An Introduction to the Single Variable New Calculus.

John Gabriel.

Forward

This book is intended to be read by anyone who claims to be any of the following: mathematician, educator of mathematics, STEM professional, mathematics historian, university student or high school student. It is for those who are serious about understanding mathematics and especially calculus.

Engineers and STEM researchers who need to learn calculus rigorously and quickly will find the New Calculus unbeatable.

Regardless of your background, you will understand and learn more mathematics than you learned in all your school years.

Contents:

Chapter 1: A brief history of most relevant events

Before the Ancient Greeks, there was only a vague notion of the number concept. People didn't need or use numbers much at all. It was easy to keep track of one's livestock or gold bars, by allocating a stone or a stick for each head of cattle or gold bar. Thus, if a theft were suspected, an audit could easily be accomplished using a journal of stones. This primitive method of counting resurfaced during the time of Georg Cantor, when there was an attempt to abandon the rigorous foundations laid by the Ancient Greeks. Unfortunately, mainstream academics eventually succumbed to Cantor's ridiculous ideas.

Cantor imagined (erroneously) that sets could be used in place of numbers. Out of his absurd theories arose bijective cardinality and consequently different levels of infinity. Cantor's claim to fame was his definition of what is a countable set, that is:

A set is countable if its members can be named in a systematic way.

Hence, a given set is also called countable if its members can be *listed* in a systematic way.

While not Cantor's exact definition, this is no doubt how Cantor decided in his mind that the set of natural numbers is countable: its members can be listed systematically using any radix system and every member has a unique name. It should be obvious now why Cantor didn't choose another set such as the imaginary set of *real* numbers. It's not possible to name objects which don't exist. We'll learn more about this in later chapters where it is revealed that no valid construction of real numbers was ever produced.

Set theory, which is influenced by thinkers like Ludwig Wittgenstein and Bertrand Russell, is an attempt to measure without any actual concept or clear understanding of measure at all, only a notion of measure through *containment.* The very base object which is a set, is undefined and the approach is one where an object is determined to be a set if it meets the requirements (beliefs is a more appropriate noun) stated in the Zermelo-Fraenkel (ZF) axioms. In fact, till this day, mainstream academics do not agree on a common definition of mathematics even among themselves, which is the abstract science of measure and number.

According to mainstream academia, measure in its simplest form is defined as a cardinal number and more complex measure results from set unions or set mappings. The only required criteria is that elements or members are recognizably distinct from each other. There is no formal definition of set or element and the two are used interchangeably. In fact, the ZF axioms are stated in terms of a membership relation denoted by $\in$. This too is circular because membership requires that the parent object is already defined.

These ideas led to many unresolved paradoxes and contradictions that resulted in the Zermelo-Fraenkel (ZF) axioms which were unfortunately ordained the “New Foundations”, as a result of the misguided ideas of the Bourbaki group in France, circa 1938. Cantor was without any doubt the father of all mathematical cranks.

Throughout the centuries, zero progress was made in understanding the concept of number, even though many new number properties were realized, especially the study of prime numbers. As a result of prevailing academics’ inability to grasp what Euclid had attempted to write down in the Elements, the notions of axioms and postulates were established by decree rather than erudition. I suppose if one can't explain a concept, then one can always declare it to be an axiom from which to form new knowledge.

There was never a need for new foundations of mathematics because everything a mathematician needs is obtained from the Elements of Euclid.

In the chapters to come, we'll learn precisely how numbers were derived perfectly by the Ancient Greeks.

Rene Descartes restarted the age of mathematics enlightenment by returning to the knowledge of Ancient Greece. Descartes realised a Cartesian coordinate system which replaced the complicated conic coordinate system of the Greeks and made it possible for many to learn mathematics, something that would otherwise have remained out of reach for most.

Newton and Leibniz, the so-called fathers of calculus, struggled to find a rigorous method of determining the slope of a line that is tangent to a non-linear curve. They both failed and although numerous attempts were made that culminated in the flawed Cauchy-Weierstrass epsilon-delta limit theory with its numerous error prone inequalities and real analysis, a rigorous formulation of calculus was not realised until the beginning of the last half of the twentieth century, when I discovered it in the New Calculus.

Neither Newton or Leibniz wasted his time trying to find the slope of a straight line that is tangent to another straight line (which is defined circularly in mainstream calculus), because they knew how to find the slope of any straight line either as an angle (θ) or a ratio ($tan\ \theta$). Today's ignorant academics advocate the ridiculous idea of straight lines being tangent to other straight lines. It's a very good thing the fathers of calculus thought of tangent lines in the way they did, for who knows if they could have realised calculus as it developed in the misguided mainstream formulation that followed. What other justification could there have been for the finite difference quotient $\frac{f(x+h)-f(x)}{h}$ and the fact that no finite difference quotient $\frac{f(x+h_i)-f(x)}{h_i}$ exists which equals the derivative $f'(x)$? The so-called "big idea" itself, that is, the *limit*, uses this quotient. If not for the tangent line as understood by the Ancient Greeks, then what else could have been the reason? What makes secant lines special about a point? - The fact that their slopes approach that of the tangent line.

Except for the New Calculus, there has been zero progress in mainstream calculus for more than 150 years.

Archimedes is often hailed by the mainstream as the first to realise integral calculus. The reasons given are not convincing because Archimedes never recognized any numbers besides the *rational* numbers. In order to see the truth of this claim, one need look no further than propositions 3 and 4 (On Spirals, The Works of Archimedes) to understand the actual Archimedean property, which in fact has nothing to do with real numbers. The correct statement of the property is:

Given any magnitude x, *whether commensurate or incommensurate with any other magnitude, there exist two rational numbers* m *and* n, *with* $m < n$, *such that* $m < x < n$.

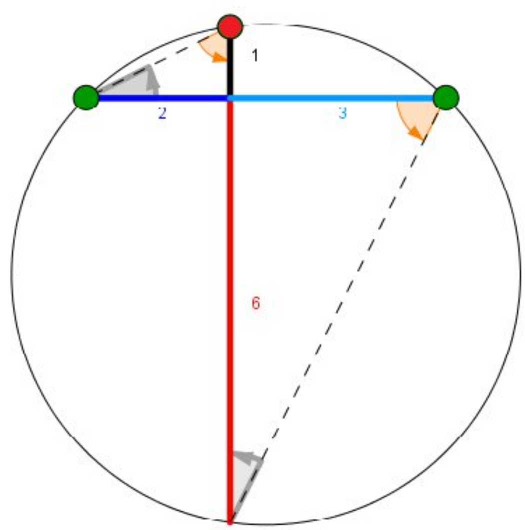
There are absurd notions in the mainstream thought, that Archimedes anticipated limits in his method of exhaustion, but the facts tell a different story. His method always involved the use of his property and an argument by contradiction to determine the measure of an area or a volume. The argument typically involves the comparison of two objects, for example p and q. Assume $p < q$ and reach a contradiction. Then assume $p > q$ and once again reach a contradiction in which case p must be equal to q. Some might say this is similar to the squeeze theorem or the limit of a function, but Archimedes didn't deal at all with functions as we know them today. In fact, neither a circle nor a sphere is originally defined as a function (mapping) at all.

Calculus was later developed on the building blocks of functions and numbers, but functions are described by formulas and not only by numbers. For example, the area of any circle is described by a function $A(r) = \pi r^2$ which is always a parabola. The algebra hides the fact that π is a symbol for a ***constant*** size and decidedly not a number. π is never used as a number in algebra, only as a symbol denoting a (rational) number or a rational number approximation, because there is no measure of the constant ratio (π is not commensurate with any other magnitude) known as $c{:}d$ where c denotes circumference and d denotes diameter. In all cases, π acts like an *unknown* in algebra. For example,

$$2 \times x \times 3 = 6 \times x \quad \text{and} \quad 3 \times \pi \times 2 = 6 \times \pi$$

are equivalent in algebra, for both state that $6 = 2 \times 3$ or $6 = 3 \times 2$ or $2 = 6 \div 3$ or $3 = 6 \div 2$, and the size/magnitude known as x or π plays no role whatsoever. Geometrically, exact arithmetic involving any ratios of magnitudes is possible *regardless of magnitude* (See Appendix

B which explains how you got arithmetic from geometry). The four basic operations of arithmetic (difference, sum, quotient and product) come to us from the geometric arithmetic of ratios where no numbers are used at all. The diagram that follows explains these facts. By choosing one side (not hypotenuse) of either right-angled triangle to be the unit, division and multiplication of any magnitude is well defined by moving either of the green points horizontally and noting that a circle always passes through all three points shown (green and red).



$$\frac{2}{1} = \frac{6}{3}$$

By using a circle with similar triangles as shown, exact division and multiplication is possible assuming one of the triangle sides acts as a unit.

Thus, geometrically these operations are very well defined for any magnitude. The astute reader will also notice that the above diagram is an ancient calculator whereby the four basic operations of arithmetic can be performed to 100% accuracy without any knowledge of number or mathematics, save a few constructios using compass and straight edge.

In algebra however, the story is quite different. For starters, the magnitudes have to be given names, that is, they need to be measured. In the sample diagram there are the measures $1, 2, 3$ and 6 where the black line segment is chosen to be the unit.

Next, the operations of division and multiplication need to be defined, that is, what does $6 \div 3$, etc, actually mean? This and more to be explained in How we got algebra.

Before Newton and Leibniz, many efforts were made to find ways of determining tangent line slopes, but all of them failed. One method that was more rigorous than Newton's method, was Descartes' method of constructing a circle that is tangent to a given curve. However, it is only possible to use in very simple cases.

Fermat's *adequality* method was absolute rot because it uses exactly the same ideas as modern calculus vis a vis the limit of a difference quotient. The method may look different in appearance but the underlying mechanics are the same.

There have been numerous other attempts to formulate a calculus without the use of limits, but all use the limit concept, if only indirectly.

The only rigorous formulation is the New Calculus which uses no ill-formed concepts such as infinity, infinitesimals and limit theory. One can write many books on calculus and its development, but no book ever written contains the well-formed knowledge you will realise in this book.

Chapter 2: What it means for a concept to be well defined

Never in the history of academia has anything been published on what it means for a concept to be well defined, never mind the requirements or a systematic method for determining well-formedness.

This is evident when one looks at all the paradoxes and contradictions that arise in theory - especially such as first order logic, axioms, etc. One needs only to investigate the bogus concept of infinity which cannot be reified in any way, shape or form to see how it has infected all mathematics.

In this chapter, I will show you a simple method of how you can determine whether a concept (A better word for concept in this context, is a *noumenon*, that is, a well-formed concept that exists independently of the human mind or any other mind) is well formed or not, in just four simple steps.

Determining if a concept is well formed or not:

Here are my **four** essential requirements for any concept to be well defined:

In order to be well defined, a concept

1. Must be reifiable **(R)** either intangibly or tangibly.

2. Must be defined in terms of attributes **(A)** which it possesses, not those it lacks.

3. Must not lead to any logical contradictions **(C)**.

4. Must exist in a perfect Platonic form. What this means, is that it exists independently **(I)** of the human mind or any other mind, as a noumenon of course.

The method is easy to remember using the acronym **RACI**.

RACI: **R**eifiable - **A**ttributes - **C**ontradictions - **I**ndependence

A simple proof that any concept is well formed, can be confirmed if an alien from another world realises the concept in the same way. For example, the attempted measurement of a circle's circumference using its diameter, must be realised by an alien in exactly the same way. π cannot be realised in any other way. Hence it is a perfect concept or noumenon as explained shortly.

Reification:

If you can't reify a concept, then it may possibly not exist outside your mind. If a group of mainstream academics get together and claim an infinite sum is possible, even among themselves, they do not think of it the same way. The fallacious idea that $0.999\ldots$ and 1 are both representations of 1, is a fine example. Some academics think that it is actually possible to sum the series: $0.9 + 0.09 + 0.009 + \ldots$

But let’s not go too far, no mainstream academic even understands what mathematics is all about any longer.

Others (such as Rudin, author of ubiquitous mainstream Real Analysis text book called Principles of Mathematical Analysis) realise that only a limit is possible. Still, others believe that it's a good idea to give a sequence a value in terms of its limit, even when the limit is unknown.

To *reify*, means to produce an instance of the concept, so that someone who knows nothing about it, can understand it exactly the same way as any another. Even though the $0.999\ldots$ fallacy has been around for so many decades, ask yourself how it is that so many students and even educators have different views on it, with most forums split almost evenly among those who acknowledge the fallacy and those who don't. The ill-formed concept $S = Lim\ S$, is always a problem for students because it is derived from an ill-formed definition by Euler, who in his Elements of Algebra, defined an infinite sum S, to be equal to its limit $(Lim\ S)$, provided of course, the infinite sum converges.

To accept that $0.333... = \frac{1}{3}$, that is,

$$0.333... = \frac{1}{3} = \lim_{n\to\infty} \frac{1}{3} \left[1 - \frac{1}{10^n}\right]$$

one first has to believe that an n exists such that $\frac{1}{10^n} = 0$. The very thing is impossible, not to mention absurd! Leonhard Euler was one particular mathematics academic who believed that such an n is possible. Moreover, the number $\frac{1}{3}$ has no measure in base 10 or what is commonly called decimal.

Theatrical (not theoretical physicists, because those went extinct shortly after Einstein) physicists took this idea to a whole new level where convergence is no longer relevant. In fact, the foundation of String Theory rests on the delusion that

$$1 + 2 + 3 + 4 + ... = -\frac{1}{12}$$

You can reify a concept without someone else being able to understand it for many other reasons; some include intelligence, ignorance, etc. However, I am talking about all things being equal, in which case the concept can be acknowledged as having been reified.

If you can't get past reification, then your concept is no doubt nonsense. Some examples are:

Irrational numbers.

Infinitesimals.

Limits, when in fact, there is no valid construction of "real" numbers.

Infinity.

Einstein's theories.

Hawking's theories.

Definitions that are self-referential.

Attributes:

If a concept is not defined in terms of attributes it possesses, then you may as well be talking about innumerably many other concepts. For example, to define an irrational number as a number that is not rational is clearly nonsense, because in order to be rational, there must be a numerator/antecedent and denominator/consequent, both which are themselves already established numbers. If these are not numbers, then a gorilla and banana are also irrational numbers because neither can be expressed as a ratio of numbers.

Thus, there is endless ambiguity. Attributes are the most important second step after reification. An attribute describes the boundaries or limits, the extent of the instantiated object from the concept.

Mathematicians are like artists, the objects arising from concepts in a mathematician's mind are only as appealing as they are well defined.

Clearly, concepts are unusable if they cannot be well defined. A good example is $0.999\ldots$ - it has no use and nothing worthwhile can be done with such an ill-formed definition, that is, $S = Lim\, S$. This is what I call the Eulerian Blunder, because it was Euler who defined $S = Lim\, S$.

Contradictions:

Once a concept has been reified and defined (there are limitations to being well defined and this is why one needs to have checks for contradictions, until the concept becomes *axiomatic* over a long period of vetting) in terms of attributes it possesses, it has to be vetted. This is done by always verifying that any results stemming from its use do not produce logical contradictions.

Independence:

Finally, the last criterion which is sufficient for a well-formed concept, is that it must exist outside of the human mind or any other mind. For example, if aliens realise the *incommensurable* constant π, they will realise it in the only logical way: the attempted measure of a circle's circumference magnitude using the diameter magnitude as a unit.

Perfect concepts or noumena (singular is noumenon), exist whether life exists or not. That is what the Greeks discovered when they studied geometry. The concepts of geometry are all without any exception perfect concepts (Platonic).

Not even "God" can measure the constants (not numbers!) π and $\sqrt{2}$, because neither of these concepts require anyone to think of them. They have existed inanimately as **perfect concepts**, in past perpetuity and continue to exist indefinitely. Like the geometry of the Ancient Greeks, these noumena (perfect knowledge) exist independently of the human mind or any other mind.

If the concept you realise meets all four of these requirements, then you can be assured that it is a well-formed concept or a noumenon.

Chapter 3: Mainstream misconceptions about mathematics

The misconceptions in mainstream mathematics are too numerous to discuss. In this chapter, we'll look at some of the most common.

The most serious misconception is that of *numbers*. Before me, there was no definition of number, only vague ideas of what is meant by the concept of number. This misconception has its roots in the failure of academics to understand the Elements of Euclid and what Euclid attempted to write down in a perfect way. In a subsequent chapter, the concept of numbers and how we got numbers will be discussed in great detail.

The next major misconception is that Euclid's five requirements are axioms or postulates. In the following chapter, we'll see how all five requirements are derived systematically from nothing.

However, the failure of mainstream academics to comprehend the concept of number is due partly to this misconception, because number is derived from the geometry that one realises from these five requirements. For over 2300 years *no academic* before me was able to realise these facts.

Since *number* is the building block of mathematics, it's no surprise that the ideas of the past few hundred years are a compendium of gibberish that has resulted in the chaos one finds in mainstream mathematics which is no longer a science of measure and number, but mythology.

Behind all the ideas of mathematics lies Platonic philosophy and the mathematics of Ancient Greece which developed directly from it. The main concepts one deals with in geometry are location and distance – the very concepts that are discovered from pondering an empty universe. To plot a point, that is, “.”, and call it such, is only a visualisation of the location idea.

. is not a point.

____ is not a line.

o is not a circle.

The objects as drawn above, are mere visualisations or instantiations of the perfect ideas representing the geometric objects known as location and path. Location is the concept realised by asking the question “Where?” and path is the concept realised by asking the question “How do I go from one location to another?

For example, a line is one of innumerably many distances that are possible between any two locations or points in the universe. A straight line is that unique shortest distance between any two locations. A path is thus a systematic way of moving between locations.

Therefore, the most important attribute of a line is its length or the distance it covers. A line never consists of points because in reality, points are like flags or road signs which are not part of a road. A point indicates a distance along a line, but is not part of a line. This fact is true of any geometric object. The triangle is the shortest distance joining any three distinct locations in the universe, the quadrilateral is the shortest distance joining any four distinct locations in the universe and the circle is the shortest distance joining all the locations equidistant from a given centre in a plane.

The second most important concept in mathematics is that of *arithmetic mean* or more correctly named *level term* or *level number* or *level magnitude*. Every number is not only a (the adjective *rational* being redundant because number implies rationality) number, but also an arithmetic mean or level number. The misplaced mainstream ideas of area and volume are a result of failing to understand this vital concept. For example, no definite integral would be possible without an arithmetic mean. We'll see in a later chapter how the mean value theorem is proved using the fact that it describes exactly an arithmetic mean. I was the first in human history to realise this fact.

It therefore behooves us to understand this basic concept of arithmetic mean which permeates all of mathematics and science. We deal with this concept in detail in a chapter dedicated entirely to its explanation.

Had the arithmetic mean been understood correctly, area would have been defined as the product of two arithmetic means and volume the product of three arithmetic means.

Without the arithmetic mean which is a *ratio* of special fractions called *natural numbers* in the vernacular, none of the following would ever have been realised:

- Slope or gradient
- Finite difference
- Ratio
- Probability
- Derivative
- Integral

etc.

Thus, number and arithmetic mean are the two most important concepts in mathematics. A good understanding is not possible without a sound knowledge of geometry.

Infinite series

There is no such thing as an infinite series – neither potential nor actual. By writing $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$, we mean a partial sum which may or may not necessarily converge. You'll hear ignorant academics talk about the Cauchy criterion for infinite series; in fact it is a misnomer. A logical way to state it would be as follows:

Cauchy criterion for partial sum:

A series $\sum_{k=1}^{n} a_k$ converges if $\forall\, \varepsilon > 0, \exists\, n \in N$ such that for any

$n > m,\ \ \sum_{k=m+1}^{n} a_k < \varepsilon.$

The above statement has nothing to do with infinity because infinity does not exist in any form or shape and cannot be reified, which means it is an ill-formed concept.

Observe that $\sum_{k=m+1}^{n} a_k$ is the tail part of the series. In order to talk about an infinite sum, we must have no tail part left and this is the reason why we use $\varepsilon > 0$ and not $\varepsilon = 0$, because the latter requires a super task which is impossible.

As a student, one is expected to prove such a statement using the partial sum of the series or some other method. However, to see why this requirement works is easy with an example. Consider the series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Such a series can be shown to diverge using the comparison test. But without having to learn convergence theory, what it means is that you can choose an ε which you assume to be greater than any given tail part and then conclude that your assumption is incorrect by proof. The important thing to remember is that no proof contains anything about infinity. All proofs are inference based and are finite processes which have nothing to do with infinity.

Unfortunately, ignorant mainstream academics, define the sum of a convergent series as being equal to the limit of the series:

The sum is defined as the limit of its partial sums:

$$S = S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k$$

S is the sum and $\sum_{k=1}^{n} a_k$ is the limit in which the infinite series $\sum_{k=1}^{\infty} a_k$ itself is assumed. What most mainstream academics don't understand is that the *infinite series* is not the *limit*.

For example, given $\frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$ means the sum $\frac{7}{8}$ is equal to the series $\frac{4}{8} + \frac{2}{8} + \frac{1}{8}$. However, $\frac{4}{8} + \frac{2}{8} + \frac{1}{8} + \dots = 1$ does not mean 1 is equal to the series $\frac{4}{8} + \frac{2}{8} + \frac{1}{8} + \dots$, because such a series is not possible. In fact, the limit does not care if the terms are all there or even there at all. The limit also does not care if all the partial sums are possible or not.

Such dysfunctional thinking started with Isaac Newton who unfortunately chose to call his partial sums "infinite series".

An infinite series is equivalent to a series with a general term and possibly a sum to n terms. It is represented by a partial sum followed by an ellipsis. **There is nothing infinite about it.**

An infinite series sum is equivalent to the limit of a convergent partial sum. **There is nothing infinite about it.** It's meaningless to talk about $1 + 2 + 3 + 4 + \ldots$ because such a sum has no limit unless you are a theatrical physicist (fake theoretical physicist).

Chapter 4: There are no axioms or postulates in mathematics

The first Book of Euclid's Elements is the least understood of all thirteen books. Appearing in Book 1 straight after the definitions, we have five requirements (*Αιτήματα: (h)ey-tea-ma-ta*) which are interpreted as postulates by the mainstream academics of the last two thousand years. The first requirement contains the Ancient Greek word *Ηιτησϑω ((h)e-tis-tho)* whose meaning is uncertain. Sir Thomas Heath translated this word *Ηιτησϑω* as "Let it have been postulated". The modern Greek word *υποϑέτω ((h)e-poe-the-to)*, means to assume or hypothesise or even guess. However, there is no obvious connection between the ancient and the modern words.

