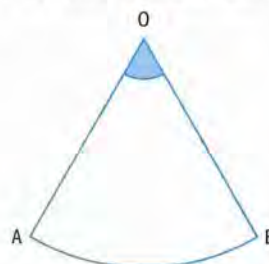


- Find the length AY. (2 marks)
- Find the size of angle $\widehat{XAY}$. (2 marks)

- 16 P1:** A cylinder has radius 1.8 cm and height 14.5 cm.

- Calculate the volume of the cylinder, giving your answer to 3 sf. (2 marks)
- Calculate the total surface area of the cylinder, giving your answer to 3 sf. (2 marks)

- 17 P1:** A pendant is made in the shape of a circular sector OAB, with angle $\widehat{AOB} = 40^\circ$ and radius 7 cm.



Find the perimeter of the pendant.

(4 marks)

- 18 P2:** Simplify the following algebraic expressions.

a $\frac{10x^3 \times 3x^4}{2x^{-6}}$ (3 marks)

b $\frac{x^2 \times 4x^{-3}}{x^{-1}}$ (3 marks)

c $\sqrt{3x^3 \times 12x^0 \times 4x^5}$ (3 marks)

d $\frac{(x^{-2})^5}{(x^3)^{-4}}$ (3 marks)

- 19 P1:** Elea is 1.6 m tall. She stands on the same horizontal level as the base of a tree. The height of the tree is 23.5 m, and it is 100 m away from Elea. Find the angle of elevation of Elea's line of sight to the top of the tree. (4 marks)

- 20 P1:** A mathematics student decides to use an approximation to π of $\frac{355}{113}$. Calculate his percentage error in using this value, giving your answer in standard form. (3 marks)

- 21 P1:** Sharon stands on the top of a cliff 90 m high. The angle of elevation from Sharon to a flying kittiwake is 15° . The angle of depression from Sharon to a yacht on the sea is 19° . Given that the kittiwake is flying directly above the yacht, find the distance between the yacht and the kittiwake. (6 marks)

- 22 P1:** A straight ladder of length 7.1 m rests against a vertical wall. A person climbing the ladder should be "safe" as long as the foot of the ladder makes an angle of between 70° and 80° with the horizontal ground. Determine the minimum and maximum heights that the ladder can safely lie against the wall. (4 marks)

- 23 P1:** The speed of light is approximately $3 \times 10^8 \text{ m s}^{-1}$. The distance from the Earth to the Sun is $1.496 \times 10^8 \text{ km}$.

The distance from the Earth to Proxima Centauri, the nearest star, is $4.014 \times 10^{13} \text{ km}$.

- a** Find the time it takes (in minutes) for light from the Sun to reach the Earth. (3 marks)
- b** Find the time (in years) it takes for light from Proxima Centauri to reach the Earth. Assume that one year is equivalent to 365 days. (3 marks)
- c** Given that light from the Andromeda galaxy takes approximately 2.5 million years to reach us, find how far (in km) Andromeda is from the Earth. (3 marks)
- 24 P1:** Einstein's equation relating mass and energy is given by the formula $E = mc^2$, where m is the mass of an object and $c = 3 \times 10^8 \text{ m s}^{-1}$, the speed of light.
- a** Find how much energy is present in a 1 kg mass. ($1 \text{ kg m s}^{-1} = 1 \text{ joule}$) (2 marks)
- b** Given that a standard light bulb uses 60 joules of energy every second, calculate how long (in years) such a light bulb could be powered by a 1 kg mass, if all the mass was converted to energy. (3 marks)
- 25 P1:** In an electric circuit, the resistance (R ohms) of a component may be calculated using the formula $R = \frac{V}{I}$, where V is the potential difference across the component and I is the current through the component. The potential difference is measured as 6 V to the nearest volt, and the current is measured as 0.2 A to the nearest tenth of an ampere.
- a** Calculate lower and upper bounds for the resistance. (3 marks)
- b** The actual value of the resistor is 30Ω . Calculate the maximum possible percentage error that could be obtained. (2 marks)
- 26 P1:** From the top of a cliff 120 m high, Josie sees two small boats in the distance.



One boat lies at an angle of depression of 39° and the other at an angle of depression of 31° . Given that Josie and both ships lie in the same vertical plane, how far apart are the ships?

(6 marks)

- 27 P2:** The formula for the volume of a sphere is given as $V = \frac{4\pi r^3}{3}$, where r is the radius of the sphere.