The word *Ηιτησϑω* is used only *once* in all thirteen books and its meaning is unclear. It is doubtful that Euclid hypothesised anything because the whole idea of the Elements was to write down the foundations of mathematics in a systematic and coherent way. The base concepts such as point, line, circle, etc., were clearly understood by the Ancient Greek scholars. There was no need for Euclid to derive all of these from nothing, however it certainly would have helped.

A postulate or axiom today is simply a *definition* that has been vetted over time. While a definition cannot be described as right or wrong, it can be well formed or ill formed. Thus, by calling a definition 'right' or 'good', we mean it is well formed as per the guidelines described in chapter two.

The words postulate and axiom were not realised in Euclid's time, therefore it is preposterous to assume that Euclid meant 'axiom' and not ‘requirement’ or ‘claim’. In many instances, Euclid omitted proofs by writing “I say, …” and then stating a result or theorem. There are no axioms or postulates in sound mathematics. It’s ironic that an advanced automation engine such as ChatGPT can comprehend these facts, but not a professor of mathematics. Such a shame!

Book 5, Proposition 12 forms the foundation of all fraction (numbers), arithmetic and algebra. It can be easily demonstrated using similar triangles and one can see how this is done in appendix B.

If Newton understood this proposition, you probably wouldn’t be reading this, because the derivative is easily defined using the property of this proposition as demonstrated in the New Calculus, that is,

$$f'(x) = \frac{f(x+n)-f(x-m)}{m+n}$$

where m and n are horizontal distances from the point of tangency $(x, f(x))$ to the endpoints of a secant line that is parallel to the tangent line at the same point and $x - m < x < x + n$.

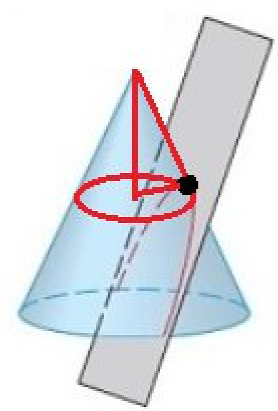
But let’s not get ahead of ourselves. To say that the Ancient Greeks were intelligent beyond belief, is an understatement. Often, they would omit more than half of a given sentence when the meaning and inference were clear from context. In certain instances, even the adjectives used to disambiguate, were dropped as qualifiers. There are many examples in the Elements. These language rudiments of Ancient Greek can be discussed in lengthy volumes, but that is not the purpose of this book.

The five requirements (not postulates or axioms) can be systematically derived from nothing, that is, as well-formed concepts. There is no need to accept anything on faith and no need to call any well-formed concept an “axiom”. Contrary to mainstream academic thought, the foundation object in geometry, which is the ‘point’, is a very well-defined concept. A point is simply the idea of location or place. It arises instantly when the question “Where?” is asked. In a void universe, location is impossible to reify. There is no frame of reference or a means of describing (using coordinates) any given location exactly.

The Greeks overcome this by using a conic coordinate system whereby any location or conic curve can be described in space using the cone. For example, if we are observing a cone and the apex is on top with reference to us standing upright, then a random parabola is given by a cone rotation, distance from cone apex to disk on which parabola apex rests and distance between cone apex and parabola apex. A particular point would be described by the intersection of the geometric objects and those cone geometric objects acting as locators on the cone itself.

But to arrive at the cone, it is first necessary to produce the objects from which the cone is constructed, that is, a point, line and circle.

The locators (lines and circle) for the black point which is the parabola apex are indicated in red in the diagram that follows:



The clarity of Ancient Greek thought has been unmatched by any who came before or after them. In the history of mathematics, no such clarity has been realised in the last 2300 years.

Preliminary notions:

It is imperative first to understand that noumena (or well-formed concepts) exist independently of the human mind or any other mind. In fact, these noumena do not even require a Creator or God for those of you who might be religious. A noumenon (first coined by Immanuel Kant) is a perfect concept that exists inanimately and requires no prior existence or thought. To start with an example:

Consider that the symptom of the incommensurable magnitude known as the **constant** π exists, whether the Ancient Greeks thought of it or not. π is a noumenon, because it is a well-formed idea or concept that arises when an attempt is made at measuring a circle's circumference using its diameter.

Whoa! You may exclaim, but where did points, lines and circles come from? All geometry which exists independent of any mind, is based on the base geometric object called a point.

What is the point?

A point is simply an idea of location or place. It asks the question "*Where?*".

The period or dot that is normally used to represent a point, that is, '.', is only a visualisation or instantiation of the noumenon. The next logical idea that arises, is the relation of one point to another, that is, how to get from one location to another. It begs the question "*How to get there?*". In other words, what are the *directions*. A rudimentary definition would be to posit two locations: ‘here’ and ‘not here’. Motion can be defined as the change in location.

A systematic approach.

At this stage, we can't give precise directions yet. But we do observe that between any two locations/points, there are innumerably many *paths*. We also observe that any given path describes a collection of directions from one point to the other. So, a path describes the process or means of moving from one point to another. Naturally, it involves at least one or more directions, that is, a change in direction is possible. If we think of two individuals as points and a third person moving from one individual to the other without any change in direction, we arrive at the idea of a shortest distance. From this, we can formulate our second geometric object called the straight line:

A straight line describes a path between one point and another, such that the direction remains unchanged.

Or a path consists of directions and the path with the least directions (a primitive tally using a one to one correspondence between the different path directions) must be the shortest one, that is, the straight line.

Direction is well defined: It is the "How do I get there?" concept.

Definition of direction: ***The orientation of a given object that moves along a path or course.***

Orientation can initially be defined very simply as *facing toward* or *facing away* from a given object which is used as a reference point.

Pause to think about the *definition of straight line*. Does it require acceptance of any prior beliefs (axioms)? Does it require any vague or mystical dogmas? The answer is no. Therefore, we are able to define a straight line perfectly with only two points, that is, **a straight line is the shortest path or distance between two points.** Clearly, any change in direction between the two points whilst traversing from one to the other, will result in a distance that is *longer*.

To see the general logic behind this principle, consider ***any path***, and then observe that ***any deviation*** from the path will result in a longer path. See? It's very easy to define. This idea works on all surfaces. For example, you may have heard the expression "As the crow flies" - this actually means the shortest distance or the "most direct" distance, even though "most direct" is a redundancy where the meaning of direction is concerned and in the case of the crow, the shortest distance is a geodesic. We simply say the direction resulting in the shortest path, that is, *no change or deviation* from a given course.

From the geometric object called the straight line, it is easy to **systematically** derive all the remaining so-called "postulates" (actually requirements).

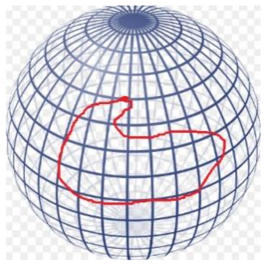
The second requirement states that a straight line can be extended indefinitely (as near or far as desired) in either direction. The definition and proof of this requirement is amazingly simple. All one has to do, to prove the second requirement, is to consider a portion of any given straight line, and since we know that a straight line exists between any two given points from the first requirement, we are done.

First, we define a ***complete path*** as that path which starts at one point and ends up at the same point after it has traversed other points. This path describes a distance between the points since points have no size.

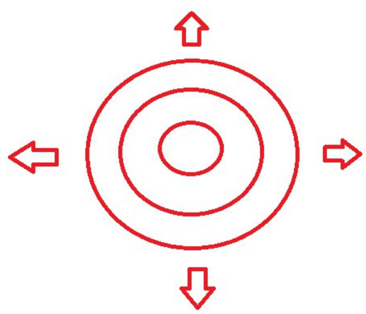
The third requirement is stated as follows:

A circle is that shortest complete path from which the distances to a given point called a centre, are all described by the same straight line and a straight line (*) from one point on a circle, if extended toward the centre, will meet the circle path again.

Consider that ***all*** the paths on a sphere's surface would be circles if the word *shortest* and the last qualification (*) were omitted. In astronomy, any such path is called a *great circle*. In the following diagram, any point on the red path shown on the surface is the same distance from the circle centre.



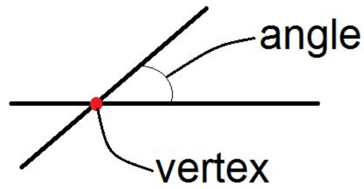
From this, we can immediately define a **plane** as the indefinite area spanned by a circle of increasing radius (size in area).



In only a few pages, we have already derived the first three requirements from nothing. We can now also define a sphere's surface as that geometric object which describes all circles with the same radius. Furthermore, we can define a sphere as the solid created from the encompassed space (volume) spanned by a rotating circle.

The fourth requirement states that all right angles are equal. Before we can prove this fact, we first have to define what an angle is and several other geometric objects.

An *angle* is formed at the intersection of two lines which is called a *vertex*.

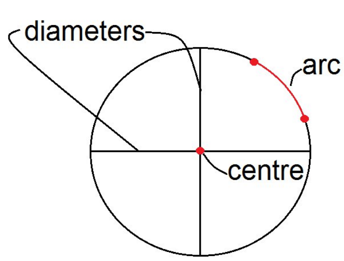


For any angle to be measured, the vertex must coincide with the centre of any circle and one of the lines must coincide with a diameter, which is a line segment through the circle centre whose endpoints lie on the circle path.

A *ratio of magnitudes* is literally the comparison of any two given magnitudes. A magnitude can be length, mass, area, volume or any other scalar quantity. Since we are dealing with line segments, it follows that given any two line segments AB and CD, the ratio AB : CD is the comparison of the lengths of AB and CD. AB is called the antecedent and CD is called the consequent.

Bear in mind that at this stage we have no numbers yet. Thus, length is just some given magnitude or quantity or size. Choose whichever word you like best, because they all mean the same thing: the concept of size, dimension or extent.

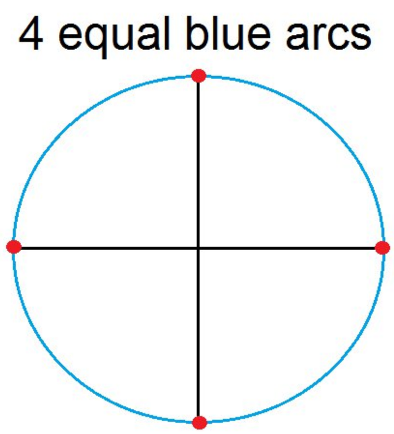
An *arc* is any length portion of the circle path also known as the periphery or circumference.



The *measure of an angle* is defined as a ratio of magnitudes, where the antecedent is the *length of a circle arc subtended by the angle,* to the consequent which is its *radius length,* where the *radius* is the straight line from the centre to any part of the circle path. The vertex is located at the centre of any circle, so that the angle has a corresponding arc on the circle.

Definition:

A *right angle* is realised by observing that *four equal arcs* correspond to four angles in any given circle.



Proof:

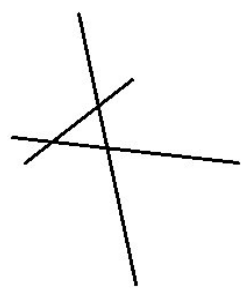
Given that any diameter, a line through the centre, partitions the circumference of a circle into exactly two equal paths or parts by the ***property of symmetry*** **(*)**, we know that any diameter subtends an arc that is half the circumference. By compass and straightedge construction, we can show that a bisector of the diameter can be constructed at the centre of any circle. A *bisector* by definition divides the diameter into two equal parts called radii. Such a bisector further also divides the half circle circumference into two equal parts. Note that the arcs so formed on either side of such a bisector, are ***equal by symmetry***, thus we call each of these angles in the half circle *right angles*. Moreover, we call the bisector line a *perpendicular* to the diameter, because it forms two right angles whose vertices are at the centre – one on either side of the diameter.

(*) Symmetry is the property or attribute of being made up of exactly the same paths, lines or parts in general.

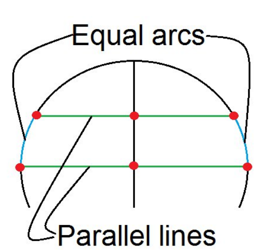
Before we can get to the fifth and last requirement, we need to define parallel lines.

So, we now have defined the concepts of centre, vertex, angle, arc, ratio, right angle, radius, diameter, arc, bisector, perpendicular and introduced the notion of symmetry, that is, the **quality of being made up of exactly similar paths or parts**.

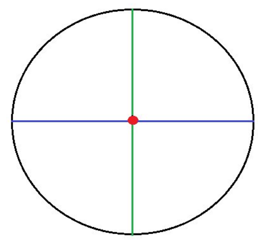
A *transversal line* is a line that intersects (crosses) two different lines. The lines being intersected, may themselves intersect each other without respect to the transversal. All the lines in the figure below are transversal lines.



Parallel lines are chords or lines that are subtended on equal arcs between the chord and the diameter, and are said to be *parallel to the diameter.* In the figure that follows, the green lines are parallel and the blue arcs equal.



Henceforth, I will be using the established fact that two diameters are perpendicular to each other and whose point of intersection is the circle centre. In the diagram below, green and blue diameters are perpendicular because there are four equal arcs.



A chord that rests on equal arcs from a given diameter is *parallel* to the diameter which it does not intersect and *perpendicular* to the diameter which it does intersect. Either of the diameters can serve the function of transversal.

Cointerior angles lie between the two lines (chord and a diameter) on the same side of the transversal.

An *alternate angle* always lies between the parallel lines (one is the chord and the other is the diameter) and is equal to the cointerior angle located at the other parallel line (diameter) on the opposite side of the transversal.

Proof that sum of cointerior angles is two right angles:

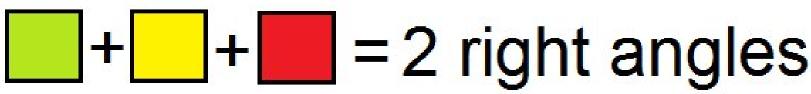
Given that there is only one diameter/transversal that is perpendicular to the other diameter, it follows that a parallel chord (line) will also intersect the transversal (diameter) at right angles.

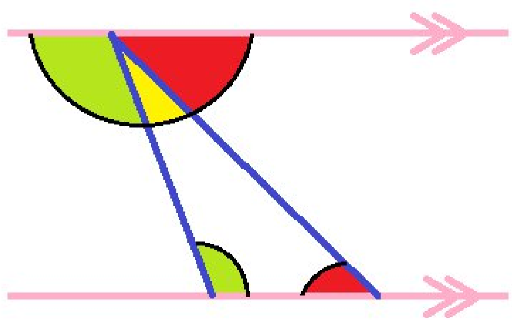
Therefore, the sum of angles formed where a transversal (diameter) cuts two parallel lines (chord and perpendicular diameter) is two right angles, because the parallel line (chord) is straight and by symmetry, the alternate angles are equal. Hence the proof is complete.

All that remains to show, is that lines (chord and diameter) are parallel, provided the sum of two cointerior angles is equal to two right angles.

Proof:

If we take any other diameter as a transversal, one that is not perpendicular to the chord or remaining diameter, then we see that the alternate angles are equal by symmetry about the transversal.





Hence, the sum of two cointerior angles is always two right angles regardless of how the transversal cuts the parallel lines.

One more thing to do before we look at requirement 5, that is, we need to define a *triangle* and establish a theorem regarding the *sum of its angles*. A triangle in a plane can be defined as the shortest distance joining three distinct points. We can let two of these points lie on one parallel line and the other on the remaining parallel line.

We can prove that the sum of the angles in a triangle is two right angles.

Proof:

This is easily proved using alternate angles and *transitivity of equality*. The property known as transitivity of equality states that if $p = q$ and $q = s$, then $p = s$.

Let the lone angle with vertex A on the parallel line be denoted by x and the other two angles with vertices B and C respectively on the remaining parallel line be denoted by y and z. Then AB and AC are transversals, and so the alternate angles that are supplementary **(**)** to x are equal to the corresponding alternate angles on the other parallel line, that is, the angles that are supplementary to y and z.

Since the triangle consists of all the angles that are supplementary on the parallel line where the lone angle is located, it follows that their sum must be two right angles.

(**) *Supplementary angles* are angles with the same vertex such that their sum is two right angles.

Measure of an angle:

I defined ratio as the comparison of two magnitudes. From this I define the measure of an angle as the ratio of its arc length to the given radius, that is,

arc length : radius

The measure of an angle is not necessary to complete the requirements since angles are considered as a factor or multiple of the magnitude already defined as a right angle, but is included here for good measure.

The measure of an angle is given the dimension *radian,* because it is a product of some magnitude and 2π.

Measure $= \frac{2k\pi r}{r} = 2k\pi$ where k is a magnitude. One radian is the measure of that angle whose arc length equals to the length of the radius.

Now we are ready to state the fifth and last requirement in very simple language:

The sum of cointerior angles on the same side of a transversal is constant.

What this means, is that it does not matter if the lines are parallel or not, and it also does not matter how the transversal cuts the lines. This is the essence of requirement 5, not the statement of the parallel postulate, you may have heard repeated over and over again by mainstream academics.

Proof:

If the lines are parallel, then we are done, since we already know that the cointerior angles have a sum of two right angles.

If the lines are not parallel, then we know that a triangle is formed on the side of the transversal where they (non-parallel lines) meet. Given that the angle where the lines meet does not change, it follows that the sum of the remaining angles is constant on either side of the transversal. Q.E.D.

The fifth requirement is stated in the Elements in terms of only the cointerior angles on one side of the transversal, the side where the lines intersect, that is, the sum of the cointerior angles is less than two right angles. But this is obvious from the proof that the sum of triangles is equal to two right angles. The way I have stated the requirement addresses the sum of cointerior angles being constant on either side of the transversal.

I have painstakingly shown you in just a few pages how you can systematically derive all of the requirements from nothing by constructing all the 5 requirements, beginning with the point. It should be clear that there are no axioms or postulates in Greek mathematics.

The next step is the derivation of the abstract concept known as number.

Chapter 5: How we got numbers

In this chapter, I'll show you how to derive the concept of number from nothing, as it should have been derived. In order to do this, I build on the brilliance of the Ancient Greeks whose clarity of thought was unmatched by any who came before or after them.

After Euclid and before me, ***not a single mathematics academic***, **ever understood what is a number.** The fact that academics called the requirements axioms or postulates is proof of this, because in order to derive numbers, these requirements must have been established. Mainstream academics failed dismally to understand what Euclid was attempting to write down – the perfect derivation of numbers. The Chat bot ChatGPT can actually understand the derivation but once again, mainstream mathematics professors are too stupid.

First, let me begin by saying that the foundations of mathematics have nothing to do with set theory or Georg Cantor, whom I am convinced made a grave blunder in mathematical research and student education. Cantor was unable to have a clear understanding of number and associated properties such as arithmetic using the binary operations. His work, I would argue, should have been taken into more scrutiny before being taught and used as a viable construct of research.

Secondly, well-formed concepts exist as noumena, independently of the human mind or any other mind. This truth was explained by the world's greatest philosopher - Plato. Platonism is the theory that ideas/concepts or other abstract objects are objective, timeless entities, independent of the physical world and of the symbols used to represent them.

Thirdly, a well-defined concept is imperative in mathematics or any discipline. It is a very dangerous thing to rely on one’s intuition. Many academics such as set theorists, topology majors and teachers of real analysis, are guilty of relying on their intuitions which arise from ideas that are ill-formed.

Unless a concept can be reified, either intangibly or tangibly, you may as well dismiss it as junk knowledge. Anything you build from it, will eventually be filled with paradoxes and contradictions, only to become so complex, one can only wonder if such knowledge should be passed onto future generations.

To understand numbers, one must start with Euclid. It might surprise you that measurement came before numbers. How so? Well, the Greeks started off with the concept of magnitude.

A magnitude is the idea of size, dimension or extent. (Elements, Bk. V, Definition 1)

A magnitude is decidedly not a number. It can be a length, mass, volume or any other measurable size, dimension or extent.

The Greeks used line segment lengths usually denoted as AB, CD, et cetera. For this to be accomplished, it was necessary to have the 5 requirements in place already.

We begin to consider any two magnitudes by comparing the same. This comparison is denoted by a ratio of magnitudes which came long before numbers or ratios of numbers. Given line segments AB and CD, we write $AB : CD$ which literally means AB compared with CD. Since neither magnitude is a number, we can only perform qualitative measurement or trichotomy, that is, if we can tell visually (qualitatively) that AB is not equal to CD, then we can go a step further to conclude which line segment is longer or shorter. That's all we can do if we stop here. For example, we can't tell what the *difference* (the most primitive arithmetic operator) is in any more precise terms.

Much later, the idea of fraction was born from ratio, that is, if p and q are magnitudes, then $p : q$ means that p is always measured by the magnitude q following the colon. A new representation was designed for the measure that describes a number, that is $\frac{p}{q}$ which does not mean "p divided by q", but rather " p is measured by q".

AB : CD is called a ratio of magnitudes. The idea of *unit* was discovered when the Greeks compared two magnitudes of the same size, that is, AB : AB. The outcome of such a comparison is equality or zero difference. To find out how different are other magnitudes from AB, we can designate AB to be the *standard* or *unit* of measurement. Hence the abstraction of the unit was discovered. Note that the size of AB is immaterial.

In the case of magnitudes such as constants π and $\sqrt{2}$, the unit is not required, because measurement takes place with the diameter of a circle and the leg of a right-angled isosceles triangle respectively. Circles with diameters are symptoms of π, just as right-angled isosceles triangles are symptoms of $\sqrt{2}$. It is impossible to measure any magnitude, given only its symptoms.

A number is a name given to a measure that describes a ratio of magnitudes.

Provided any two magnitudes are commensurate with AB, these can now be quantitatively measured, that is, we can tell the difference (if any) exactly in terms of units. As you can see, the discovery of the unit was a quantum leap in the efforts to measure more precisely. After arithmetic and algebra were established in Book V of the Elements using magnitudes in geometry, the ideas were extended in Book VII Definition 1, to numbers through the abstract unit.

This knowledge led to the ideas of multiples and factors. Given a unit AB, a multiple of AB is measured exactly by AB and AB is called a factor of that multiple. From what you've read so far, it becomes clear that we can construct the natural numbers from any given unit.

What this means is that provided the magnitude is a multiple of a unit, we can define the natural numbers as ratios of such a multiple to the unit. I bet you never imagined that ratios came before the natural numbers! In contrast, modern mathematics states that natural numbers are considered the starting point. We can name these numbers by assigning symbols as we please, for example a, b, c or 1, 2, 3 and so on (Elements, Bk. VII).

But what happens if a magnitude is part of a unit? We let those equal parts of the unit be units, and the measurement of the unit by those equal parts of a magnitude, to be the number, that is, a fraction and thereby arrive at a representation for the magnitude. All numbers are fractions.

The key to this approach is to divide the unit into the right amount of equal parts and the answer is simply that natural number, which results in the measurement of those equal parts. That number is the antecedent part of the fraction or the numerator as commonly referred to in today's lingo. The consequent part is the number of equal parts in the unit or the denominator.

From the natural numbers, we define fractions as a ratio of natural numbers.

Example: $2:3$ or $3:2$ or $22:7$, etc. For the ratio $m:n$, we call m the antecedent part, and n the consequent part. If the antecedent is less than the consequent, then we have a proper fraction, that is, a number x such that $|x|<1$ (x lies between -1 and 1), otherwise we have an improper fraction. All numbers are by definition **fractions**.

One ought to bear in mind that the *abstract* unit is **dimensionless**, not like the units one finds in a **table of standard units** or a **table of physical measures**. Furthermore, the physical unit is not the same as the abstract unit even though it inherits all its properties therefrom. Rather, the abstract unit is used to generate all the rational numbers (fractions). The word *rational* is redundant in front of the word *number* because to be a number implies rationality. The abstract unit starts off as a qualitative comparison of equal magnitudes. The **abstract magnitude** u is chosen **randomly** as a standard measure. So u : u is the unit. Then the natural numbers are formed as multiples of the unit, that is, k : u where k is a multiple of the unit. Next, the rational numbers are from ratios of ratios where the consequent ratio is always u : u. Rather than write k : u : u : u, we simply write k.

Also, if m : u and n : u are two multiples of the unit, then m:u :n:u is called a fractional ratio of magnitudes. Now, we want to deal only with numbers and since we know the unit, we omit the u and write m:n or n:m.

So how do we differentiate a ratio from a fraction? We introduce the vinculum symbol - that horizontal line which separates the numerator from the denominator, eg. $\frac{m}{n}$ or $\frac{n}{m}$. Thus, we write $\frac{2}{3}$ or $\frac{3}{2}$. The vinculum $(-)$ does not mean division, because we are defining the new symbols of fractions. Any division used has already taken place in the prior processes. Geometrically, it's very easy to divide any line segment into any number of equal parts. Algebraically, this is a different story - enter the obelus or division symbol, ÷, which denotes repeated subtraction (numerator minus denominator) if the numerator is greater than the denominator and terminates once the remainder is less than the denominator.

If _ : _ _ is a ratio, then its measure, that is, measure (_ : _ _) = $\frac{1}{2}$.

Contrary to popular academic misconception, nothing happens when the numerator is less than the denominator, example: $1 \div 3 = \frac{1}{3}$. All that happens in algebra, is that the obelus dots are discarded, the 1 goes to the top of the vinculum and the 3 goes to the bottom of the vinculum. In other words, nothing happens in algebra. The vinculum does not mean division (as per the obelus binary operator $\div$) when the numerator is less than the denominator. In fact, it needn't mean division vice-versa either, but then the operations of arithmetic on fractions become slightly more complex.

Division (ala obelus) is repeated subtraction and applies only when the first operand is greater than the second, that is, given $\frac{p}{q}$, p must be equal or greater than q.

For example, in order to convince me that $1 \div 3$ equals to $\frac{1}{3}$ using algebra, you would need to use the same process of repeated subtraction. The fact is that you can't, because there is only a remainder, that is, the numerator is already smaller than the denominator, so no subtraction takes place at all, i.e. no division à la obelus.

The truth of these facts is once again confirmed by realising that most academics have never understood division or even polynomial division, except by rote fashion. They also forget that any binary arithmetic operation is finite and cannot continue indefinitely.

The derivation of the rational numbers is complete. Notice that I made no reference to beliefs that are the essence of axioms and postulates. There are no axioms or postulates in mathematics.

Any magnitude that can't be measured by the unit or another magnitude other than itself, is called an *incommensurable magnitude*, not an *irrational number* as defined by mainstream academia. A magnitude is not a number!

There was no accident when Euclid defined a magnitude in Book V and number in Book VII, that is, an incommensurable magnitude (or magnitude that is incommensurate with any other magnitude) has no common measure with any other magnitude. As such, it cannot be called a number, because it can never be measured or described exactly by any other magnitude or established number. Examples are the constants π, e, $\sqrt{2}$, etc. These magnitudes are called incommensurable. Euclid called these *irrational magnitudes*, not *irrational numbers* (Elements Bk. X)

A number is the measure of a ratio.

π is a ***constant*** that is discovered from a ratio of magnitudes which has no common measure with any other ratio:

circle periphery : diameter

Through the Pythagorean theorem, we discover that $\sqrt{2}$ is also not a number.

But you may say, how is it that we can well define a square with area of 2 square units? The fact is that both $\sqrt{2}$ and 2 are magnitudes. It is possible to perform all arithmetic operations using magnitudes in geometry without any use of numbers.

One of the greatest mathematicians called Gauss agreed with me:

…3 is not as close to the true value of π as is 3.14, and 3.14159 is still closer. By adding additional places to the right of the decimal, it is possible to approximate the true value of π as closely as one likes. But Gauss insisted that one could not assume all the terms of the decimal expansion to be given to determine π exactly. To do so would involve an infinite number of terms, and thus comprise an actually infinite set of numbers, which Gauss refused to allow in rigorous mathematics [Dauben 1977, 861.]

What Gauss was also saying here that is not immediately clear to most, is that there is no such thing as an *infinite set*.

One can only approximately describe incommensurable magnitudes. It has been proved that there is no valid construction of irrational numbers, hence no real numbers too. Neither classes of equivalent Cauchy sequences nor Dedekind Cuts are valid constructions.

To begin, one should realise that any magnitude that cannot be measured exactly in terms of rational numbers, is not a number of any kind. It is the measure of a magnitude that results in a number. Not the partial measure of a magnitude, or even a point in the 'real' number line as Dedekind imagined which supposedly corresponds to a real number.

As an exercise, find the measure of the following ratios:

_ _ _ : _ _ _ _ = ?

_ _ : _ _ _ _ _ _ = ?

_ _ _ : _ = ?

What does it mean to reify a point on the number line?

Consider that all rational numbers can be constructed from a chosen magnitude which is normally called the 'unit'. Suppose we choose ____ (four underscores) as the standard unit. It's not necessary to use line segments. One could also use areas, masses, volumes or any other magnitude, but line segments are simple and that's why they are the easiest to use!

How do we reify the origin (zero) on a number line?

We simply place the units adjacently, that is, |____|____, where the vertical line is actually invisible and represents the marker or point. It takes up no space at all; just as well, because points have no size and we would run into contradictions otherwise. Thus, we can calibrate the first part of our number line with the symbol 0 meaning no magnitude. The first vertical line represents 0. The second vertical line represents the unit or 1. So, to reify the number two, we draw |____|____|____ and the third vertical line represents 2. Please note that on an actual number line, there are no spaces between the vertical lines and the line segments. In fact, the vertical bars take up no space whatsoever - they are the points or markers.

We can easily represent any rational number on the number line in the same way. For example, to represent $\frac{3}{4}$, we simply place four equal line segments (magnitudes) adjacently and call the sum of them all a unit. That is, |____|____|____|____| so that the second vertical line represents $\frac{1}{4}$, the third represents $\frac{2}{4}$ and the fourth vertical line represents $\frac{3}{4}$.

So, to reify a point involves accomplishing all of the following:

a. Constructing the line segments

b. Assigning the markers (as in calibration)

c. Providing a measure for each marker

We've looked at (a) and (b), but we still need to understand what is the meaning of (c).

A number is the measure of a magnitude.

Symbols are not numbers, but objects used in representing numbers. In the previous examples, we call the unit by '1'. Consequently, we call two units placed adjacently by '2' and so forth. Given that ____ is the unit, we can represent 2 and $\frac{1}{2}$ respectively as follows:

$\frac{2}{1} = |\_\_\_\_\_|\_\_\_\_\_| : |\_\_\_\_\_|$

$\frac{1}{2} = |\_\_\_\_\_| : |\_\_\_\_\_|\_\_\_\_\_|$

In each case, the abstract magnitude we imagine is well defined in terms of the given unit. Therefore, (c) requires that *any* given magnitude be expressible in terms of the chosen unit, that is, there exists a number that describes its measure in terms of the unit. On the number line, the measures are used to qualify the markers or points.

So, reification is a 3-step process. As long as we have our chosen unit and can represent all magnitudes in terms thereof, we can calibrate our number line with any chosen unit.

What about those magnitudes that refuse to be measured by the unit?

Consider that $\sqrt{2}$ is such an (incommensurable) magnitude. For starters, there is no unit we can choose to measure the hypotenuse of a right-angled isosceles triangle.

We realise the symptom of $\sqrt{2}$ from a right-angled isosceles triangle just as we realise the symptom of π from any circle circumference whose measure is attempted by the circle diameter.

We can never reify $\sqrt{2}$ or π on the number line, therefore it is impossible to calibrate any number line with a marker and a number that describes the measure of these incommensurable magnitudes. The symbols $\sqrt{2}$ and π are decidedly not measurements! They denote magnitudes or more accurately **constants** whose measure is not possible.

Mainstream academics are misinformed in their thinking that a Taylor series describes π or $\sqrt{2}$, for no matter how many digits are produced, the result is never representative of any incommensurable magnitude.

In fact, there are innumerably many other numbers with the same first n digits as π or $\sqrt{2}$.

Dedekind's claim to fame is that he could describe any incommensurable magnitude (erroneously called 'irrational' number) by two sets of rational numbers. Consider the following:

$D = \{d \in Q : d^2 < 2 \lor d < 0\}$ or $D = \{d \in Q : d < \sqrt{2}\}$

$U = \{u \in Q : u > \sqrt{2}\}$

Dedekind claimed foolishly that $\sqrt{2} = \{D, U\}$.

But in what way does Dedekind *describe* the incommensurable magnitude $\sqrt{2}$? Does he describe it exactly in terms of rational numbers? Can $\sqrt{2}$ be measured using the definition $\sqrt{2} = \{D, U\}$?

The following questions arise immediately in the mind of an astute reader:

1. The lower set D has no least upper bound that is recognised, except as $\sqrt{2}$. How can one write (or measure in a radix system such as base 10) $\sqrt{2}$ as a finite sum of known numbers, that is, rational numbers?
2. The upper set U has no greatest lower bound that is recognised, except as $\sqrt{2}$. How can one write $\sqrt{2}$ as a finite sum of known numbers, that is, rational numbers?
3. Is it a good idea to claim that $\sqrt{2}$ corresponds to a point on the 'real' number line, when we know that such a point cannot be reified. That is, while it is possible to mark off $\sqrt{2}$ geometrically, the invisible marker (point) cannot be associated with any known

number, except the symbol $\sqrt{2}$. But as yet, $\sqrt{2}$ has not been shown to be a number of any kind.

We've already learned that $\sqrt{2}$ cannot be reified on any number line.

It is easily seen that $\sqrt{2}$ is really all that Dedekind had to describe the **incommensurable magnitude**, that is, given by the hypotenuse of a right-angled isosceles triangle whose equal legs are of length 1 unit. The two sets $(-\infty,\sqrt{2})$ and $(\sqrt{2},\infty)$ tell one absolutely nothing about $\sqrt{2}$.

Rather than deal with the entire number line, in the following example, I choose without any loss of generality, only the interval $(0,1)$ to attempt my definition of 'irrational' number using Dedekind's idea of a cut. In the example, I try to find a unique partition of the number line such that $\frac{\sqrt{2}}{2}$ is well defined. Observe that $0<\frac{\sqrt{2}}{2}<1$.

Let's begin by arranging all the rational numbers with denominators not greater than 6 in the interval $[0,1]$ in ascending order:

$$R_6=\left(0,\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},1\right)$$

The elements of R_6 are irreducible fractions. Supposedly, we can sandwich an arbitrary 'irrational' number, say $\frac{\sqrt{2}}{2}$, in the interval $[0,1]$ between two neighbouring elements of R_6, that is, between the elements $\frac{2}{3}$ and $\frac{3}{4}$.

For example, $\frac{\sqrt{2}}{2}$ is between $\frac{2}{3}$ and $\frac{3}{4}$: $\quad \frac{2}{3}<\frac{\sqrt{2}}{2}<\frac{3}{4}$.

So, the ‘irrational’ number $\frac{\sqrt{2}}{2}$ divides R_6 into $S_6(9)$ and $L_6(9)$ as follows:

$$S_6(9) = \left(0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}\right)$$

$$L_6(9) = \left(\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1\right)$$

$\frac{2}{3}$ is the ninth element of R_6. The set $S_6(9)$ contains all the elements less than $\frac{\sqrt{2}}{2}$ and set $L_6(9)$ contains all the elements greater than $\frac{\sqrt{2}}{2}$.

Using the previous example, we can generalise it as follows:

$$R_n = (0, Q_2, \ldots, Q_m, Q_{m+1}, \ldots 1)$$

with

$$S_n(m) = (0, Q_2, \ldots, Q_m)$$

and

$$L_n(m) = (Q_{m+1}, \ldots 1)$$

Thus, the cut can be written as: $[S_n(m), L_n(m)]$.

As n increases, the cut presumably approaches some ‘irrational’ (or ‘real’) number k:

$lim_{n \to \infty} [S_n(m), L_n(m)] = k$ [D]

Notice that there are an innumerable number of partitions (Dedekind cuts) that sandwich the ‘irrational’ number k. Since n can never be infinite, there is no unique cut.

Furthermore, a cut does not define an ‘irrational’ number, because no finite m can be found such that [D] is unique.

And if no finite m can be found, then which of the innumerable ‘irrational’ numbers is k?

m is not infinite, it is defined as an ***integer*** in definition [D] which is the definition!

To convince yourself these claims are true, find R_7 as an exercise, then find m for $S_7(m), L_7(m)$.

For the same reason, a Cauchy sequence does not define a real number, because from the lower set of a D. Cut, one can obtain innumerably many equivalent Cauchy sequences exhibiting the same logic flaws as Dedekind cuts.

Let’s take the cut $(2,3) \cup [3,4]$ or $(-\infty,3) \cup [3,\infty)$.

These are equivalent without any loss of generality since the point referenced remains unchanged.

The main objection from mainstream academics is to this definition. Ironically, it simplifies Dedekind’s definition and nothing is lost. But let’s see how to find a Cauchy sequence.

Choose the general term to be

$$S_n = S_{n-1} + \frac{1}{2^{n-1}}$$

Let $S_1 = 2$

Then

$S_2 = 2 + \frac{1}{2} = 2.5$

$S_3 = 2.5 + \frac{1}{4} = 2.75$

$S_4 = 2.75 + \frac{1}{8} = 2.875$

So, the Cauchy sequence extracted from the lower set is

$$[a_1] = \{2.5;2.75;2.875;...\}$$

It's very easy to verify that the limit of this sequence is 3 and we are done.

By using the following definition, one can extract innumerably many other Cauchy sequences which converge to 3 also:

$$S_n = S_{n-1} + \frac{1}{r^{n-1}}$$

where $r > 1,\ n,\ r$ are natural numbers and $[a_r]$ is a Cauchy sequence.

To further convince yourself that D. Cuts are nonsense, try to use the above method in a similar way for the cut $(3,\pi) \cup (\pi, 4)$.

If you are astute, then you will notice that you would need a distance that converges to $|\pi - 3|$, but this difference is assumed to be possible in the very definition of the number being attempted. Do you think circularity in any definition adds to its well-formedness?

The gist of this comment is that if you have two numbers that are close to each other, there is no unique way of representing the D. Cuts as proved in my example. That is, both cuts would have an indeterminate m and span the same interval.

Objections:

"You aren't using the definition!"

Oh yes, I am! However, I am not bound by any malformed definition.

"A cut is a partition of rational numbers into two non-empty sets A and B, such that all elements of A are less than all elements of B, and A contains no greatest element."

A cut $[3,\pi) \cup [\pi, 4]$ is a partition of rational numbers into two non-empty sets $A = [3,\pi)$ and $B = [\pi, 4]$ such that all elements of A are less than all elements of B, and A contains no greatest element.

Since $[3,4]$ is a subset of the rational numbers, whatever applies to it, will also apply to $(-\infty, \pi) \cup [\pi, \infty]$.

If anyone claims that a radix representation is equal to a given number in a certain base, then one must provide an equivalent fraction in that base.

For example, $\frac{1}{4} = \frac{2}{10} + \frac{5}{100} = \frac{25}{100}$ [expressed in base 10]

So if you claim that $\frac{1}{3} = 0.3333\ldots$, then you must provide an equivalent fraction in base 10 as demonstrated with $\frac{1}{4}$. Same for $0.999\ldots$ whatever this means.

If you can't provide an equivalent fraction, then the radix representation does not represent that fraction.

Mainstream academics do not understand that you can't have **three different rules for numbers**. They claim:

number = Limit of equivalent Cauchy sequences

RULE 1.
In the case of $\frac{1}{4}$, the limit is the sum.

RULE 2.
In the case of $\frac{1}{3}$, the limit is not the sum.

RULE 3.

However, in the case of π and $\sqrt{2}$, no one has a clue what the limit is, except that it's some symbol or its partial sums can be approximated by some formula.

Rule 3 is particularly amusing, because it assumes the completed infinite radix representation (commonly known by the gibberish “infinite decimal expansion”) is possible. Modern academics, in the spirit of Euler, define numbers by this myth.

Interesting to note, is that if one accepts the infinite unique representation, then whatever is the limit, it is not equal to the completed infinite representation, because if it were, then guess what? **All "irrational numbers" would be *rational numbers*.**

In conclusion, the objects you think of as ‘irrational’ numbers cannot be defined as Dedekind cuts. In fact, Dedekind cuts really attempt a definition via use of a function. But associating the value of a function with a particular cut does not make it a number! For example, I could define the incommensurable magnitude e as follows:

$$D = \{d \in Q:\ (1+d)^{\frac{1}{d}} < e\ \ \forall\ \ d > 0\}$$

$$U = \{u \in Q: (1+d)^{\frac{1}{d}} > e\ \ \forall\ d < 0\}$$

where $f(d) = (1+d)^{\frac{1}{d}}$ and $f(0) = e$. So, while $f(0)$ corresponds to e, it is not a measurement of the incommensurable magnitude e.

For the same reasons, an equivalence class of Cauchy sequences does not define an 'irrational' or 'real' number. In fact, the Cauchy definition is far worse, because it is circular and presumes the existence of an 'irrational' number.

Did Dedekind discover anything new at all? The surprising answer to this, is that he took the idea straight from Archimedes (see proposition 3 and 4, On Spirals), who used it to approximate π. Much later, Riemann committed some plagiarism of his own by usurping the same ideas of Archimedes in his formulation of the definite integral.

All arithmetic operations are defined geometrically.

Long before numbers, the operations of arithmetic were all defined geometrically using the symmetrical geometric object known as the circle. These operations are all different types of measurement.

A difference $(-)$ is that measure by which two magnitudes can be made the same.

To compare any magnitudes quantitatively in order to determine the difference, the numbers **must** be expressed in terms of the **abstract unit** 1.

Finding the difference of numbers composed of multitudes of units is straight forward:

If the numbers are $4/1$ and $7/1$, the difference which can make these the same is 3/1:

$7 - 3 = 4$ and $4 - (-3) = 7$

If the numbers are $\frac{5}{3}$ $\left(\text{or } \frac{10}{6}\right)$ and $\frac{7}{2}$ $\left(\text{or } \frac{21}{6}\right)$, then we still find the difference by comparing numbers that are expressed in terms of units, that is, 10 and 21.

The difference is therefore $\frac{11}{6}$.

$$\frac{21}{6} - \frac{11}{6} = \frac{10}{6} \qquad \frac{10}{6} - \left(-\frac{11}{6}\right) = \frac{21}{6}$$

because $\frac{7}{2} = \frac{7+7+7}{2+2+2} = \frac{21}{6}$ **and** $\frac{5}{3} = \frac{5+5}{3+3} = \frac{10}{6}$

Equivalent fractions are derived in Euclid’s Elements using similar triangles and the fact that if $\frac{p}{q} = \frac{r}{t}$, then

$\frac{p}{q} = \frac{p+r}{q+t} = \frac{r}{t}$.

A sum $(+)$ is that measure by which two differences (magnitudes) are considered as one. Another way of seeing this, is that the two magnitudes are both required to measure the sum.

For example, the sum of $\frac{10}{6}$ and $\frac{11}{6}$:

$$\frac{10}{6}-\left(-\frac{11}{6}\right)=\frac{21}{6}$$

Adding a number to another means removing the need for it. Hence, $\left(-\frac{11}{6}\right)$ is what is needed with $\frac{10}{6}$ to measure $\frac{21}{6}$. Taking the sum therefore removes the need:

$$\frac{10}{6}+\frac{11}{6}=\frac{21}{6}$$

Thus, between positive numbers, the arithmetic operator of sum $(+)$ is still a difference.

A quotient $(/)$ is that measure which is realised by using differences (magnitudes) composed of whole units. In realising a measure, the unit is typically used. However, the unit which is also a divisor, can be of different lengths. Return to this paragraph after you have completed reading the chapter to acquire a better understanding. A quotient is simply a ratio of magnitudes.

Every number is by definition a quotient.

However, the obelus or division operator $(\div)$, is that measure which applies only to cases where the numerator is greater than the denominator. For example, $p \div q = \frac{p}{q}$ regardless of the size of p and q. However, if $p < q$, then nothing happens in algebra because the division operator is a finite process. Moreover, $\frac{p}{q}$ is a number or quotient, and no division is pending. In cases where $\frac{p}{q}$ is not in irreducible form, the cancellation process has nothing to do with division but everything to do with proportion.

Consider the number $\frac{20}{12}$. It is not in irreducible form, but can be reduced by the proportion theorem of Euclid. Since $\frac{5}{3}$ is in proportion to $\frac{20}{12}$, it is true that

$$\frac{20}{12} = \frac{20-5-5-5}{12-3-3-3} = \frac{5}{3}$$

5 and 3 are relatively prime to each other even though they are prime numbers, hence $\frac{5}{3}$ cannot be reduced further, except perhaps into a sum of units and parts of a unit, that is, $1 + \frac{2}{3}$.

Since the division operator is a measure, it can be used to represent numbers in a given radix system. For example, $\frac{1}{4}$ in base 10, is represented by 0.25 which follows from a finite division:

$$0.25 = 25 \div 100 = \frac{25}{100} = \frac{25-1(twenty\ four\ times)}{100-4(twenty\ four\ times)} = \frac{1}{4}$$

Because $\frac{25}{100}$ is proportional to $\frac{1}{4}$.