- a** If the radius of a particular sphere is 3.5 cm, measured to the nearest tenth of a centimetre, find lower and upper bounds for its volume. (4 marks)
- b** Another sphere is designed to have a volume of 500 cm^3 , to the nearest 10 cm^3 . Find the boundaries between which its radius must lie. (6 marks)

- 28 P1:** The shape of an *E. coli* bacterium may be approximated by a cylinder of diameter $1.3 \times 10^{-6} \text{ m}$ and length $4.5 \times 10^{-6} \text{ m}$.

- a** Find the volume of the bacterium. (2 marks)
- b** For practical purposes, the volume is assumed to be $2 \times 10^{-17} \text{ m}^3$. Find the percentage error when assuming this value. (2 marks)

- 29 P1:** The following table gives the masses (in kg) of the various particles which make up an atom.

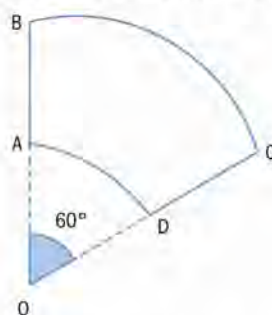
Particle	Mass (kg)
Neutron	1.675×10^{-27}
Proton	1.673×10^{-27}
Electron	9.109×10^{-31}

- a** Find the average (mean) mass of all three particles, giving your answer in standard form to 3 sf. (2 marks)
- b** A helium atom consists of two protons, two neutrons, and two electrons.

Find the mass of a helium atom, giving your answer in standard form to 3 sf. (2 marks)

- c** Determine the ratio of electron mass to neutron mass, giving your answer in the form $1:x$ where x is accurate to 3 sf. (2 marks)
- d** Calculate the percentage error in taking the mass of an electron to be $1 \times 10^{-30} \text{ kg}$. (2 marks)

- 30 P1:** In the diagram shown above, $ABCD$ is a plane figure, where both BC and AD are arc lengths of a circle centred at O . Length $OA = 5 \text{ cm}$ and $AB = 8 \text{ cm}$.



Find the perimeter of the figure $ABCD$. (6 marks)

- 31 P1:** For parallel resistors in an electrical circuit, the total resistance R_{TOT} is given by the formula $\frac{1}{R_{\text{TOT}}} = \frac{1}{R_1} + \frac{1}{R_2}$. $R_1 = 7.2 \Omega$, measured to the nearest tenth of an ohm, and $R_2 = 3.6 \Omega$, also measured to the nearest tenth of an ohm.

- a** Find the lower and upper bounds for the total resistance. (6 marks)
- b** If the actual total resistance is 2.40Ω , find the range of percentage errors that could be obtained for R_{TOT} . (4 marks)

Back of an envelope

Approaches to learning: Research, Thinking skills: Evaluate, Critiquing, Applying

Exploration criteria: Personal engagement (C), Reflection (D), Use of mathematics (E)

IB topic: Approximation, Estimation and Errors



How much water do you use?

How much water could you save?

If you combine simple approximations, such as the ones in this chapter, and elementary arithmetic, you can find reasonably good estimates to answer all sorts of interesting questions.

Did you know?

This type of calculation is often called a “back of the envelope” calculation because it is so simple that you could write it down on the back of an envelope!

How much water do you use?

Discuss:

- Which of your daily activities use water?
- How much water does each use?
- In an average week what does your household use water for? Write down all the uses you can think of.
- How many times does your family do each activity in an average week?

Research how many **litres** of water are used for each activity that you and your family do.



Use the figures collected so far to estimate the average number of litres of water used by your household in an average week.

Comment on your findings so far.

How reliable are these estimates?

What may have caused some errors? Are these errors significant? How could you adjust your figures to account for this?

Now calculate the estimated average amount of water used by households in your class in a **week**.

How does your household's water usage compare to others in your class?

You could also now estimate the average amount of water used by a household in one **year** in **your school**.

What information would you need for this?

Comment on your findings.

Estimate how many households there are in your country.

How might you do this?

Estimate the amount of water used for domestic tasks in your country in **one day, one week** and **one year**.

Check your estimates against official figures.

How accurate were you?

What is your percentage error?

How could you make your results more accurate?

What assumptions have you made in calculating your figures?

How do you think the official figures are calculated?

How much water can you save?

Discuss:

- Which activity uses the most water in your household?
- Is this what you expected?
- What could you do to reduce the amount of water used in your household?

Select one of the things you could do.

Estimate how much water your household could save in **one year** if it did this.

Estimate how much water could be saved in **your country** in **one year** if every household in your country did this.

Extension

Devise your own "back of the envelope" style question.



What other questions could you answer in a similar way?

2 Representing space: non-right angled trigonometry and volumes

This chapter deals with finding measures of angles and distances in right and non-right triangles, and calculating the area of 2D and the volume of 3D geometric shapes. These techniques are used in astronomy to measure the distance to nearby stars, in cartography to measure distances between landmarks and in navigation to determine the courses of ships and aircraft.

How are the exact locations of landmarks established in mapmaking?

Concepts

- Space
- Representation



Microconcepts

- Area of a triangle and other 2D figures
- Area of a sector
- 3D solids: right cylinder; pyramid; right cone; sphere; hemisphere
- An angle between two lines, and between a line and a plane
- Volume
- Surface area of 3D figures



What part of your lawn can a sprinkler water and how does it depend on the angle of the sprinkler's rotation?



How can the distance to a nearby star be measured?

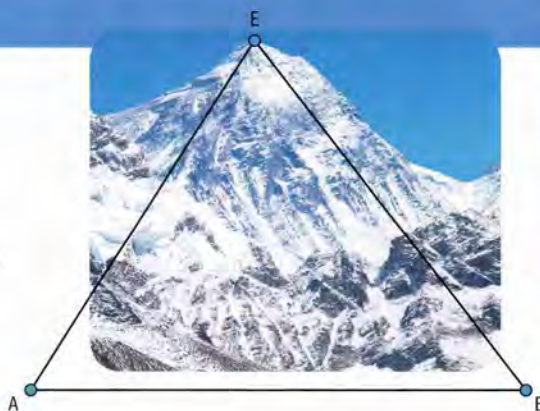


In 1856 The Great Trigonometric Survey of India measured the height of Mount Everest (known in Nepali as Sagarmatha and in Tibetan as Chomolungma). The surveyors measured the distance between two points at sea level and then measured the angle from each of these two points to the top of the mountain.

The summit of Mount Everest is labeled E, the two points A and B are roughly at sea level and are 33 km apart.

If $\hat{E} = 90^\circ$, what would be the maximum possible height of Mount Everest?

- Why do you think it took so long to determine the elevation of Mount Everest?
- Why do you think surveyors prefer to measure angles, but not lengths of sides?
- What assumptions are made?



Developing inquiry skills

Write down any similar inquiry questions you might ask if you were asked to find the height of a tree, the distance between two towns or the distance between two stars.

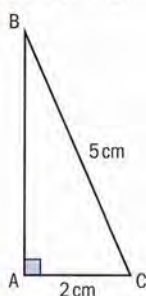
What questions might you need to ask in these scenarios that differ from the scenario where you are estimating the height of Mount Everest?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

- 1 Use the properties of triangles, including the Pythagorean theorem.



eg Find the length AB in this triangle.

$$AB^2 = BC^2 - AC^2$$

$$AB^2 = 5^2 - 2^2$$

$$AB^2 = 21$$

$$AB = \sqrt{21}$$

$$= 4.58 \text{ cm}$$

- 2 Trigonometric ratios in right triangles

eg $\sin \hat{B} = \frac{2}{5}$, $\cos \hat{B} = \frac{\sqrt{21}}{5}$, $\tan \hat{B} = \frac{2}{\sqrt{21}}$

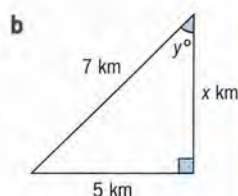
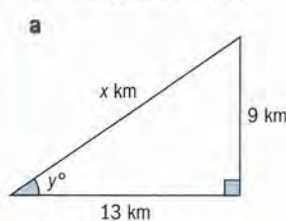
- 3 Find the area of a triangle if a side and the height towards it are given.
- 4 Determine the third angle in a triangle if two angles are known.

Skills check

[Click here for help with this skills check](#)



- 1 In each of these right triangles, find the length of side x.

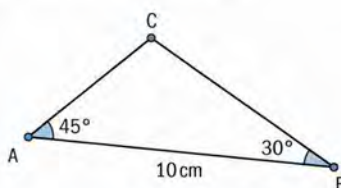


- 2 For the triangles in question 1, find the size of angle y.
- 3 Find the area of a triangle with base 6 cm and height 12.5 cm.
- 4 In $\triangle ABC$, $\hat{B} = 135^\circ$ and $\hat{C} = 25^\circ$. Find $\hat{A}$.

2.1 Trigonometry of non-right triangles

Right triangle trigonometry can be used to solve problems involving right triangles. But many geometry problems involve non-right triangles.

For example, $\triangle ABC$ where $AB = 10$ cm, $\hat{A} = 45^\circ$ and $\hat{B} = 30^\circ$ is one such triangle.