To find $\frac{1}{4}$ in base 10:

$$\frac{1}{4} \times \frac{10}{10} = \frac{10}{4} \times \frac{1}{10} = \left(2 + \frac{1}{2}\right) \times \frac{1}{10} = \left(2 \times \frac{1}{10}\right) + \left(\frac{1}{2} \times \frac{10}{100}\right)$$

$= \left(2 \times \frac{1}{10}\right) + \left(\frac{10}{2} \times \frac{1}{100}\right)$

$= \left(2 \times \frac{1}{10}\right) + \left(5 \times \frac{1}{100}\right) = 0.25 = \frac{2}{10} + \frac{5}{100} = \frac{25}{100}$

Any radix system r requires that a given number $\frac{p}{q}$ is measured using only the coefficients $(c_i \;\; and \;\; f_i)$ and factors $\left(r^n \;\; and \; \frac{1}{r^n}\right)$ in the polynomial

$$\frac{p}{q} \;=\; \ldots + c_n r^n + c_{n-1} r^{n-1} + \ldots c_1 r + c_0 \;\; . \frac{f_1}{r} + \frac{f_2}{r^2} + \frac{f_3}{r^3} + \ldots$$

Thus, given $r = 10$, it follows that $\frac{1}{4} = 0 + \frac{2}{10} + \frac{5}{100}$ or simply 0.25

In order for $\frac{p}{q}$ to be represented in radix system r, the prime factors of q must all be prime factors of r also.

The above is an **important number theorem** ignored by the BIG STUPID (mainstream academics).

Note that $\frac{1}{3}$ cannot be represented in base 10.

To summarise, the division operator $(\div)$ is a repeating subtraction process terminating when the remainder is less than the difference (divisor).

Example: $18 \;\div\; 5 = \;?$

i. $18 - 5 = 13$

ii. $13 - 5 = 8$

iii. $8 - 5 = 3$

So, 3×5 and $\frac{3}{5} \times 5$ (three fifths of 5) are required to measure 18. That is, 5 alone cannot measure 18. And so, the division result is $5 + \frac{3}{5}$.

In this process, 5 acts as the measuring "unit". Also note that the process stops once the remainder (3) is less than the measuring unit (also known as divisor 5 in the example).

A product $(\times)$ is that measure which is realised by using differences (magnitudes) composed of one or more equal parts of the unit. In all cases, the operands of a product both measure the product individually. For example, $3 \times 2 = 6$ means that 6 is measured by 3 and 6 is also measured by 2. Return to this paragraph once you have completed reading the chapter to gain a better understanding. A product is simply a measure using a reciprocal quotient.

The product $\frac{2}{1} \times \frac{3}{1}$ can be determined in one of two ways.

$\frac{2}{1} \div \frac{1}{3} = \frac{\frac{2}{1}}{\frac{1}{3}} = \frac{\frac{2}{1}+\frac{2}{1}+\frac{2}{1}}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \frac{6}{1}$ because $\frac{\frac{2}{1}}{\frac{1}{3}}$ is proportional to itself.

OR

$\frac{3}{1} \div \frac{1}{2} = \frac{\frac{3}{1}}{\frac{1}{2}} = \frac{\frac{3}{1}+\frac{3}{1}}{\frac{1}{2}+\frac{1}{2}} = \frac{6}{1}$ because $\frac{\frac{3}{1}}{\frac{1}{2}}$ is proportional to itself.

Note that in every case, the operation is done in **units** and the most primitive operator is the **difference** operator. Now is the time to study Appendix B.

Derivation of arithmetic operators.

The Elements of Euclid is least understood with respect to the description of the derivation of number from the measure of magnitude (size).

But even more mysterious to mainstream academics is which operator came first, that is, which of the four basic arithmetic operators is the most primitive from which all the others can be derived.

If your intuition tells you that addition is the most primitive, then you are wrong. There is no way to derive the operation of subtraction from addition. In fact, the most primitive operator is the difference or subtraction operator.

In the expression $\frac{p}{d} - \frac{q}{d}$, the difference operator is $-$

The expression is evaluated by simply taking the difference of p and q over d, where p, q and d are all integers.

We can derive the addition operator as follows:

$$\frac{p}{d} - -\frac{q}{d} \qquad \text{where} \quad - - \quad \text{is denoted by} \quad +$$

The expression $\frac{p}{d} + \frac{q}{d}$ is evaluated by taking the sum of p and q.

Now that this is in place, we can subtract and add fractions with different denominators.

To evaluate the expression $\frac{p}{r} - \frac{q}{s}$, we need to find equivalent fractions whose denominators are the same.

The solution is simple. We simply add $\frac{p}{r}$ to itself $s - 1$ times and we also add $\frac{q}{s}$ to itself $r - 1$ times. Let's see an example.

$$\frac{2}{3} - \frac{3}{5} = \frac{2+2+2+2+2}{3+3+3+3+3} - \frac{3+3+3}{5+5+5} = \frac{10}{15} - \frac{9}{15} = \frac{1}{15}$$

Addition works the same way:

$$\frac{2}{3} + \frac{3}{5} = \frac{2+2+2+2+2}{3+3+3+3+3} + \frac{3+3+3}{5+5+5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$$

Now we've seen that we can do arithmetic involving differences and sums. So, what is the next operator that we derive?

Division is derived from the previous two operators of difference and sum as follows:

$$\frac{p}{q} \div \frac{r}{s} = \frac{\frac{p}{q}}{\frac{r}{s}}$$

If $\frac{p}{q} < \frac{r}{s}$, there is nothing left to do. For example,

$$\frac{1}{1} \div \frac{3}{1} = \frac{\frac{1}{1}}{\frac{3}{1}} = \frac{1}{3} \quad \text{because } 1 < 3.$$

If however $\frac{p}{q} > \frac{r}{s}$, then we use $\frac{r}{s}$ to measure $\frac{p}{q}$ by repeated subtraction, stopping only when the remainder is less than $\frac{r}{s}$.

Let's see a simple example.

$$10 \div 4$$

I will omit the denominators which are 1. That is, the above is actually equivalent to

$$\frac{10}{1} \div \frac{4}{1}$$

So, $10 - 4 - 4 = 2$, that is, $\frac{10}{4} = 2$ remainder 2. Since $2 < 4$, we write $2 + \frac{2}{4}$ or simply $2\frac{2}{4}$.

If the denominators are 1 as in the previous slide, the process of division is simple.

What about $\frac{2}{3} \div \frac{4}{5}$?

Well, $\frac{2}{3} \div \frac{4}{5} = \frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2+2+2+2+2}{3+3+3+3+3}}{\frac{4+4+4}{5+5+5}} = \frac{\frac{10}{15}}{\frac{12}{15}} = \frac{10}{12}$

$\frac{10}{12}$ is not in its irreducible form, but this is not a problem once we derive multiplication which is the last of the four basic arithmetic operators.

Now, we can derive multiplication from division as follows:

$$\frac{p}{q} \times \frac{r}{s} = \frac{p}{q} \div \frac{1}{\frac{r}{s}} \quad \textbf{OR} \quad \frac{p}{q} \times \frac{r}{s} = \frac{r}{s} \div \frac{1}{\frac{p}{q}}$$

Both division expressions must be defined and result in the same value.

Example:

$$\frac{2}{1} \times \frac{3}{1} = \frac{2}{1} \div \frac{1}{3} = \frac{\frac{2}{1}}{\frac{1}{3}} = \frac{\frac{2+2+2}{1+1+1}}{\frac{1}{3}} = \frac{\frac{6}{3}}{\frac{1}{3}} = \frac{6}{1} = 6$$

The same product/multiplication can be accomplished as follows:

$$\frac{2}{1}\times\frac{3}{1}=\frac{3}{1}\ \div\ \frac{1}{2}=\frac{\frac{3}{1}}{\frac{1}{2}}=\ \frac{\frac{3+3}{1+1}}{\frac{1}{2}}\ =\frac{\frac{6}{2}}{\frac{1}{2}}=\frac{6}{1}=6$$

You might be wondering how one gets from $\frac{\frac{6}{2}}{\frac{1}{2}}$ to $\frac{6}{1}$. Well, the denominators are the same, which means the measure is common to both fractions, that is, both are measured in halves or a unit of two equal parts.

If we write $\frac{6}{1}$, this is the same as $\frac{\frac{6}{1}}{\frac{1}{1}}$, hence we can drop the denominators if they are the same. For example, when we write $2\div 3$, we really mean $\frac{\frac{2}{1}}{\frac{3}{1}}$ since $2\div 3=\frac{2}{3}$ or equivalently $\frac{2}{1}\div\frac{3}{1}=\frac{\frac{2}{1}}{\frac{3}{1}}$ by the definition of division.

Now that we have derived all the four basic arithmetic operations, we are ready to discuss other forms.

In a previous slide, we saw a division example that produced a quotient of $\frac{10}{12}$. To reduce such a quotient, we once again need a proportional fraction, that is, one which can be obtained by finding common factors.

We see that the highest common factor of 10 and 12 is 2 and we can find this using the Euclidean algorithm which is described in Book 7.

The Euclidean algorithm is a repeated measure where we try to measure the larger number (12) with the smaller (10).

$$12=10\,(1)+2$$
$$10=2\,(5)+0$$

We place the larger number 12 on the left and express it in terms of the smaller number on the right. It turns out that the largest number that can measure both 10 and 12 is 2.

So, $\frac{10}{12} = \frac{5\times2}{6\times2} = \frac{\frac{5}{\frac{1}{2}}}{\frac{6}{\frac{1}{2}}} = \frac{5}{6}$ because the denominator $\frac{1}{2}$ measures both the fractions. $\frac{5}{6}$ cannot be further reduced because the highest **integer** factor that **divides** or **measures** both, is 1.

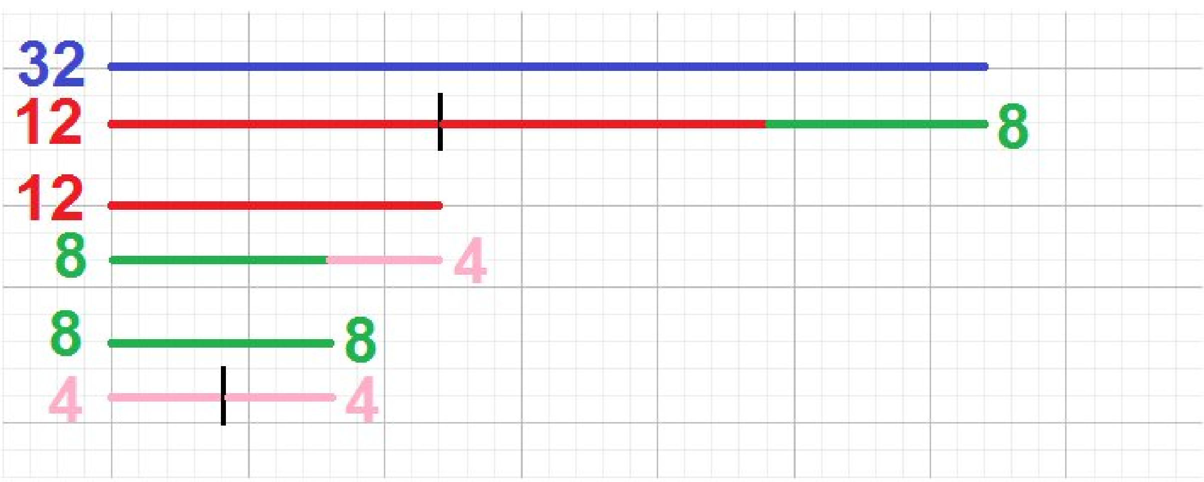
Let's find the greatest factor that measures 12 and 32.

1. $32 = 12\,(2) + 8$

$12 = 8\,(1) + 4$

$8 = 4\,(2) + 0$

So highest common factor is 4.



Let's find the greatest factor that measures 111 and 421.

2. $421 = 111\,(3) + 88$

$111 = 88\,(1) + 23$

$$88 = 23\,(3) + 19$$

$$23 = 19\,(1) + 4$$

$$19 = 4\,(4) + 3$$

$$4 = 3\,(1) + 1$$

$$3 = 1\,(3) + 0$$

So highest common factor is 1. The remaining number theorems in Euclid's Elements include prime numbers and many other interesting facts which require a separate course of study.

Chapter 6: How we got algebra

The Ancient Greeks used many equations even though they didn't use the same symbols of modern algebra. For example, one well known equation is the result of the Pythagorean theorem: $a^2 = b^2 + c^2$.

Algebra is established in Book V of Euclid's Elements from which all arithmetic operations are derived using only magnitudes (**not numbers!**). Book V also covers the theory of proportional ratios and numbers.

Some other well-known equations:

$p = h \times w$ (Generic planar area known as a plane number)

$s = h \times w \times b$ (Generic cubic volume known as a solid number)

$\pi = \frac{c}{2r}$ (The circle constant)

$A = \pi r^2$ (Area of circle)

$y = x^2$ (Equation of parabola)

The equations of all the conics and many trigonometric identities were known too.

Before algebra could be realised, it was imperative that a means was possible to denote magnitudes or sizes, whether or not describable by a number. The best approach was through the use of numbers and symbols.

Algebra was developed in order to communicate the ideas of geometry without having to refer to diagrams (visualisations or instantiations) all the time

It is clear that in algebra both numbers and symbols (for incommensurable magnitudes) are treated the same. There is no difference between how x and π are treated, and these are for all intents and purposes unknowns.

The four basic arithmetic operations that are used by equations in algebra are described in Book 5, Proposition 12 of the Elements. This proposition deals with magnitudes, but the results are applied also to numbers that are the measure of magnitudes.

Zero is not a number, never mind a rational number. If you try to express 0 as a rational number, then you must express it using other numbers, not zero itself. There is no $\frac{p}{q}$ with p and q not equal to 0 such that $0 \;= \frac{p}{q}$. On the other hand, there is no k such that $0 \times k = 1$.

Hence, the measure of zero using any k equal partitions of the unit, that is, $\frac{1}{k}$, is simply not possible. However, this is true for every other (rational) number. The Ancient Greeks rejected zero because it is not a magnitude, but a symbol for no magnitude or a non-magnitude. **The inclusion of a zero magnitude would render most of the propositions in Book V invalid. Zero means 'no number'.**

While useful as a placeholder, zero is decidedly not a number. In fact, it is not even required at all in mathematics, but it is useful for disambiguation in representation. Unfortunately, due to ignorant mainstream academics, the myth that zero is a number is now firmly entrenched in mainstream mathematics academia.

For example, consider the base ten radix system or decimal system represented as a template:

$$...Thousands \quad Hundreds \quad Tens \quad Units \ . \ Tenths \ Hundredths \quad Thousandths..$$

We can represent a rational number in a finite and unique way using only the digits $1\ through\ 9$, any $\frac{p}{q}$ with p and q integers if the prime factors of q are also factors of ten. To represent ten, we simply place the digit 1 in the Tens column of the template. Nothing else is required.

To represent $\frac{1}{4}$, we place 2 in the Tenths column and 5 in the Hundredths column. We cannot represent $\frac{1}{3}$ in base ten, because 3 is not a prime factor of ten. Similarly, we cannot represent $\frac{4}{18}$ in base ten, because ten does not contain all the prime factors of 18. Therefore, the only fractions that can be represented in base ten are those whose denominator contains only the prime factors 2 and 5.

Without the template, if we simply wrote 1, the meaning would not be clear. Is it one, ten, one hundred, one tenth, one hundredth, etc? So adding the correct number of zeroes removes the ambiguity. Writing 0.003 means three thousandths and not 300, 30, 3, $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{10000}$ and so on.

Therefore, 0 is not required at all in mathematics, but it is extremely useful as a place-holder, for disambiguation and also in describing the properties of equations, such as roots.

To qualify as a number, a given number $\frac{p}{q}$ must have the property that 1 can be divided into q equal parts. Only the *rational numbers* qualify as numbers.

$\frac{p}{10}$ means p equal parts of 1 that has been divided into 10 parts.

$\frac{p}{0.1}$ means $\frac{10p}{1}$ which means $10p$ parts of 1.

Therefore 0 is not a rational number. It's not a number at all. Zero measures nothing except perhaps itself in an absurd way. There is nothing remarkable about anything measuring itself as we can only understand the measure of a given magnitude by using other magnitudes, such as a standard magnitude that is called a *unit*.

Zero is neither positive nor negative, for to be either, there must exist a unique additive inverse, but zero is its own additive inverse which is absurd. In any case, the additive inverse requires that a given number has a sign. The sign of zero cannot be both negative and positive.

In algebra, one has to remember that "arithmetic" with zero must be impotent, that is, it cannot change the *value* or *sense* of the equation or render it undefined. For example:

Let $a = b$

$a^2 = ab$

$a^2 - b^2 = ab - b^2$

$(a - b)(a + b) = b(a - b)$

$a + b = b$

$2b = b \;\therefore 2 = 1$

The factor $a - b$ is equal to zero and division or any arithmetic by a non-number is not possible.

We can see that addition and subtraction by 0 never affects an equation. However, multiplication and division will inevitably change the sense and value of the equation.

$a = a$

$a + 0 = a + 0$

$a - 0 = a - 0$

$a \times 0 = a \times 0 \ \rightarrow 0 = 0$ which is not the same as $a = a$.

Here, the sense of the equation relates a to a. Multiplication by 0, relates 0 to 0 and changes the value of both sides, thus the sense and value are altered.

$a \div 0 = a \times \frac{1}{0} = \frac{a}{0}$ which is meaningless. Therefore, 0 is safe to use, as long as it doesn't change the sense and value of the equation. What is meant by sense and value? Sense describes the relationship between two sides of an equation and value tells one the measure of either side of the equation.

In the example $a^2 = ab$, a^2 is related to ab for any value – this is the sense of the equation. If the equation is manipulated incorrectly as shown, this relationship is destroyed, that is, $2b = b$ or $2a = a$. The value describes the size of a^2 or ab or b^2.

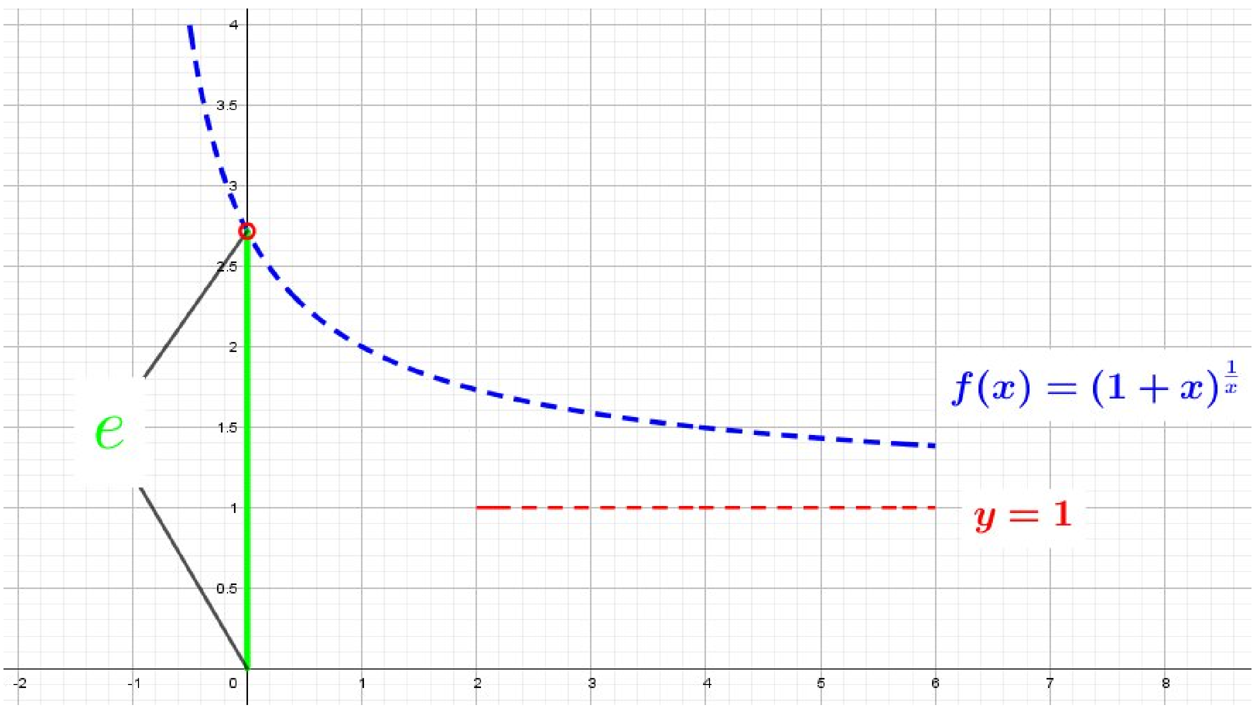
Thus, whilst 0 is immensely useful, it brings along its own bag of snares because it does not behave as do all numbers. How can it? It's not a number.

The main idea of equations is to remember that whatever operation is performed on either side, it must not affect the value of either side.

Also, it must not affect the sense, that is, the relationship must remain intact (unchanged), which is only possible if the operation is carried out on both sides.

The magnitude e.

This magnitude can be realised as the distance of the straight green line between the origin and the red point as shown in the illustration that follows:



The function e is easily derived from a binomial.

$$f\,(x{,}n) = (1 + xn)^{\frac{1}{n}} = 1 + \left(\tfrac{1}{n}\ 1\right)xn + \left(\tfrac{1}{n}\ 2\right)(xn)^2 + \dots$$

$$\left(\tfrac{1}{n}\ 1\right) = \frac{\frac{1}{n}!}{1!\left(\frac{1}{n}-1\right)!} = \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)!}{\left(\frac{1}{n}-1\right)!} = \frac{1}{n}$$

$$\binom{\frac{1}{n}}{2} = \frac{\frac{1}{n}!}{2!\left(\frac{1}{n}-2\right)!} = \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)!}{2!\left(\frac{1}{n}-2\right)!} = \frac{\frac{1}{n}\left(\frac{1-n}{n}\right)}{2!} = \frac{\left(\frac{1-n}{n^2}\right)}{2!} = \frac{\frac{1}{n^2}-\frac{1}{n}}{2!}$$

So,

$$f\,(x,n) = (1+xn)^{\frac{1}{n}} = 1 + \frac{1}{n}xn + \frac{\frac{1}{n^2}-\frac{1}{n}}{2!}(xn)^2 + \ldots$$

$$f\,(x,n) = (1+xn)^{\frac{1}{n}} = 1 + x + \frac{1-n}{2!}x^2 + \ldots$$

$$f\,(x,n) = (1+xn)^{\frac{1}{n}} = 1 + x + \frac{x^2}{2!} - \frac{x^2 n}{2!} + \ldots$$

$$\therefore f\,(x,0) = 1 + x + \frac{x^2}{2!} + \ldots$$

But $e^x = f(x,0)$ and so

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

From the previous function, we see that e is well defined by

$e = e^1 = f(1,0).$

Contrary to mainstream thought, there is no hole in the function f at the red point $(0,e)$ and it's also easy to see that $e^0 = 1 = f(0,0)$.

A difference $(-)$ is that measure by which two magnitudes can be made the same.

To compare any magnitudes quantitatively in order to determine the difference, the numbers **must** be expressed in terms of the **abstract unit** 1.

Finding the difference of numbers composed of multitudes of units is straight forward:

If the numbers are $4/1$ and $7/1$, the difference which can make these the same is $3/1$:

$7 - 3 = 4$ and $4 -(-3) = 7$

If the numbers are $\frac{5}{3}$ $\left(or \frac{10}{6}\right)$ and $\frac{7}{2}$ $\left(or \frac{21}{6}\right)$, then we still find the difference by comparing numbers that are expressed in terms of units, that is, 10 and 21.

The difference is therefore $\frac{11}{6}$.

$$\frac{21}{6} - \frac{11}{6} = \frac{10}{6} \qquad \frac{10}{6} -\left(-\frac{11}{6}\right) = \frac{21}{6}$$

because $\frac{7}{2} = \frac{7+7+7}{2+2+2} = \frac{21}{6}$ **and** $\frac{5}{3} = \frac{5+5}{3+3} = \frac{10}{6}$

Equivalent fractions are proved in Euclid's Elements using similar triangles and the fact that if $\frac{p}{q} = \frac{r}{t}$, then

$\frac{p}{q} = \frac{p+r}{q+t} = \frac{r}{t}$.

Use similar triangles to convince yourself these claims are true.

A sum $(+)$ is that measure by which two differences (magnitudes) are considered as one.

For example, the sum of $\frac{10}{6}$ and $\frac{11}{6}$:

$$\frac{10}{6} -\left(-\frac{11}{6}\right) = \frac{21}{6}$$

Adding a number to another means removing the need for it. Hence, $\left(-\frac{11}{6}\right)$ is what is needed with $\frac{10}{6}$ to measure $\frac{21}{6}$. Taking the sum therefore removes the need:

$$\frac{10}{6}+\frac{11}{6}=\frac{21}{6}$$

Thus, between positive numbers, the arithmetic operator of sum $(+)$ is still a difference.

A quotient $(/)$ is that measure which is realised by using differences (magnitudes) composed of whole units.

Every number is by definition a quotient.

However, the obelus or division operator $(\div)$, is that measure which applies only to cases where the numerator is greater than the denominator. For example, $p \div q = \frac{p}{q}$ regardless of the size of p and q. However, if $p < q$, then nothing happens in algebra because the division operator is a finite process. Moreover, $\frac{p}{q}$ is a number or quotient, and no division is pending. In cases where $\frac{p}{q}$ is not in irreducible form, the cancellation process has nothing to do with division but everything to do with proportion.

Consider the number $\frac{20}{12}$. It is not in irreducible form, but can be reduced by the proportion theorem of Euclid. Since $\frac{5}{3}$ is in proportion to $\frac{20}{12}$, it is true that

$$\frac{20}{12}=\frac{20-5-5-5}{12-3-3-3}=\frac{5}{3}$$

5 and 3 are relatively prime to each other even though they are prime numbers, hence $\frac{5}{3}$ cannot be reduced further, except perhaps into a sum of units and parts of a unit, that is, $1 + \frac{2}{3}$.

In order for $\frac{p}{q}$ to be represented in radix system r, the prime factors of q must all be prime factors of r also.

The above is an **important number theorem** ignored by the BIG STUPID (mainstream academics).

Note that $\frac{1}{3}$ cannot be represented in base 10.

To summarise, the division operator $(\div)$ is a repeating subtraction process terminating when the remainder is less than the difference (divisor).

Example: $18 \div 5 = \ ?$

i. $18 - 5 = 13$

ii. $13 - 5 = 8$

iii. $8 - 5 = 3$

So, 3×5 and $\frac{3}{5} \times 5$ (three fifths of 5) are required to measure 18. That is, 5 alone cannot measure 18. And so, the division result is $5 + \frac{3}{5}$.

In this process, 5 acts as the measuring "unit". Also note that the process stops once the remainder (3) is less than the measuring unit (also known as divisor 5 in the example).

A product $(\times)$ is that measure which is realized by using differences (magnitudes) composed of one or more equal parts of the unit.

The product $\frac{2}{1} \times \frac{3}{1}$ can be determined in one of two ways.

$\frac{2}{1} \div \frac{1}{3} = \frac{\frac{2}{1}}{\frac{1}{3}} = \frac{\frac{2}{1}+\frac{2}{1}+\frac{2}{1}}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \frac{6}{1}$ because $\frac{\frac{2}{1}}{\frac{1}{3}}$ is proportional to itself.

OR

$\frac{3}{1} \div \frac{1}{2} = \frac{\frac{3}{1}}{\frac{1}{2}} = \frac{\frac{3}{1}+\frac{3}{1}}{\frac{1}{2}+\frac{1}{2}} = \frac{6}{1}$ because $\frac{\frac{3}{1}}{\frac{1}{2}}$ is proportional to itself.

Note that in every case, the operation is done in **units** and the most primitive operator is the **difference** operator.

Chapter 7: The arithmetic mean

For as long as I can remember, educators have been calculating class averages also known as arithmetic means. A class average is generally the arithmetic mean of all the students' scores. I have never understood how a class average can tell anyone anything that is reliable or useful from any perspective. "Whoa!" you may say. Well, let's see exactly what the *arithmetic mean* tells us.

Suppose that a class average is 90%, then all this tells one, is the following:

If the students shared their marks, so that each student scored the same, then each student would have a score of 90%.

But can students ever share their marks? Of course not. This is ridiculous. Someone asked the following question on that internet site called Quora:

If my class average in biology is 90 *and if I get* 85*, then does this mean I am below average in biology? Does this mean that I'm bad or below average in biology?*

In fact, a class average tells one nothing about the students' abilities or how well they performed.

Suppose in a class of ten students, the following scores are obtained:

$80, 80, 80, 85, 90, 95, 95, 95, 100, 100$ for a total of 900

Clearly, the student who obtained 85%, is better than thirty percent of his class.

No educator I have ever met, has understood that an arithmetic mean makes sense only when ***redistribution*** also makes sense. In such cases, there is no need to check for *data outliers* or other *data anomalies* because the inference is clear and correct. The word ***mean*** implies

middle but ironically the arithmetic mean has nothing to do with the ***middle of anything***!

For example, it is silly to calculate the arithmetic mean of the heights of a group of people. What does this mean? Can you chop off parts of each individual and redistribute them so that all in the group are of the same height? Preposterous! In such a scenario, the only useful information would be the range of heights.

You might be tempted to think that the class average helps you as an educator to determine the progress of your students. **You would be wrong!**

Consider the following scores in a class of ten students:

$30, 30, 30, 30, 30, 30, 30, 90, 100, 100$ for a total of 500

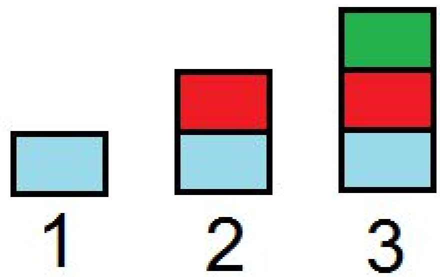
An ignorant inference is that students are all performing at the average mark because the class average is 50%, but 70% of the students scored below the average!

An arithmetic mean leads to correct inference, if and only if, ***redistribution makes sense****.*

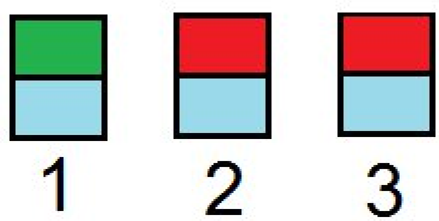
Definition of Arithmetic Mean:

Given any set of numbers or magnitudes, the arithmetic mean indicates the value that each number would have, if all the numbers in the set were made to be equal through a process of redistribution.

Suppose that numbers are represented by blocks as in the following diagram:



It's easy to see that if the green block is moved from pile 3 to pile 1, then all three piles will be level and equal:



The arithmetic mean is clearly given by: $\frac{1+2+3}{3} = \frac{6}{3} = 2$. As you'll see later, it turns out this *process of redistribution* used in determining the arithmetic mean, is precisely what happens when you use integral calculus to determine the area between a curve and some axis! We'll see examples of this later.

Chapter 8: The lack of rigour in mainstream calculus and its flaws

In her book, *The Origins of Cauchy's Rigorous Calculus*, Judith Grabiner starts the preface with:

"Augustin-Louis Cauchy gave the first reasonably successful rigorous foundation for calculus. Beginning with a precise definition of limit, he initiated the nineteenth century theories of convergence, continuity, derivative, and integral."

Neither Cauchy nor anyone else ever provided a rigorous formulation of calculus before I produced the first and only rigorous formulation in human history. The problems with calculus had nothing to do with a precise definition of limit, because neither the derivative or integral require the limit concept or related theory as has been proved in the New Calculus and to be demonstrated in the chapters to come.

The misguided and deceptive story of rigour began with **Cauchy's definitions**. Cauchy used questionable or ill-formed terminology such as *finite value*, *infinity, infinitesimal, limit*, *infinitely small* and *decreasing indefinitely*.

Cauchy's definition of limit:

When the values successively attributed to the same variable approach indefinitely a fixed value, eventually differing from it by as little as one could wish, that fixed value is called the limit of all the others.

The fact that a *value* of a difference quotient or some partial sum of a sequence is a magnitude, is proved by convergence without even knowing anything about the limit which is the value being investigated.

However, the measure of the *value,* may not even be possible, except perhaps as a recognisable symbol, that is, the constants π, e or $\sqrt{2}$. It's certainly untrue that one can make it differ by as little as one could wish, because if one wishes **no difference**, then this is not possible in the case of a magnitude that is incommensurate with any other such as π, e or $\sqrt{2}$.

The famous mathematics historian Carl Boyer had this to say:

"Cauchy had stated in his Cours d'analyse that irrational numbers are to be regarded as the limits of sequences of rational numbers. Since a limit is defined as a number to which the terms of the sequence approach in such a way that ultimately the difference between this number and the terms of the sequence can be made less than any given number, the existence of the irrational number depends, in the definition of limit, upon the known existence, and ***hence the prior definition, of the very quantity whose definition is being attempted****.*

That is, one cannot define the number $\sqrt{2}$ as the limit of the sequence 1, 1.4, 1.41,1.414, ... because to prove that this sequence has a limit one must assume, in view of the definitions of limits and convergence, the existence of this number as previously demonstrated or defined. Cauchy appears not to have noticed the circularity of the reasoning in this connection, but tacitly assumed that every sequence converging within itself has a limit."

The History of Calculus and its Conceptual Development'
(Page. 281) Carl B. Boyer

Boyer realised the circularity of Cauchy's arguments. Oddly enough, no one else in the mainstream seemed to notice.

Cauchy's definition of infinitesimal:

When the successive absolute values of a variable decrease indefinitely in such a way as to become less than any given quantity, that variable becomes what is called an infinitesimal. Such a variable has zero for its limit.

As far as Cauchy was concerned, an *infinitesimal* is actually a *variable expression*, rather than an actual number, which becomes very small. So small in fact, that it is less than any given quantity which is greater than 0. To believe this, you need faith that there is a number that immediately succeeds 0 and is less than every other number. The singular is plainly absurd, but mainstream academics such as Abraham Robinson have "succeeded" in creating special subsets (called ultra-filters) of the interval $(0,1)$ in which only infinitesimals can reside. To ask where the infinitesimal numbers begin and end is heresy, and one can be excommunicated from the Church of Academia (mainstream academia).

The *infinitesimal*, like its older sibling *infinity*, is clearly and provably a rubbish concept, which has no place in mathematics or any other field of rational thought. Some of Cauchy's colleagues and compatriots knew this and the struggle for rigour continued.

Cauchy's definition of continuity:

Let $f(x)$ be a function of a variable x, and let us suppose that, for every value of x between two given limits, this function always has a unique and finite value. If, beginning from one value of x lying between these limits, we assign to the variable x an infinitely small increment h, the *function itself increases by the difference $f(x+h) - f(x)$, which depends simultaneously on the new variable h and on the value of x. Given this, the function $f(x)$ will be a continuous function of this variable within the two limits assigned to the variable x if, for every value of x between these limits, the absolute value of the difference $f(x+h) - f(x)$ decreases indefinitely with that of h. In other words, the function $f(x)$ will remain continuous with respect to x between the given limits if, between these limits, an infinitely small increment of the variable always produces an infinitely small increment of the function itself.*

Note that h in the above definition is the same as δ in the FOL (first order logic) "*veri-finition*" that follows.

Cauchy's continuity definition was restated in symbols as follows by Weierstrass:

$\forall\varepsilon, \exists\delta{:}\forall x \; (\; |x-c| < \delta \; \Rightarrow |f\,(x) - L| < \varepsilon \;)$

where L is the limit at the point c.

If L is not a rational number, then L is denoted by some symbol (in this case by L itself!) and the difference $|f\,(x) - L|$ cannot be 0 for any imagined $f(x)$. However, it is required in the modern definition that $f\,(c) = L$, in addition to $\forall\varepsilon, \exists\delta{:}\forall x \; (\; |x-c| < \delta \; \Rightarrow |f\,(x) - L| < \varepsilon \;)$.

In fact, the first order logic statement

$$\forall\varepsilon, \exists\delta{:}\forall x \; (\; |x-c| < \delta \; \Rightarrow |f\,(x) - L| < \varepsilon \;)$$

takes care of this requirement, because no condition is placed on either ε or δ with respect to being greater than 0.

The web of ignorance becomes more intricate when Weierstrass defines a limit formally:

$\forall\varepsilon > 0,\ \exists\delta > 0{:}\forall x\ (\ 0 < \ |x - c| < \delta\ \Rightarrow\ \ 0 < |f\,(x) - L| < \varepsilon\ \ )$

Let's see an example of how mainstream academics use this so-called "rigorous definition". Consider the function $f\,(x) = 2x + 1$. We are required to verify that $f(x)$ has the limit 5 when $x = 2$. So, we start with the distance between some arbitrary y ordinate and 5 as follows:

$$|2x + 1 - 5| < \varepsilon$$

$$\rightarrow\ \ |2x - 4| < \varepsilon$$

$$\rightarrow\ 2|x - 2| < \varepsilon$$

$$\rightarrow\ \ |x - 2| < \frac{\varepsilon}{2}\ \rightarrow\ -\frac{\varepsilon}{2} < |x - 2| < \frac{\varepsilon}{2}$$

So, any $\delta = \frac{\varepsilon}{2}$ will do. Now your professor plays the "epsilon-delta" game with you. It goes something like this: Give me any ε and I will find you a δ that works for it. The wary student now picks $\varepsilon = 1$. Thus, $\delta = \frac{1}{2}$.

And so, $|x - 2| < \frac{1}{2} \rightarrow\ 2\,|x - 2| < 1 \rightarrow\ \ |2x - 4| < 1$

$\rightarrow |2x + 1 - 5| < 1$

Then,

$|x - 2| < \frac{1}{2} \rightarrow\ |2x + 1 - 5| < 1$

which is what we wished to prove. The astute reader will notice that this process not only assumes that the function $f(x)$ is defined (hence continuous!) on the interval $(1,3)$, but the alleged "proof" uses the concept of *infinity* to show that one can get closer to the limit *indefinitely*, but never actually arrive at the limit. Therefore, the ill-formed concept of infinity is still being used contrary to the raucous objections of those in mainstream academia.

These facts are the main reason Weierstrass needed $\varepsilon > 0$ and $\delta > 0$. Firstly, because we can't make ε as small as we please, that is, it can't be 0, secondly, the function may not be defined at the limit when $x = c$ and thirdly, the function may not be defined at some $x = k$ in the given interval $m < k < n$. That is, given $(x) \times \frac{x-k}{x-k}$, it is true that $f(x)$ is no longer defined at k, and c could be anywhere in the interval $(m{,}n)$.

Thus, we can't say that given any ε that δ will be within $|x - c|$ distance of c, where ε is within $|f(x) - L|$ of L, simply because it is not possible to check every distance in the interval $(m{,}n)$.

That $|x - 2| < \frac{1}{2} \rightarrow |2x + 1 - 5| < 1$ is a consequence of the fact that $f(x) = 2x + 1$ is continuous on $(1{,}3)$ and continuity must be **assumed**. There is no certainty that f is continuous on the given interval, that is, the very definition of continuity is circular, for it assumes itself!

Therefore, despite all the efforts of mainstream academia to discard infinity and infinitesimals, both concepts are needed and used indirectly in showing that if ε is within $|f(x) - L|$ distance of L, then δ will be within $|x - c|$ distance of c.

Modern academics needed holes in functions in order to justify their derivative definition:

$$f'(x) = \frac{f(x + \delta) - f(x)}{\delta}$$

Since the application of this definition results in:

$\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x\ (\ 0 < \ |x - c| < \delta\ \Rightarrow\ 0 < |\Delta f\ (c,\delta) - L| < \varepsilon\ )$

where $\Delta f\ (c,\delta) = \frac{f(c+\delta)-f(c)}{\delta}$, mainstream academics erroneously think they are safe from having assumed that L is in fact the limit or the required derivative $f'(c)$. One academic called Professor Wolfgang Mueckenheim thinks that the Cauchy criterion rigourises calculus, but this is obviously false because verifying a sequence converges to some limit L, does not tell us how to find L in a systematic way, as is the case with the derivative definition in terms of limits.

The Weierstrass definition is obviously flawed for many reasons, but the one that stands out most is the fact that L which is equal to $f'(c)$ is being used in its own *verifinition*. Weierstrass's definition is really a verifinition, not an actual definition.

A verifinition is a definition that is used to prove that a given guess, in this case L, is in fact the derivative. One must wonder at this stage what exactly did Cauchy and Weierstrass do which made calculus rigorous? They still lacked any sound method to find L. The ubiquitous *first principles method* (FPM) is an outright sham and fraudulent too.

FPM not only uses a spurious method, but contradicts itself in every way. The δ in $f'\ (x) = \frac{f(x+\delta)-f(x)}{\delta}$ which is the FPM, cannot be equal to 0 at any time, but to arrive at the *guess* L, one must set $\delta = 0$.

Let's see how this buffoonery "works":

Let $\Delta f(x,\delta)$ be the difference quotient defined as follows:

$$\Delta f\ (x,\delta) = \frac{f\ (x + \delta) - f(x)}{\delta}$$

which reduces to $\Delta f\,(x) = f'\,(x) + Q(x{,}\delta)$ and $Q(x{,}\delta)$ is some expression in any combination of x and δ.

Then,

$$f'\,(x) = \frac{f\,(x+\delta) - f(x)}{\delta} \;= \lim_{\delta\to 0} \Delta f\,(x{,}\delta) = f'\,(x) + Q(x{,}\delta)$$

Now in order for $f'(x)$ to equal to $f'\,(x) + Q(x{,}\delta)$, it is required that $Q\,(x{,}\delta) = 0$, but this is possible only if $\delta = 0$, that is,

$$f'\,(x) = \frac{f\,(x+\delta) - f(x)}{\delta} \;= \Delta f\,(x{,}0) = \frac{f\,(x+0) - f(x)}{0} = \frac{0}{0}$$
$$= f'\,(x) + 0$$

Any astute reader will immediately see that the FPM is a kludge that was contrived by Cauchy and sold to the mainstream by Weierstrass.

Thus, we can conclude there is no systematic way of finding the derivative in mainstream calculus. One must use the kludgy FPM.

This illogical thinking and ill-formed reasoning didn't stop with the derivative, but was carried through to the definition of the integral also. In particular, the Riemann integral is a kludge of the same kind.

There are many different forms of the Riemann integral, but essentially they are all equivalent to the following definition in terms of the product of two arithmetic means:

$$\int_a^b f\,(x)\,dx = \sum_{i=0}^{n-1} f\left(a + \frac{(b-a)\,i}{n}\right) \times \frac{b-a}{n}$$

The same definition can be rewritten as:

$$\int_a^b f(x)\,dx = \sum_{i=a}^{b} f(x_i + i\delta) \ \times \delta$$

where $\delta = \frac{b-a}{n}$ and $x_1 = a,\ x_k = a + k\delta,$ etc.

The actual integral sum is thus a function of two variables expressed as follows:

$$Sum\,(x,\delta) = \int_a^b f(x)\,dx + R(x,\delta)$$

where $R(x,\delta)$ is the remainder or error as the partial sums $\sum_{i=a}^{b} f\,(x_i + i\delta) \times \delta$ approach the actual $Sum(x,0)$:

$$Sum\,(x,\delta) = \int_a^b f(x)\,dx = \sum_{i=a}^{b} f\,(x_i + i\delta) \times \delta$$

Therefore,

$$Sum\,(x,0) = \int_a^b f(x)\,dx = \sum_{i=a}^{b} f\,(x_i + i0) \times 0 = 0$$

And once again, the actual sum is not possible, but must be imagined as some *limit* which can be proven to exist using convergence. Yet the very limit is possible through the knowledge of the Mean Value Theorem, provided the function f has a primitive or antecedent function. If not, then the method results in numeric integration and is never exact, unless the limit is known to be a rational number or a recognisable magnitude such as the constants $\pi,\ e,\ \sqrt{2}$, etc.

δ cannot be 0 and yet the limit is not possible otherwise. Ironically, the FOL (first order logic) verifinition described earlier, requires $\delta > 0$. The mean value theorem is the product of two arithmetic means and the fundamental theorem of calculus is derived in one step.

Piecewise functions?

Mainstream academics barely understand their own ill-formed theories. For example, a function is defined as follows:

In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

Ignoring the fact that a set is an ill-formed concept without any definition, the above statement would rule out the ill-formed concept of “piecewise function”, because a piecewise function contains more than one set. A function with a discontinuity is not considered to be a piecewise function. Calculus does not generally apply to functions that have discontinuities or are not smooth over a given interval.

Holes and more holes.

It is important in mainstream calculus that a hole can be produced at any point of a function. Given any function $f(x)$, it is believed by the ignorant mainstream, that one can undefine $f(x)$ at $x = k$, simply by creating a “new” function by the following operation:

$$\text{new\_function} = f(x) \times \frac{x-k}{x-k}$$

The reason for holes as explained earlier, is so that the Weierstrass limit verifinition cannot be faulted, that is, a limit exists without the function necessarily being defined at the limit. **It is a myth that any function has a limit at a given point without being continuous also at that point.** The limit veri-finition of Weierstrass is exactly the same as the Cauchy continuity definition with only one extra condition added to allow for a function to be undefined at the limit point, that is, a hole or discontinuity.

Multiplying any function by 1 does not change the *function rule* which is based on **one** set relation, not many as erroneously believed by mainstreamers. More absurd is how mainstream academics can proceed to multiply by 1 and then find it unacceptable to divide by 1, thus arriving back at the original $f(x)$. It seems that once the new function is obtained, reducing the function as one would a fraction through cancellation of factors, is a cardinal sin. It should be clear now that this fraudulent activity was meant to facilitate Cauchy's "rigorous" calculus.