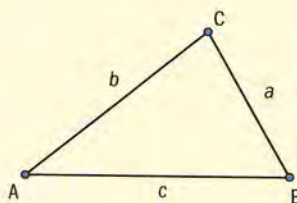


Reflect How can you determine the measures of the unknown angle and sides?

Investigation 1

- 1 A surveyor measures a plot of land in the form of a triangle $\triangle ABC$ as shown in the diagram. She measures $a = 5$ m, $b = 7$ m and $c = 8$ m, and $\hat{A} = 38.2^\circ$, $\hat{B} = 60.0^\circ$ and $\hat{C} = 81.8^\circ$, given to 1 dp.

Place all relevant information at the appropriate places in the diagram above.

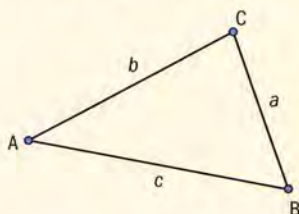


- 2 Calculate the ratios $\frac{a}{\sin \hat{A}}$, $\frac{b}{\sin \hat{B}}$ and $\frac{c}{\sin \hat{C}}$. Give your answers to 1 dp and place them in the table below.

$\triangle ABC$	$\frac{a}{\sin \hat{A}}$	$\frac{b}{\sin \hat{B}}$	$\frac{c}{\sin \hat{C}}$

Compare the three ratios. What do you notice?

- 3 $\triangle ABC$ is shown in the diagram below.



Draw the height of $[CH]$ from vertex C towards side $[AB]$. Label the height h .



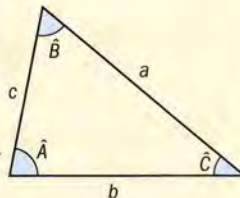


- 4 The height [CH] divides $\triangle ABC$ into two right triangles: $\triangle ACH$ and $\triangle BCH$.
Note that $\hat{A}$ belongs to $\triangle ACH$, and $\hat{B}$ belongs to $\triangle BCH$.
Write an equation in terms of $\sin \hat{A}$ and h using right triangle trigonometry:
 $\sin \hat{A} =$
- 5 Write an equation in terms of $\sin \hat{B}$ and h using right triangle trigonometry:
 $\sin \hat{B} =$
- 6 Write each of the equations from parts 4 and 5 in the form.
 $h = \dots$
- 7 Equate the right-hand side expressions from the two equations in part 6.
- 8 Write an equation equivalent to your equation in part 7, regrouping first with $\sin \hat{A}$ and then with $\sin \hat{B}$.
- 9 **Conceptual** Do you think that if you had drawn the height from B to AC instead, you would have come to the same conclusion about the ratios $\frac{c}{\sin \hat{C}}$ and $\frac{a}{\sin \hat{A}}$? Why or why not?
- 10 **Conceptual** What is the relationship between the sine rule and the sine ratio in a right triangle?

The sine rule

For any $\triangle ABC$, where a is the length of the side opposite $\hat{A}$, b is the length opposite $\hat{B}$, and c is the length of the side opposite $\hat{C}$:

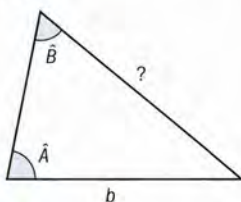
$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} \text{ or } \frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$



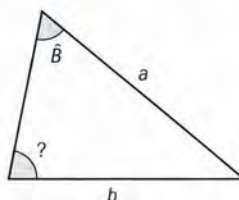
TOK

If sine, cosine and tangent are ratios in a right triangle, how can we use them on angles greater than 90 degrees?

The $\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$ form of the sine rule is used to calculate the length of a side given two angles and an opposite side.



The $\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$ form of the sine rule is used to calculate an angle given two sides and an opposite angle.

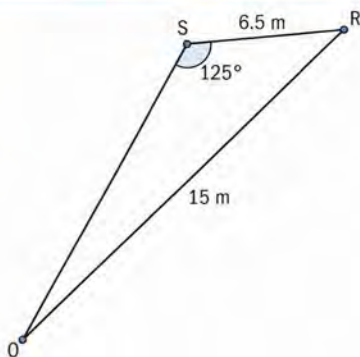


The sine rule is true for any triangle, including right-angled triangles.