These misguided ideas have infected mainstream calculus to such an extent, that students have to learn nonsense like $lim_{x\to 0}\frac{x}{x}=1$ when in actual fact, the expression $\frac{sin\ x}{x}$ is derived by multiplying the following function $f(x)$ by $\frac{x}{x}$:

$$f(x)=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots$$

$\frac{sin\ x}{x}=f(x)\times\frac{x}{x}$ whence it's clear that $f(0)=1$ and so $1=\frac{sin\ 0}{0}$.

Last but not least, it gives incompetent mainstream calculus teachers more impetus to design meaningless exam questions involving limits and derivatives on piecewise functions.

Before we learn about the New Calculus definition of integral, let's first understand what the mean value theorem says!

Fixing the broken mainstream formulation of calculus.

The following is a special update from my unpublished work called *What you had to know in mathematics but your educators could not tell you*.

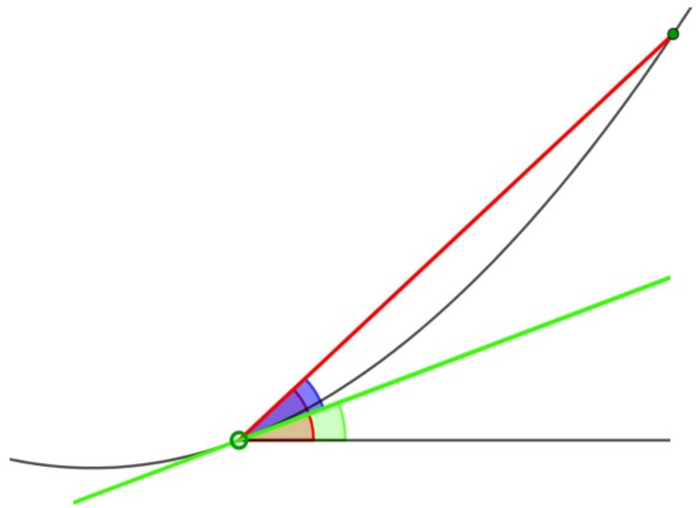
The update is included to complete the claims made previously, that is, no ill-formed concepts such as infinity, infinitesimals or limit theory are required in calculus and **geometry is the only reason** calculus actually works.

The theory of limits is deeply flawed for several reasons, the most important being that there is no valid construction of *real numbers*. Neither classes of equivalent converging sequences (Cauchy) nor Dedekind Cuts are valid constructions.

The New Calculus is the first and only rigorous formulation of calculus in human history and excludes all ill-formed concepts such as infinity, infinitesimals and limit theory. It is based entirely on sound geometry.

I shall demonstrate how you can formulate your mainstream calculus using a strictly geometric approach, albeit not as efficient as the New Calculus, which is simple to learn, elegant and powerful.

Let’s see how we can find the derivative of a function without the use of limit theory.



$\frac{f(x+h)-f(x)}{h} = f'(x) + Q(x,h)$ is the slope formula.

$\frac{f(x+h)-f(x)}{h}$ is the slope of a non-parallel **secant line**.

$f'(x)$ is the slope of the **tangent line** at $\boldsymbol{x}$.

$Q(x,h)$ is the **difference** in slope between

$\frac{f(x+h)-f(x)}{h}$ and $f'(x)$.

The slope formula is expressed as $\frac{rise}{run}$.

$$\frac{f(x+h)-f(x)}{h} = f'(x) + Q(x,h)$$

$$\frac{\left(x^n+(n\ 1\ )x^{n-1}h+(n\ 2\ )x^{n-2}h^2+\ldots+h^n\right)-x^n}{h} = f'(x) + Q(x,h)$$

$(n\ 1)\ x^{n-1} + (n\ 2)\ x^{n-2} h + \ldots + h^{n-1} = f'(x) + Q(x,h)$

Now subtract $Q(x,h)$ from both sides where

$Q\,(x,h) = (n\ 2)\ x^{n-2} h + \ldots + h^{n-1}$

Thus,

$(n\ 1)\ x^{n-1} = f'(x)$

Or $f'(x) = nx^{n-1}$ since $n = (n\ 1)$

Let's see a simple example with actual numbers to see how it works. Suppose that $f\,(x) = x^3,\ x = 1$ and $h = 2$.

From the slope function $\frac{f(x+h)-f(x)}{h}$ we have:

$\frac{(1+2)^3-1^3}{2} = 13 = 3(1)^2 + 3(1)(2) + 2^2 = 3 + 6 + 4$

$\leftrightarrow 13 = 3 + 10$

Subtract $Q\,(1,2) = 10$ from both sides of the above and you have $3 = 3$ as required. Notice that regardless of what value is chosen for h, the derivative is found after subtracting $Q\,(x,h)$ from both sides.

As you can see, finding a derivative is entirely based on sound geometry without the need for the flawed theory of limits!

In your flawed mainstream formulation of calculus, you need $Q\,(x,h)$ to be equal to 0. However, $Q\,(x,h)$ is never equal to 0 because then the slope or finite difference is indeterminate, that is, it takes the form $\frac{0}{0}$.

To establish the definite integral using this new approach is much harder because it is not as optimal as the New Calculus, that is, one has to include the extraneous term or expression $Q\,(x,h)$ that results from using a non-parallel secant line in the fixed mainstream derivative definition as shown earlier.

First note that **any area** is in actual fact a **product of two arithmetic means**. For example, a rectangular area is the product of its sides, which are arithmetic means of all the vertical line lengths and horizontal line lengths in the rectangle respectively.

To calculate an irregular bounded area between a curve and the x-axis, we need to determine the arithmetic mean of all the y-ordinates of the function in the interval and then multiply it by the interval width, so as to find the area – just as we would for a rectangle.

Let's see how we can find the definite integral of a function without the use of limit theory, in other words, the fundamental theorem of calculus which is directly obtained from the mean value theorem.

You probably have never learned this before, so I'll quickly show you why it's true and explain it thoroughly in the following chapters.

From the mean value theorem:

$$\frac{f(x+h)-f(x)}{h} = f'(c)$$

We obtain the fundamental theorem of calculus:

$$f\,(x+h) - f\,(x) = f'\,(c) \times h = \int_x^{x+h} f'\,(x)\,dx$$

The mean value theorem which is about an arithmetic mean, i.e. $f'(c)$ is easy to prove and you will see how it is done in the following proof.

We begin with an interval $(x, x+h)$ divided into n equal parts as follows:

$$x \quad x+\frac{h}{n} \quad x+\frac{2h}{n} \quad \ldots \quad x+\frac{(n-1)h}{n} \quad x+h$$

To find the arithmetic mean of all the y ordinates of $f'(x)$, we observe the following:

$$f'(x) + Q\left(x, \frac{h}{n}\right) = \frac{f\left(x+\frac{h}{n}\right) - f(x)}{\frac{h}{n}}$$

$$f'\left(x+\frac{h}{n}\right) + Q\left(x+\frac{h}{n}, \frac{h}{n}\right) = \frac{f\left(x+\frac{2h}{n}\right) - f\left(x+\frac{h}{n}\right)}{\frac{h}{n}}$$

…

$$f'\left(x+\frac{(n-1)h}{n}\right) + Q\left(x+\frac{(n-1)h}{n}, \frac{h}{n}\right) = \frac{f\left(x+\frac{(n-1)h}{n}+\frac{h}{n}\right) - f\left(x+\frac{(n-1)h}{n}\right)}{\frac{h}{n}}$$

Note that the right hand side sum telescopes, and all the purple terms cancel out to give $\frac{f(x+h)-f(x)}{\frac{h}{n}}$.

Thus, summing the left hand side and the right hand side, we get:

$$\sum_{i=0}^{n-1}\left[f'\left(x+\frac{hi}{n}\right) + Q\left(x+\frac{hi}{n}, \frac{h}{n}\right)\right] = \frac{f(x+h)-f(x)}{\frac{h}{n}}$$

Let $Q(x,h) = \sum_{i=0}^{n-1}\left[Q\left(x+\frac{hi}{n}, \frac{h}{n}\right)\right]$

$$\sum_{i=0}^{n-1}\left[f'\left(x+\frac{hi}{n}\right)\right] + Q(x,h) = \frac{f(x+h)-f(x)}{\frac{h}{n}}$$

Dividing by n gives the **arithmetic mean**:

$$\frac{1}{n}\times\sum_{i=0}^{n-1}\left[f'\left(x+\frac{hi}{n}\right)\right]+\frac{Q(x,h)}{n}=\frac{f(x+h)-f(x)}{h}$$

Now we multiply by h to get the area:

[PC] $$\frac{h}{n}\times\sum_{i=0}^{n-1}\left[f'\left(x+\frac{hi}{n}\right)\right]+\frac{h\cdot Q(x,h)}{n}=f(x+h)-f(x)$$

OR

[MC] $$\frac{h}{n}\times\sum_{i=0}^{n-1}\left[f'\left(x+\frac{hi}{n}\right)\right]=f(x+h)-f(x)-\frac{h\cdot Q(x,h)}{n}$$

The above result **[PC]** is the **fundamental theorem of calculus.** Note that the result is obtained by a **FINITE number of steps**, that is, any integer value of $n>0$ will be sufficient to find the integral.

One might ask why we need to subtract $\frac{h\cdot Q(x,h)}{n}$ from $f(x+h)-f(x)$**.** The reason for this is immediately obvious from the first geometric identity:

$$\frac{f(x+h)-f(x)}{h}=f'(x)+Q(x,h)$$

If we want the arithmetic mean of all the y-ordinates of the function $f'(x)$, then we must determine the arithmetic mean in terms of $f(x+h)-f(x)-\frac{h\cdot Q(x,h)}{n}$, otherwise we are not considering $f'(x)$, but the slopes of non-parallel secant lines given by $\frac{f(x+h)-f(x)}{h}$. Hence, we use **[MC]** to find the area.

Let's see an example of [MC] where $f\,(x) = x^2$, $h = 2$, $n = 1$ and we need the area from $x = 1$ to $x = 3$.

$$2 \times f'\left(1 + \frac{2(0)}{1}\right) = f\,(3) - f(1) \; - \frac{2 \cdot 2}{1}$$

$\rightarrow \; 2 \times 2 = 9 - 1 \; - 4$

$\rightarrow \; 4 = 4$

Let's see yet another example of [MC] where $f\,(x) = x^2$, $h = 2$, $n = 2$ and we need the area from $x = 1$ to $x = 3$.

$$1 \times [f'\,(1) + f'(2)] = f\,(3) - f(1) \; - (1 + 1)$$

$\rightarrow \; 2 + 4 = 9 - 1 \; - 2$

$\rightarrow \; 6 = 6$

As a final example consider $f\,(x) = x^3$, $h = 2$, $n = 3$ and we need the area from $x = 1$ to $x = 3$.

$$\frac{h}{n} \times \sum_{i=0}^{n-1} \left[f'\left(x + \frac{hi}{n} \right) \right] + \frac{h \cdot Q(x,h)}{n} = f(x+h) - f(x)$$

$$\frac{2}{3} \times \left[f'(1) + f'\left(\frac{5}{3}\right) + f'\left(\frac{7}{3}\right) \right] + \frac{2 \cdot \left[3(1)\left(\frac{2}{3}\right) + 3\left(\frac{5}{3}\right)\left(\frac{2}{3}\right) + 3\left(\frac{7}{3}\right)\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 \right]}{3}$$

$$= f(3) - f(1)$$

$$\frac{2}{3} \times \left[\frac{9}{3} + \frac{25}{3} + \frac{49}{3} \right] + \frac{204}{27} = 27 - 1$$

$$\frac{2}{3} \times \left[\frac{83}{3} \right] + \frac{204}{27} = 26$$

$$\frac{166}{9} + \frac{204}{27} = 26$$

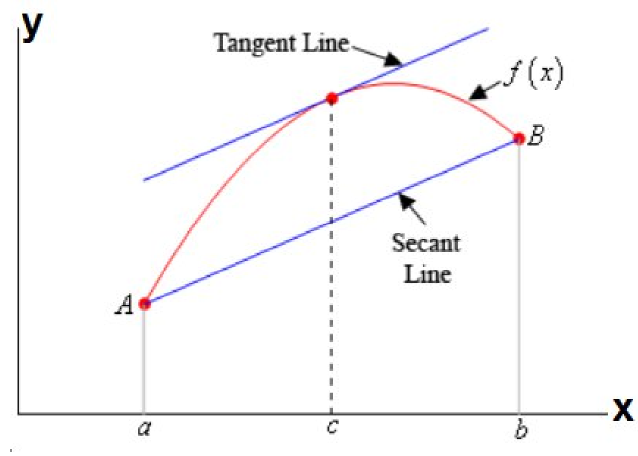
$$\frac{498}{27} + \frac{204}{27} = 26 \quad \leftrightarrow \quad \frac{702}{27} = 26$$

This process is far simpler in the **New Calculus,** because parallel secant lines (as opposed to non-parallel secant lines ala Newton and Leibniz) are used and there is no extraneous term or expression $Q\,(x,h)$ that led to the **behemoth** known as the ***theory of limits***. Limit theory is not required for either the derivative or integral.

I have generously revealed this rigorous knowledge based only on sound geometry to the world’s thousands of ignorant mathematics professors and teachers in the hope that they will cease peddling their idiocies to aspiring young mathematicians.

Now that you have seen how your mainstream calculus can be fixed, let’s study the proof of the mean value theorem – the most important theorem in calculus from which all the others are derived. We’ll see a proof using the flawed mainstream formulation and a constructive proof using the New Calculus later.

Chapter 9: The Mean Value Theorem



The purpose of this chapter is to explain why the mean value theorem works, in a systematic way using the flawed apparatus available in mainstream calculus with a patch (positional derivative explained at the end of this chapter) which I conceived to make this possible. The proof using the New Calculus requires no patch.

Statement: If f is continuous on the closed interval $[x,x+\omega]$ where $x<x+\omega$, and smooth on the open interval $(x,x+\omega)$, then there exists a point c in $(x,x+\omega)$ such that

$$f'(c)=\frac{f(x+\omega)-f(x)}{\omega} \quad \textbf{[MVT]}$$

Preliminary: Some well-formed definitions

A function is continuous over a given interval if there are no disjoint paths in that interval or if it is defined (*) *everywhere in that interval. A* *path is a distance between two points which can be systematically described.*

A tangent line is a finite straight line, such that it meets a curve in only one point, extends to both sides of the point and crosses the curve nowhere.

A function is smooth over a given interval if it is continuous over that interval AND only one tangent line is possible at any point in the interval. Inflection points are excluded because no tangent line is possible at points of inflection, only half-tangent lines.

(*) Since real numbers do not exist because there is no valid construction of the same, you are required to think in terms only of length magnitudes whose measure may be possible or not. Thus, the function is defined everywhere in terms of length magnitudes, provided there are no disjoint paths. Essentially a function path is described by a distance or length magnitude.

We can think of the x coordinate magnitudes as $x+\frac{k\omega}{n}$ where ω is the interval width, k is a rational number denoting the index of the x ordinate and n the number of equal subdivisions or partitions of the interval. Hence the y coordinates are then given by $f\left(x+\frac{k\omega}{n}\right)$. But, you may say, not all the ordinates are addressed this way. Well, you will see that this does not matter as we demonstrate the proof which involves a reducible or telescoping sum.

Define the derivative as follows:

$$f'(x) = \frac{f\left(x+\frac{\omega}{n}\right)-f(x)}{\frac{\omega}{n}} \quad \text{[ND]}$$

where ω is the interval width between x and $x+\omega$, and $\frac{\omega}{n}$ is the width of each equal partition in $(x,x+\omega)$. This isn't much different from the mainstream definition which is obtained by inverting the limit sense, that is, let $h = \frac{\omega}{n}$ so that as $n \to \infty$, it follows that $h \to 0$.Then replace with h in [ND] to get the ubiquitous form:

$$f'(x) = \frac{f(x+h)-f(x)}{h}$$

Using the new definition [ND], we can define any of the positional derivatives at a point $\left(x,x+\frac{k\omega}{n}\right)$ as follows:

$$f'\left(x+\frac{k\omega}{n}\right) = \frac{f\left(x+\frac{(k+1)\omega}{n}\right)-f\left(x+\frac{k\omega}{n}\right)}{\frac{\omega}{n}} \quad \text{[PD]}$$

Now **assume** that the LHS of **[MVT]**, that is, $f'(c)$, is an arithmetic mean of ALL **(**)** the ("infinitely many") ordinates of f' in the interval $(x,x+\omega)$, then

$$f'(c) = \frac{1}{n}\sum_{k=0}^{n-1} f'\left(x+\frac{k\omega}{n}\right)$$

(**) You can verify this statement with actual examples. Try $f'(x) = 2x$ for an area enclosed between the curve and the x axis on interval $(0;2)$. Note that the area must lie entirely above the axis or entirely

below the axis. It makes no sense to find the arithmetic mean of both positive and negative distances at the same time.

So,

$f'(c) = \frac{1}{n}\sum_{k=0}^{n-1} f'\left(x + \frac{k\omega}{n}\right)$

And

$\frac{1}{n}\sum_{k=0}^{n-1} f'\left(x + \frac{k\omega}{n}\right) =$
$\lim_{n\to\infty} \frac{1}{n}\left[f'(x) + f'\left(x + \frac{\omega}{n}\right) + f'\left(x + \frac{2\omega}{n}\right) + \right.$
$\left. \dots + f'\left(x + \frac{(n-2)\omega}{n}\right) + f'\left(x + \frac{(n-1)\omega}{n}\right)\right]$ [SD]

Now replacing each positional derivative in [SD] with its expanded form [PD]:

$$\frac{1}{n}\sum_{k=0}^{n-1} f'\left(x + \frac{k\omega}{n}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n}\left[\lim_{n\to\infty} \frac{n}{\omega}\left\{f\left(x + \frac{\omega}{n}\right) - f(x) + f\left(x + \frac{2\omega}{n}\right) - f\left(x + \frac{\omega}{n}\right)\right.\right.$$
$$+ f\left(x + \frac{3\omega}{n}\right) - f\left(x + \frac{2\omega}{n}\right) + \dots + f\left(x + \frac{(n-1)\omega}{n}\right)$$
$$\left.\left. - f\left(x + \frac{(n-2)\omega}{n}\right) + f(x + w) - f\left(x + \frac{(n-1)\omega}{n}\right)\right\}\right]$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f'\left(x + \frac{k\omega}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left[\lim_{n\to\infty} \frac{n}{\omega}(f(x + \omega) - f(x))\right]$$
$$= \frac{f(x + \omega) - f(x)}{\omega} = f'(c)$$

And this is what was expected. Replacing x by a and letting $\omega = b - a$, we have the ubiquitous form:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The fundamental theorem of calculus is derived in one step from the mean value theorem:

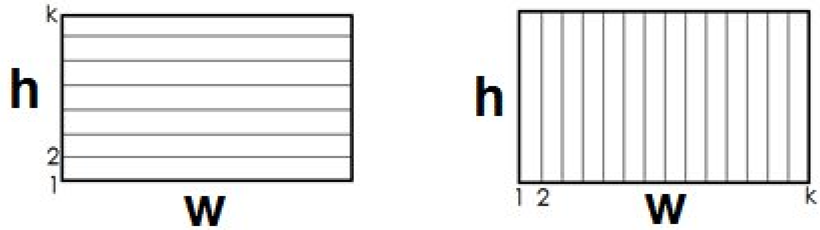
$$f'(c) = \frac{1}{b-a} \int_a^b f'(x)\ dx$$

Since area is well defined by me as the product of two arithmetic means, we have:

$$Area = f'(c) \times (b - a)$$

where $f'(c)$ is the arithmetic mean of all the vertical line lengths of $f'(x)$ in the interval (a,b) and $b - a$ is the arithmetic mean of all the horizontal line lengths or just the interval width. This is why we call the process of finding areas through definite integration 'quadrature' – we essentially normalise the irregular area so that it is a quadrilateral when we can calculate the area as a plane number (Euclid's Elements: Book VII, Definition 16).

The horizontal side length is the **arithmetic mean** of the infinitely many horizontal line lengths in a parallelogram. The vertical side length is the **arithmetic mean** of the infinitely many vertical line lengths in a parallelogram. See below:



The horizontal line length arithmetic mean is given by $\frac{kw}{k} = w$ where w is the length of each horizontal line. Similarly, the vertical line length arithmetic mean is given by $\frac{kh}{k}$ where h is the height of each vertical line. It is immediately evident that w and h are both arithmetic means of the infinitely many horizontal and vertical line lengths respectively. Area is defined as the product of these arithmetic means, that is, $A = w \times h$. Euclid would have done well had he defined the area this way.

Thus,

$$Area = \frac{f(b)-f(a)}{b-a} \times (b-a) = f(b) - f(a)$$

And so $\int_a^b f'(c)\ dx = f(b) - f(a)$ which is known ubiquitously as the fundamental theorem of calculus. Several flavours of the mean value theorem have been realised:

- Fundamental theorem of calculus
- The Average Value theorem
- The Mean Value theorem for integrals
- The Second Mean Value theorem for integrals

These are all different forms of the same Mean Value Theorem.

Academics before me never realised these facts, nor has any such information ever been published by anyone else.

The mainstream statement of the mean value theorem is an unremarkable fact about a continuous and smooth curve over a given interval. Over 2000 years ago, mathematicians knew that a line (secant) through any point could be constructed parallel to another line (tangent).

The remarkable fact is that no matter what interval or number of ordinates are considered, the arithmetic mean will always lie in the interval, even if the function is not continuous or smooth.

A parallel secant line is possible for a given tangent line so that $f'(c) = \frac{f(b)-f(a)}{b-a}$. This is a fact of any curve having the property of smoothness. Smoothness implies continuity but the converse is not true. Therefore, it makes no sense to consider the use of calculus in cases where the function is not smooth, e.g. $y = |x|$ is not smooth at $x = 0$.

More absurd is the consideration of functions where there is a known discontinuity. The mean value theorem as stated by mainstream academics will not apply when there are discontinuities. However, there are examples where the mean value theorem applies even with the mainstream conditions absent. We'll see a couple of examples soon.

The correct statement of Mean Value Theorem:

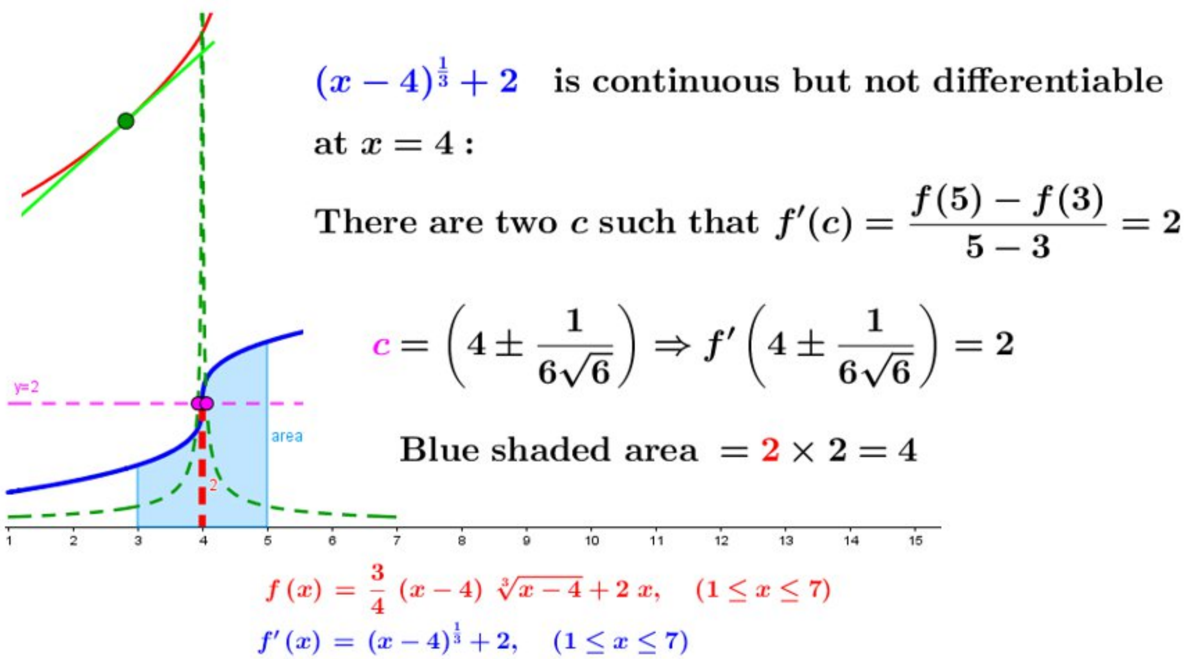
Suppose that an arithmetic mean of all the y ordinate lengths is possible for $f(x)$ on the open interval (a,b). Then there is a number c such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

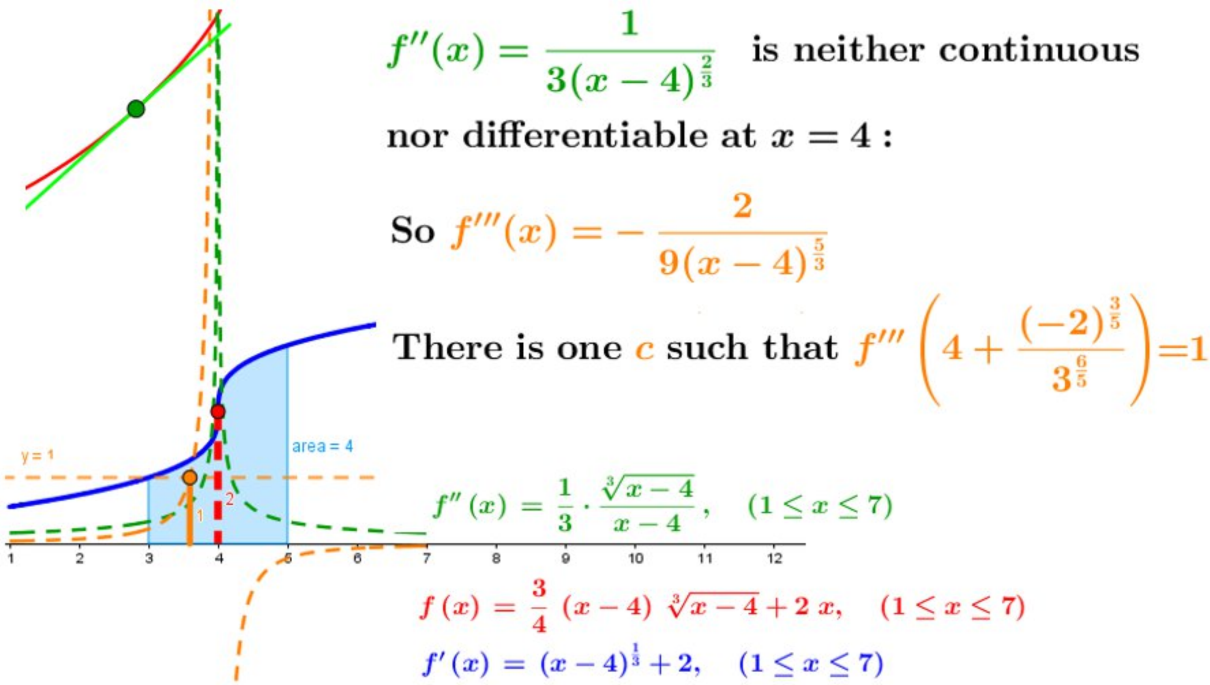
The only time this is impossible is when there is a discontinuity and no convergence in terms of area or volume (for example, Gabriel's Horn).

The correct way to teach the mean value theorem from which the fundamental theorem is derived in one step is outlined in my article.

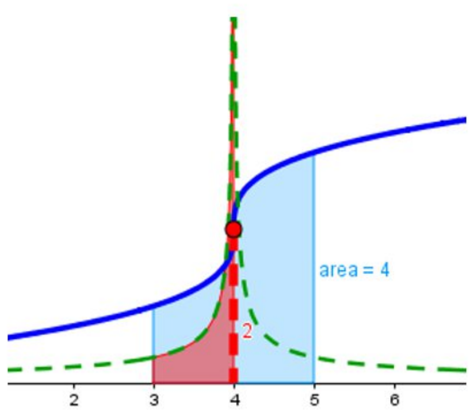
Example 1: A function is continuous but not differentiable and yet the Mean Value Theorem still applies in its correct form:



Example 2: A function is neither continuous nor differentiable, and yet the Mean Value Theorem still applies in its correct form:



Moreover, the red shading in the diagram that follows, shows half the total area which is equal to 2 and is the product of the arithmetic means 1 and 2, where 1 is the arithmetic mean of all the vertical line lengths of the dotted green curve between $x = 3$ and $x = 5$.



Mainstream confusion does not cease here, as there are many other absurd ideas.

The Positional Derivative.

I developed this concept to prove the mean value theorem constructively using the machinery of mainstream calculus. None of the following is used in the New Calculus, which is the first and only rigorous formulation of calculus in human history.

Explanation:

In any given interval, the number of ordinates are innumerable and most cannot be described by any number. The idea of indivisibles was conceived (but not used) by Archimedes and then referenced by Cavalieri many centuries later. Cavalieri didn't do much with the idea.

To prove the mean value theorem using mainstream calculus, one needs machinery that can emulate every ordinate in a given finite interval. An *indivisible* ordinate i in the interval $[x\,;x+\omega]$ can be represented symbolically as follows: $i = \left(x + \frac{\omega k}{n}\right)$.

Theoretically the sum of all the ordinates $f(i)$ can be given as:

$$\sum_{i=1}^{\infty} f(i)$$

Furthermore, the theoretical mean required for quadrature or cubature is given by:

$$\frac{\sum_{i=1}^{\infty} f(i)}{\infty}$$

Naturally, such a mean in its given form is not possible to determine because infinity is not only not a number, but a junk concept.

Mainstream calculus unsuccessfully defines integrals in terms of infinitely small rectangular areas, but this leads to a contradiction, because if a limit is to be realised, then intuitively, each area must become 0 in the Riemannian definition.

The definition of a positional derivative is as follows:

$$f'\left(x+\frac{k\omega}{n}\right) = \frac{f\left(x+\frac{(k+1)\omega}{n}\right)-f\left(x+\frac{k\omega}{n}\right)}{\frac{\omega}{n}}$$

The truth of this definition can be demonstrated with concrete examples even though every derivative is strictly symbolic.

To simplify the explanation, let $k\omega = m$ and $n = p$.

Then,

$$f'\left(x+\frac{m}{p}\right) = \frac{f\left(x+\frac{m}{p}+\frac{\omega}{n}\right)-f\left(x+\frac{m}{p}\right)}{\frac{\omega}{n}}$$

As an example, let's consider the function $f(x) = x^2$.

$$2\left(x+\frac{m}{p}\right) = \frac{\left(x+\frac{m}{p}+\frac{\omega}{n}\right)^2-\left(x+\frac{m}{p}\right)^2}{\frac{\omega}{n}}$$

$$2\left(x+\frac{m}{p}\right) = \frac{\left(x^2+\frac{2m}{p}x+\frac{m^2}{p^2}+\frac{2\omega}{n}x+\frac{2m\omega}{pn}+\frac{\omega^2}{n^2}\right)-\left(x^2+\frac{2m}{p}x+\frac{\omega^2}{n^2}\right)}{\frac{\omega}{n}}$$

This reduces to:

$$2\left(x + \frac{m}{p}\right) \ = \ 2x + \frac{2m}{p} + \frac{\omega}{n}$$

Now taking the limit as $n \to \infty$ on the right hand side, we have:

$$2\left(x + \frac{m}{p}\right) \ = \ 2x + \frac{2m}{p}$$

Replacing m with $k\omega$ and p with n, we have the expected result:

$$2\left(x + \frac{k\omega}{n}\right) \ = \ 2x + \frac{2k\omega}{n}$$

One can write many books on the flaws in mainstream mathematics theory, but what purpose would it serve? More interesting are the ideas that led to the New Calculus, the first and only rigorous formulation in human history. In the next chapter, we learn some of these ideas.

Chapter 10: The ideas that led to the discovery of the New Calculus

I was around 13 or 14 years old when I began to teach myself calculus. I read many calculus books and first used the book called *Teach yourself calculus* by P Abbott (1955) to learn the basic ideas. I completed all the exercises at the end of each chapter in less than one week. According to my teachers, I already knew more than they and all that was covered in a full semester course of calculus in the first year of university. That said, I knew that I did not understand calculus, even though I could evaluate line integrals, determine double integrals using Green's theorem and use Taylor's multi-variable expansion series.

I imagined that eventually a more knowledgeable professor of mathematics would be able to explain how the fundamental theorem came to be, let's call it a *missing link*. It was clear to me that all roads led to the MVT (mean value theorem), but how, I had no clue.

In desperation I turned to the Britannica Encyclopaedia which was renowned for accuracy. Was I disappointed! Everything I had learned was stated differently and so it was equivalent to relearning everything I already knew in different terms, and then still had to figure out what I did not understand.

When the Britannica stated "*It can be easily shown...*", what this meant was several pages of proof and concepts that were not clearly explained. I grew used to the Britannica and my knowledge increased as I realised different perspectives which I had not known from previous study.
After learning of Newton's interpolation polynomial and LaGrange's polynomial, I still had no idea what calculus is all about or even how

they (Newton and Leibniz) arrived at the knowledge of the fundamental theorem. I so badly wanted to know.

A Greek engineer who was a family friend, informed me that I could probably learn more if I knew Latin and was able to acquire a copy of Newton's famous De Analysi. There was a slim chance of that happening then, as the library in my poverty-stricken area was poorly stocked, with the exception of the only set of Encyclopaedia Britannica that was meant to be shared by all who lived there. I started to teach myself Latin from a self-help book, but it was years before I laid eyes on De Analysi.

The time came and I was able to study De Analysi. I had also been to university and spoken to many mathematics professors whom I found to be a bunch of highly ignorant academics. It eventually occurred to me that Newton knew nothing more than what I had already learned.

To be certain, I had gained many perspectives, but Newton made mistakes and his theory was based on several precarious concepts. He was no longer a giant in my eyes. In fact, he was starting to become rather small. Leibniz's work is not in the same league as Newton's, even though Leibniz tried to well define the derivative. Had he known the New Calculus, he might have succeeded.

I soon realised that both Newton and Leibniz, and in fact every academic since, has never understood the mean value theorem or was able to prove it constructively. I was the first ever to provide a constructive proof using the flawed apparatus of the mainstream calculus (with a patch I created called the positional derivative) and the New Calculus, the first and only rigorous formulation of calculus in human history.

Thus, I was back where I had begun at the age of 14 - no one else knew how the fundamental theorem came to be. Many years later, I became aware that all the academics who came before me had no clue and that the masters of calculus (Newton and Leibniz) had discovered it empirically. Just for the record, I did no such thing with the New Calculus - it is a rigorous, systematic and flawless formulation. There was no experimentation, no guessing, no ill-formed concepts.

So an intensive effort began to make the connection, by finding the missing link. The missing link is the formula that directly connects the finite difference and the derivative to the integral. As stated earlier, the fundamental theorem gives one a clue, but most did not realise that the correct statement should be:

$$f\,(x+w)-f\,(x)=\int_{x}^{x+w} f'\,(x)\,dx$$

That theorem is derived in one step from the mean value theorem as explained in an earlier chapter.

There are some academics who now call the fundamental theorem of calculus, the *net change* theorem which states that the integral of an acceleration function is given by the change in velocity, that is,

$$v\,(b)-v\,(a)=\int_a^b a\,(t)\,dt$$

To fully understand the fundamental theorem, one is required to have a profound understanding of the MVT, for it is derived directly from the MVT in one step. No one realised this fact before me because the real meaning of the mean value theorem (MVT) is that it represents an arithmetic mean.

You might be surprised at what I came up with. Before my finite difference formulas, there was no direct link between Newton's interpolation polynomial (divided finite differences), Cauchy's derivative (his limit definition) and Leibniz's integral (elongated es).

Given a function f and its derivative f', the p^{th} finite difference can be calculated using my general finite difference formula in either of two forms as follows:

In terms only of derivatives:

$$\Delta^p(x) = \left[\frac{p}{n\,\omega^{p-1}} \sum_{s=0}^{n-1} \left[\sum_{q=0}^{p-1} (p-1\ q\,)\,(-\,1)^{p-1+q}\ f'^{\left(x+\frac{(q+1)\omega s}{n}\right)} \right] \right]$$

In terms of derivatives and integrals:

$$\Delta^p(x) = \frac{1}{\omega^p} \sum_{q=0}^{p-1} (-\,1)^{p-1+q} \int_{x+q\omega}^{x+(q+1)\omega} f'(x)\,dx$$

I had established the missing link, but the fundamental theorem of calculus was still without reach in terms of understanding. Most professors could not understand my formulas and many decades later, a Chinese researcher on ResearchGate internet site, came up with something similar, but not quite the same or even close to the accuracy of my formulas.

Believe it or not, even with the missing link no longer missing, the fundamental theorem cannot be understood without the knowledge of my New Calculus. One requires a deep and profound understanding of the mean value theorem. I carefully studied the "proofs" of the MVT - all of them a shameful joke that is testimony to the ignorance, arrogance, stupidity and incompetence of academics who came before me.

So, I set out to prove it using the tools of analysis. The average sum theorem was discovered long before.

Here is my average sum theorem which I discovered 30 years ago:

For a function f differentiable over an interval $(x;x+\omega)$, the following identity is true:

$$f\left(x\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(x+\frac{\omega k}{n^{p}}\right)$$

where $p>1$ and $p \in N$.

If $p=1$, then the right hand side of the identity becomes the right (or left) hand side of the mean value formula from which we derive the fundamental theorem of calculus and the definition of an integral.
If $p>1$, the identity defines the function value at a given point, that is, $f(x)$.

A lot had fallen into place, but the final piece of the puzzle was found only after I went back to the basic definitions of area, volume, etc. To fully grasp the MVT, one has to understand that area is in fact the product of two arithmetic means. Similarly, volume is the product of three arithmetic means.

For the first time in my life's journey and in the history of humanity, the mean value theorem was understood in all its glory. The New Calculus was born from this epiphany.

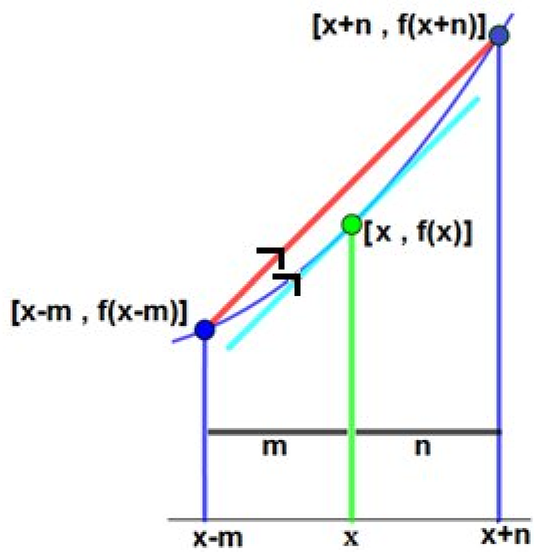
Chapter 11: The New Calculus derivative definition

The derivative in the New Calculus is the slope of a tangent line that is defined as follows:

$$f'(x) = \frac{f(x+n) - f(x-m)}{m+n}$$

where m and n are horizontal distances from the point of tangency $(x, f(x))$ to the endpoints of a secant line that is parallel to the tangent line at the same point and $x - m < x < x + n$.

The following diagram explains:



The definition can also be written as follows

$$f'(x) = \frac{f(x+n) - f(x-m)}{m+n} = f'(x) + Q(x,m,n)$$

where $Q(x,m,n) = 0$. The expression $Q(x,m,n)$ is obtained if every factor $m + n$ is cancelled from the numerator and denominator of the finite difference quotient:

$$\frac{f\,(x+n)-f(x-m)}{m+n}$$

$Q(x,m,n)$ is comprised of any combination in m and n which may or may not contain x. It is easy to prove that $m+n$ is a "real" factor of every term in the numerator. From the definition,

$$f'\,(x)\times(m+n)=f\,(x+n)-f(x-m)$$

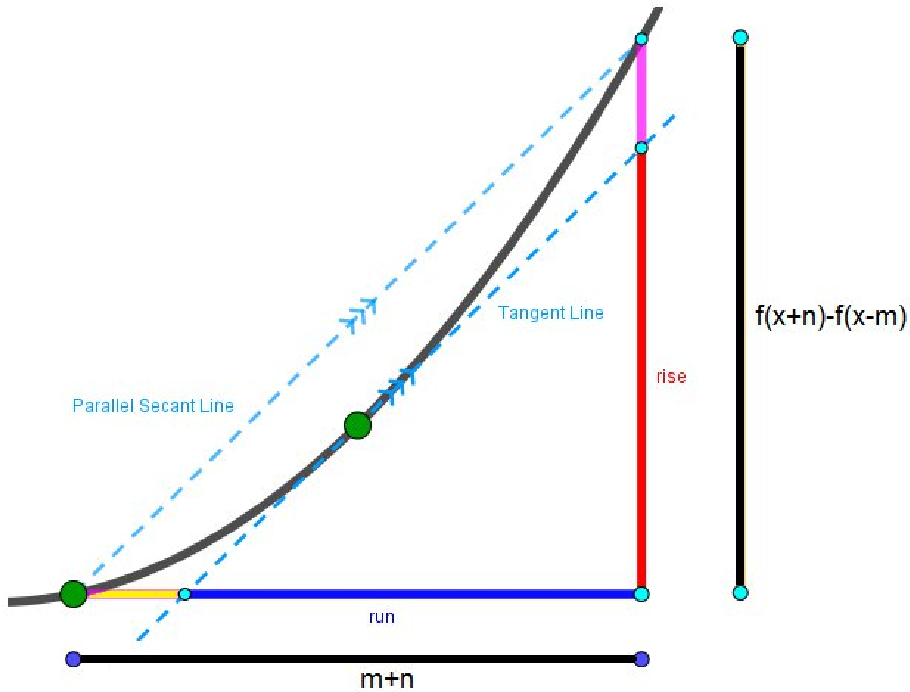
$m+n$ is a factor of the left hand side, therefore it is also a factor of the right hand side.

Proof:

If we divide the left hand side by $m+n$, the result is $f'(x)$. But $f\,(x+n)-f\,(x-m)$ is not equal to $f'(x)$. Algebra tells us that the right hand side must equal to the left hand side, which is only possible if $m+n$ is a factor. Q.E.D

From the point of tangency, it is possible to construct any number of similar triangles whose one side is a parallel secant line. This fact is true for any function that is continuous and smooth. The only exception is if the given point is a point of *inflection*, in which case a tangent line is not possible and hence a derivative is meaningless. A derivative is the slope of a special kind of straight line: one that is tangent to another curve. No straight line can be tangent to another straight line.

In the diagram that follows, two similar triangles are shown, but innumerably many such triangles are possible.



From the diagram it is easy to see that the slope of any tangent line is given by the slope of any parallel secant line:

$$\frac{rise}{run} = \frac{rise}{run} = \frac{rise + rise}{run + run} = \frac{f(x+n) - f(x-m)}{m+n}$$

Different values for m and n are possible, given that every parallel secant line has a unique (m,n) pair.

It is impossible for any parallel secant line's slope to be defined by a $(0,0)$ pair, hence $m + n$ is **never equal to** 0. However, given that $f'(x) = f'(x) + Q(x,m,n)$ and every term in $Q(x,m,n)$ has either m or n or both, setting $m = n = 0$ is equivalent to the value of $Q(x,m,n)$, which is of course always 0.

Objections by mainstream academics:

1. How do you know if a parallel secant line to any given tangent line is possible?

We can safely assume that there are innumerably many secant lines that are parallel to the tangent line and also innumerably many secant lines that are not parallel to the tangent line. That there are innumerably many parallel secant lines was proved by Euclid over 2300 years ago.

2. Why can't we use the same approach in mainstream calculus?

Since the tangent line slope $f'(x) + Q(x,h)$ does not depend on h, the sum of all the terms in h must be zero, because the expression $f'(x) + h$ must be equal to $f'(x)$, the slope of the tangent line. This is only possible if $h = 0$.

In the New Calculus, we know that the parallel secant lines have the same slope, but **none of the secant lines** in the mainstream calculus ever have the same slope. So, the statement

"$f'(x) + h$ must be equal to $f'(x)$, the slope of the tangent line",

is clearly untrue. $f'(x) + h$ is **never** equal to the slope of the tangent line, unless $h = 0$, which in the words of Anders Kaesorg (MIT graduate) is 'absolutely not allowed' (sic). Read about my Quora debate with Anders Kaesorg of MIT online.

3. But doesn't the New Calculus method still use limits?

No, because those parallel secant lines are fixed and nothing approaches anything else. It is impossible for a parallel secant line ever to coincide with the tangent line, for if hypothetically it did so, then the secant line would degenerate into a point which cannot have a slope, that is, there is no (m,n) pair $(0,0)$ which can be used in the definition of the parallel line's slope. For this reason, there is **never** division by 0 because the difference quotient $\frac{f(x+n)-f(x-m)}{m+n}$ is valid only if $m+n>0$.