Example 1



In $\triangle QRS$, $QR = 15$ m, $SR = 6.5$ m and $\hat{S} = 125^\circ$. Find $\hat{Q}$. Give your answer correct to 3 s.f.



$$\frac{\sin \hat{S}}{QR} = \frac{\sin \hat{Q}}{SR}$$

$$\frac{\sin 125^\circ}{15} = \frac{\sin \hat{Q}}{6.5}$$

$$\sin \hat{Q} = \frac{6.5 \times \sin 125^\circ}{15}$$

$$\hat{Q} = \sin^{-1}\left(\frac{6.5 \times \sin 125^\circ}{15}\right) = 20.8^\circ \text{ (3 sf)}$$

If you are not given a diagram, draw one, label the points and place the given information on your diagram.

Identify the unknown and known sides and angles before you apply the sine rule.

Round your answer to the accuracy specified in the question. If not specified, round your answer to 3 sf.

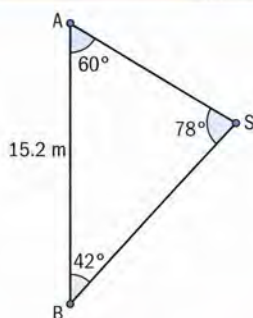
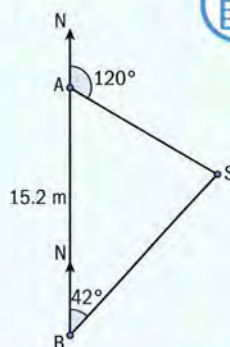
Example 2



A ship S is on a bearing of 120° from port A and 042° from port B.

The distance between ports A and B is 15.2 miles.

Find the distance between the ship and each port.



Draw a diagram to represent the positions of the ship and the ports.





$$\hat{A} = 180^\circ - 120^\circ = 60^\circ$$

$$\hat{S} = 180^\circ - (42^\circ + 60^\circ) = 78^\circ$$

$$\frac{AS}{\sin 42^\circ} = \frac{BS}{\sin 60^\circ} = \frac{15.2}{\sin 78^\circ}$$

$$AS = \sin 42^\circ \times \frac{BS}{\sin 60^\circ} = 10.4 \text{ miles (3 sf)}$$

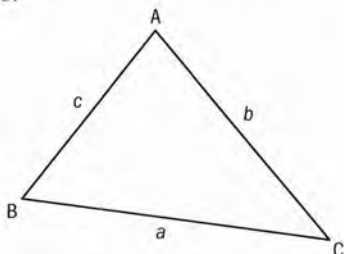
$$BS = \sin 60^\circ \times \frac{15.2}{\sin 78^\circ} = 13.5 \text{ miles (3 sf)}$$

Use angle rules to find $\hat{A}$ and $\hat{S}$.

Use the sine rule to find the lengths.

Exercise 2A

- 1 Find the unknown angle for each $\triangle ABC$. Give your answers correct to 3 significant figures.



- a Find $\hat{A}$ where $a = 12.5$ cm, $b = 15.4$ cm and $\hat{B} = 68^\circ$.

- b Find $\hat{B}$ where $b = 42.8$ cm, $c = 30.6$ cm and $\hat{C} = 41^\circ$.

- c Find $\hat{C}$ where $a = 7.2$ m, $c = 5.1$ m and $\hat{A} = 70^\circ$.

- 2 In $\triangle ABC$, $AC = 8$ cm, $\hat{C} = 101^\circ$ and $\hat{B} = 32^\circ$. Find the lengths of $[AB]$ and $[BC]$.

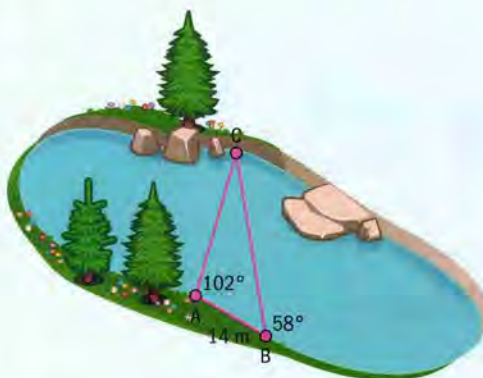
- 3 A ship is sailing due north. The captain sees a lighthouse 12 km away on a bearing of 036° . Later, the captain observes that the bearing of the lighthouse is 125° . How far did the ship travel between these two observations?

The sine rule can be used to solve practical problems involving distances which are impossible to measure.

Example 3

A park service plans to build a bridge $[AC]$ over a reservoir. A and B are two points on the shore of the reservoir, such that the distance $AB = 14$ m, $\hat{A} = 102^\circ$ and $\hat{B} = 58^\circ$.

Calculate the length of the bridge $[AC]$ to the nearest metre.



Continued on next page