4. How do you know there is a term without m or n in it?

For two reasons:

i. $f'(x)$ does not depend on m or n.
ii. It is easily proved that $m+n$ is a factor of every term in the numerator:

From $f'(x)\times(m+n)=f(x+n)-f(x-m)$, it follows that $m+n$ is a factor of the LHS which is a product of two factors $f'(x)$ and

$m+n$. $f'(x)\times(m+n)$ is measured by $m+n$ exactly $f'(x)$ times. Since $m+n$ is a factor of the left hand side (LHS), it follows that $m+n$ must also be a factor of the right hand side (RHS), that is, the RHS is also a product of two factors. Hence, $m+n$ is a factor of the expression $f(x+n)-f(x-m)$, that is, $m+n$ measures $f(x+n)-f(x-m)$ exactly $f'(x)$ times also. This means that if we divide $f(x+n)-f(x-m)$ by $m+n$, the expression so obtained must be equal to $f'(x)$. This is only possible if the sum of all the terms in m and n are 0. The equation formed by setting these terms to 0 in the New Calculus is called the auxiliary equation which is denoted by $Q(x,m,n)=0$, and is the first powerful feature new students learn about which is not possible in the mainstream calculus.

5. Why is it the case that setting all the terms in m and n to 0 is equivalent to obtaining the derivative $f'(x)$?

Because $Q(x,m,n)$ is always 0, no matter which (m,n) a parallel secant line pair is being used.

The New Calculus derivative works for every continuous and smooth function.

Example: Find the general derivative of $f\,(x) = x^3$.

Solution:

$$f'\,(x) = \frac{x^3+3x^2n+3xn^2+n^3-x^3+3x^2m-3xm^2+m^3}{m+n}$$

$$f'\,(x) = \frac{3x^2(m+n)+3x(n-m)(m+n)+(m^2-mn+n^2)(m+n)}{m+n}$$

$$f'\,(x) = 3x^2 + Q(x,m,n) = 3x^2 + 3x\,(n-m) + m^2 - mn + n^2$$

$$Q\,(x,m,n) = 0 = 3x\,(n-m) + m^2 - mn + n^2$$

$Q\,(x,m,n) = 0$ and is known as the auxiliary equation in the New Calculus. Given x and either of m or n, one can find the remaining value. More generally, given any two, one can find the remaining third.

By completing the square, we can express m in terms of x and n, or we can express n in terms of x and m:

$$m^2 - (3x+n)\,m + 3xn + n^2 = 0$$

implies

$$m\,(x,n) = \frac{3x+n \pm \sqrt{(3x+n)^2 - 4(3xn+n^2)}}{2}$$

and

$n^2 + (3x - m)n - 3xm + m^2 = 0$

implies

$n\,(x,m) = \frac{3x-m \pm \sqrt{(3x-m)^2 - 4(m^2 - 3xm)}}{2}$

There are many uses of the auxiliary equation and to discuss these would require a book dedicated to this topic alone.

Rather than go through many simple examples, let's take a look at the derivative of $sin\ x$ and verify the New Calculus definition is valid by finding an $(m\ ;n)$ pair for the angle of $\frac{\pi}{3}$ radians. We'll also find an auxiliary equation.

Solution:

There are many ways to do this, but the easiest is to use the fact that Newton derived the sine and cosine partial sum series in *De Analysi* without calculus to get:

$sin\ x\ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

and

$cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

where x is expressed in radians. These partial sum series cannot be used if x is in degrees, until x degrees are first converted to radians.

So,

$$f'(x) = \frac{\left(x+n-\frac{(x+n)^3}{3!}+\frac{(x+n)^5}{5!}-\ldots\right)-\left(x-m-\frac{(x-m)^3}{3!}+\frac{(x-m)^5}{5!}-\ldots\right)}{m+n}$$

$$f'(x) = \frac{m+n-\frac{x^2(m+n)}{2!}+\frac{x^4(m+n)}{4!}-\frac{x^6(m+n)}{6!}+\ldots+Q(x,m,n)(m+n)}{m+n}$$

$$f'(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+Q(x,m,n)$$

Since, $Q\,(x,m,n) = 0$, we have $f'\,(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots = cos\ x$

We can determine the auxiliary equation as follows:

$$cos\ x\ = \frac{sin\,(x+n)\ -sin(x-m)}{m+n} + Q(x,m,n)$$

$$\rightarrow Q\,(x,m,n) = cos\ x - \frac{sin\,(x+n)\ -sin\,(x-m)}{m+n} = 0$$

$$\rightarrow cos\ x = \frac{sin\,(x+n)\ -sin\,(x-m)}{m+n}$$

$$\rightarrow x + ncos\ x = sin\ \ (x+n)\ \ - sin\ \ (x-m)$$

$$\rightarrow \frac{sin\,(x+n)\ -sin\,(x-m)-mcos\ x}{ncos\ x}$$

Therefore, if we know any two of x, m and n, then we can always find the third.

Suppose $x = \frac{\pi}{3} = 60^{\circ}$ and $m = 0.1$.

To find $n = 0.098109$ is trivial, and can be done using any iterative method. Thus, the $(m\,;\,n)$ pair found is $(0.098109\,;\,0.1)$. This can be checked by the definition:

$$cos\ \frac{\pi}{3}\ \ = \frac{sin\left(\frac{\pi}{3}+0.098109\right)\ -sin\left(\frac{\pi}{3}-0.1\right)}{0.098109+0.1} = \frac{1}{2}$$

It's not possible to do any of these things using the flawed mainstream formulation of calculus.

It is hard to imagine anyone not knowing how to write computer programs in this day and age. The following simple C program uses the root approximation method to calculate n given x and m:

```
#include <stdio.h>

#include <math.h>

#define e    2.7182818284590452353602874713527

#define pi   3.1415926535897932384626433832795

double f(double x, double m, double init_n)

{

   double r=init_n, t=0, t1=0, n;

   do {n=r;

      t = ( sin(x+n)-sin(x-m)-m*cos(x) ) / (n*cos(x)) - 1;

      t1= (-(1/cos(x))*(sin(m-x)+sin(n+x))+m+n*cos(n+x)/cos(x))/(n*n);

      r=n-t/t1;

   } while  ( fabs(r-n) > 0.00001  );

   return r;

}

int main()

{
```

```
\\ Function is called with angle, m and an initial value for n
 double res=f((double) (pi/3), (double) 0.1, (double) 0.1);
 printf("\n%lf", res);    \\ res contains the value of n
 res=(sin(pi/3+res)-sin(pi/3-0.1))/(0.1+res);
 printf("\n%lf", res);     \\ res contains the value of cos pi/3.
 return 0;
}
```

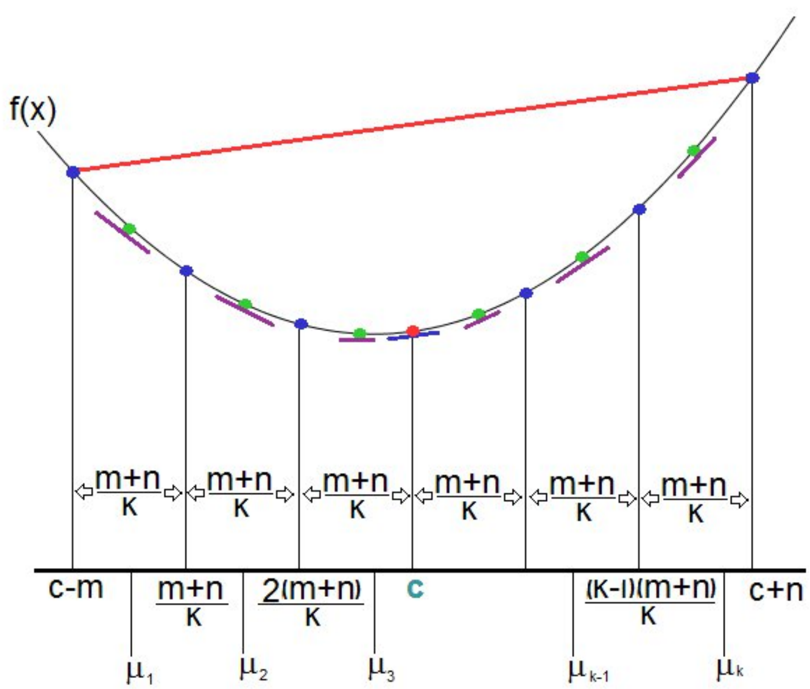
Chapter 12: The New Calculus integral definition

In order to derive the New Calculus integral, we must first study a short proof of the Mean Value Theorem using the New Calculus derivative. A proof of the theorem using the flawed mainstream formulation was given earlier.

It makes sense to use the New Calculus definition of derivative because it also shows immediately the connection between the integral and the derivative in the fundamental theorem of calculus which is derived in one step from the mean value theorem.

We begin with the **New Calculus** definition of derivative:

$$f'(c) = \frac{f(c+n)-f(c-m)}{m+n}$$



In the previous diagram, the interval $(c-m;c+n)$ is divided into equal partitions or sub-intervals of $\frac{m+n}{k}$. The point c is the abscissa (x coordinate) of the arithmetic mean $f'(c)$ of all the vertical line lengths. We will prove shortly that

$$f'(c) = \frac{f(c+n) - f(c-m)}{m+n}$$

It is required to prove that the arithmetic mean of the gradients of the **purple** tangent lines is equal to the gradient of the **blue** tangent line at **c** which is equal to the gradient of the **red** secant line.

If there were a mean abscissa μ_s in each of the sub-intervals, then for the same μ_s, we must have $f'(\mu_s)$ such that

$$f'(\mu_s) = \frac{f\left(c-m+\frac{(m+n)(s+1)}{k}\right)-f\left(c-m+\frac{(m+n)s}{k}\right)}{\frac{m+n}{k}}$$

This follows from the fact that the arithmetic mean of the arithmetic means of all the sub-intervals will be $f'(c)$.

However, none of these assumptions are actually needed, because the New Calculus derivative $f'(\mu_s)$ is by definition equivalent to the mean value theorem for the given sub-interval. At any rate, if these assumptions are incorrect, then the following reasoning will result in a contradiction.

So, the mean of all the arithmetic means is given by:

$$f'(c) = \frac{1}{k}\sum_{s=1}^{k} f'(\mu_s)$$

In the previous statement, we begin by attempting to find the arithmetic mean of all the sub-interval arithmetic means, that is, to show that

$$f'(c) = \frac{f'(\mu_1) + f'(\mu_2) + f'(\mu_3) + \dots + f'(\mu_{k-1}) + f'(\mu_k)}{k}$$

Expanding the sum for $s = 1$ to $s = k$:

$$f'(c) = \frac{1}{k}\{f'(\mu_1) + f'(\mu_2) + f'(\mu_3) + \dots + f'(\mu_{k-1}) + f'(\mu_k)\}$$

$$f'(c) = \frac{1}{k}\{\frac{f\left(c-m+\frac{m+n}{k}\right)-f(c-m)}{\frac{m+n}{k}} + \frac{f\left(c-m+\frac{2(m+n)}{k}\right)-f\left(c-m+\frac{m+n}{k}\right)}{\frac{m+n}{k}} +$$

$$+\frac{f\left(c-m+\frac{3(m+n)}{k}\right)-f\left(c-m+\frac{2(m+n)}{k}\right)}{\frac{m+n}{k}} + \dots$$

$$+\frac{f\left(c-m+\frac{(k-1)(m+n)}{k}\right)-f\left(c-m+\frac{(k-2)(m+n)}{k}\right)}{\frac{m+n}{k}}$$

$$+\frac{f(c+n)-f\left(c-m+\frac{(k-1)(m+n)}{k}\right)}{\frac{m+n}{k}}\}$$

And so, it is proven that $f'(c) = \frac{f(c+n)-f(c-m)}{m+n}$

Note that it does not matter what k we choose, because the arithmetic mean is always the same. Thus, for the purposes of quadrature, the seemingly impossible task of finding the arithmetic mean of innumerably many y ordinates, is accomplished by a reducible or telescoping sum.

I revealed that $f'(c) = \frac{1}{m+n} \int_{c-m}^{c+n} f'(x)\ dx$ for some c, such that

$c - m < c < c + n$, whence

$$\frac{f(c+n)-f(c-m)}{m+n} = \frac{1}{m+n} \int_{c-m}^{c+n} f'(x)\ dx$$

And $\int_{c-m}^{c+n} f'(x)\, dx = f(c+n) - f(c-m)$ is known ubiquitously as the fundamental theorem of calculus, which as we have shown is derived in one step from the mean value theorem. The New Calculus integral is equivalent to the mainstream, definition as follows:

$$I = \frac{m+n}{k} \sum_{s=0}^{k-1} f'(\mu_s) = \int_{c-m}^{c+n} f'(x)\, dx$$

$I = \frac{m+n}{k} \sum_{s=0}^{k-1} f'(\mu_s)$ is the New Calculus integral.

Misguided mainstream theory states $f'(c) = \frac{1}{m+n} \int_{c-m}^{c+n} f'(x)\ dx$ as:

$$F(x) = \int_{a}^{x} f(t)\, dt$$

which supposedly is meant to show the relationship between the integral and the derivative. However, $F(x) - F(a) \neq F(x)$.

But how can you blame mainstream academics who have never understood calculus or the reasons why it works?

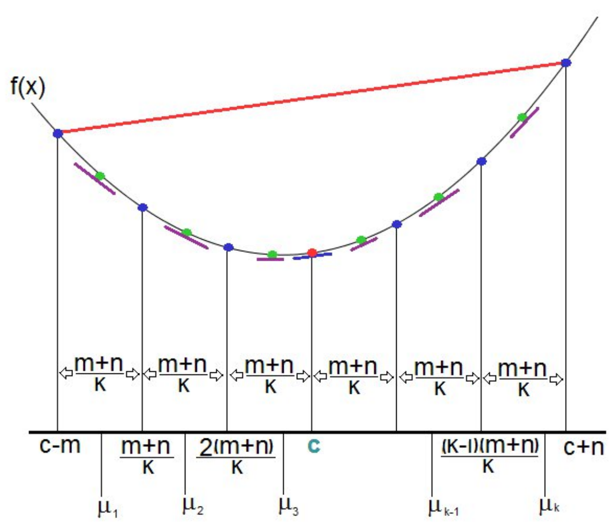
We evaluate integrals by using the mean value theorem in one of two ways:

i. We use the relationship between the primitive function $f(x)$ and its derivative $f'(x)$ as stated in the fundamental theorem of calculus, which is derived in one step from the mean value theorem.
ii. We use the fact that an integral is the product of two arithmetic means to calculate a rational number approximation, that is, numeric integration.

The New Calculus integral can be used theoretically and practically.

Let us see how we can derive the arc length formula using the New Calculus.

In the graphic that follows, the mus (μ_s) are the abscissas or x coordinates of the mean value (or arithmetic mean of all the y ordinates) in each sub-interval. The mean values are represented by the slopes of the tangent lines or the y coordinates $f(\mu\_s)$ of the point of tangency, that is, $(\mu_s\ ;f(\mu\_s))$.



To find the arc length, we use the distance formula for the segments that join the endpoints of each partition:

$$s = \sqrt{\left[f\left(c - m + \frac{(m+n)(s+1)}{k}\right) - f\left(c - m + \frac{(m+n)s}{k}\right)\right]^2 + \left(\frac{m+n}{k}\right)^2}$$

The segments lengths are given by:

$$s_k = \sqrt{\left(\frac{m+n}{k}\right)^2 \left[\frac{f\left(c-m+\frac{(m+n)(s+1)}{k}\right) - f\left(c-m+\frac{(m+n)s}{k}\right)}{\frac{m+n}{k}}\right]^2 + \left(\frac{m+n}{k}\right)^2}$$

But $f'(\mu_s) = \frac{f\left(c-m+\frac{(m+n)(s+1)}{k}\right) - f\left(c-m+\frac{(m+n)s}{k}\right)}{\frac{m+n}{k}}$

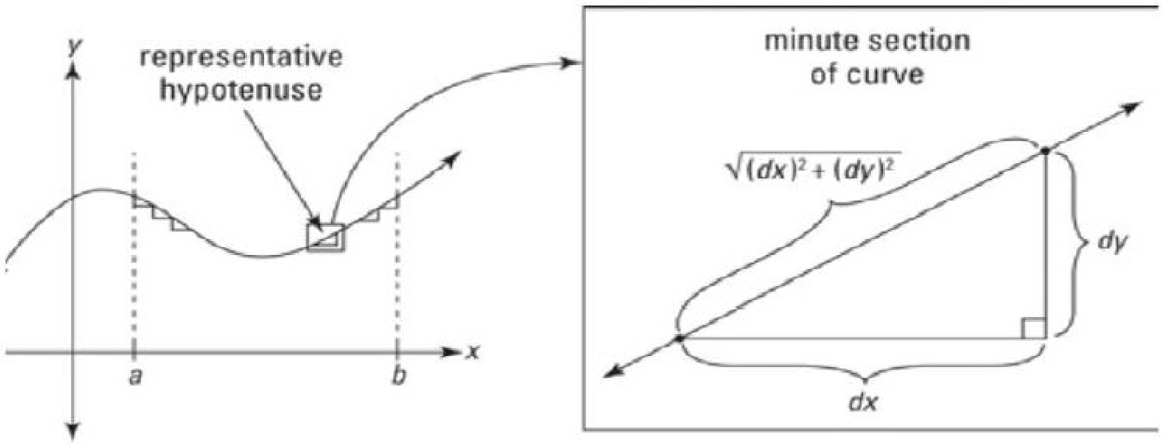
So, $s_k = \frac{m+n}{k}\sqrt{[f'(\mu_s)]^2 + 1}$

And so,

$$\frac{m+n}{k}\sum_{s=0}^{k-1}\sqrt{[f'(\mu_s)]^2 + 1} = \int_{c-m}^{c+n}\sqrt{[f'(x)]^2 + 1}\,dx$$

which is the desired result.

You normally see it done in mainstream calculus using the following approach:



In the New Calculus,

$$dy = f\left(c - m + \frac{(m+n)(s+1)}{k}\right) - f\left(c - m + \frac{(m+n)s}{k}\right)$$

and

$$dx = \frac{m+n}{k}$$

for each partition. Since no ill-formed theory is used the differentials dy and dx are well defined.

Now for an actual example.

The arc length of the function $f\,(x) = x^2$ over $(0\,;2)$ is given by:

$$\frac{m+n}{k}\sum_{s=0}^{k-1}\sqrt{4\,(\mu_s)^2+1} = \int_{c-m}^{c+n}\sqrt{4\,(\mu_s)^2+1}\;dx$$

Next, we find the arithmetic mean of all the segment lengths by integrating the distance formula obtained. In the New Calculus, this is a finite sum found using the mean of each equal partition or sub-interval.

The arc length is thus given if we let $h'\,(x) = \sqrt{4x^2+1}$ so that

$$h\,(c+n) - h\,(c-m) = \frac{m+n}{k}\sum_{s=0}^{k-1} h'\,(\mu_s)$$

Therefore, $h(x) = \frac{1}{2}x\sqrt{4x^2+1} - \frac{1}{4}ln\left(-2x+\sqrt{4x^2+1}\right)$

$m+n=2$ and we can calculate the sum for $k=1$, but it will work for any k given the appropriate mus (μ_s). In the New Calculus, every function has a closed form antiderivative or primitive function which is only possible using the Gabriel polynomial.

We find the μ_s as follows:

The mean value of $h(x)$ on $(0,2)$:

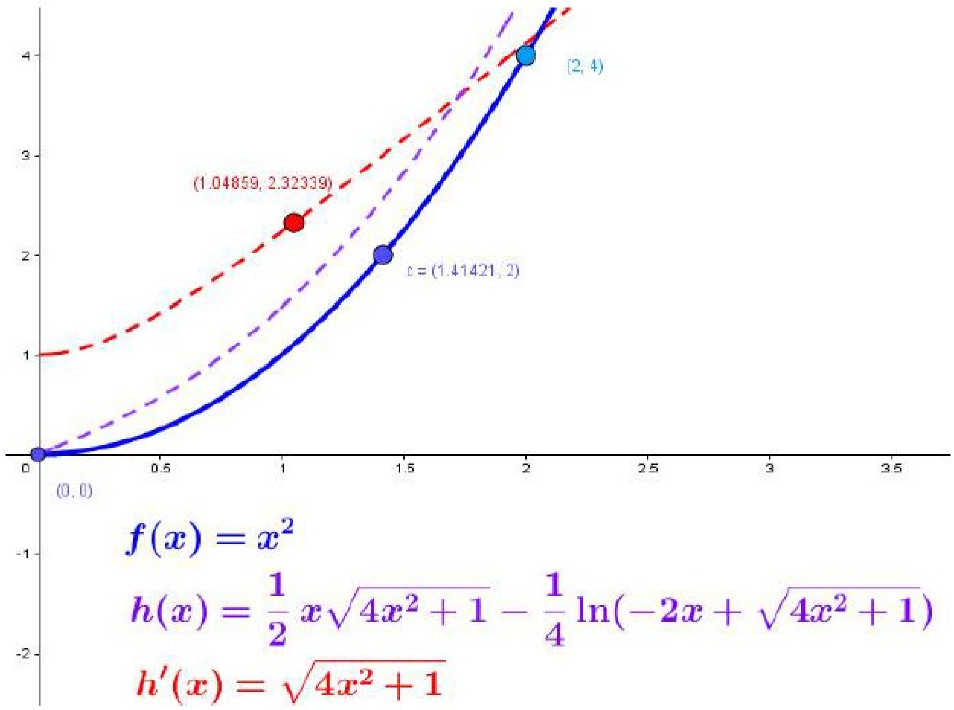
$$\frac{h(2)-h(0)}{2-0} = 2.32339$$

The mean value is an ordinate of $h'(x) = \sqrt{4x^2+1}$. Since μ_s is the abscissa of 2.32339, we find μ_0 by solving $\sqrt{4x^2+1} = 2.32339$, that is, $\mu_0 = x = 1.04859$.

$$\frac{m+n}{k}\sum_{s=0}^{k-1}\sqrt{4(\mu_0)^2+1} = \frac{2}{1}\sum_{s=0}^{k-1}\sqrt{4(1.04859)^2+1}$$

$= 2\sqrt{4(1.04859)^2+1} = 4.64678$ which is the arc length.

The graphs are shown in the following figure:



As you have seen, there is no use of infinity, infinitesimals or limits which are all ill-formed concepts.

$$f\,(c+n) - f\,(c-m) = \frac{m+n}{n}\sum_{s=0}^{k-1} f'\,(\mu_s) = \int_{c-m}^{c+n} f'\,(x)\,dx$$

It is also easy to see how the mainstream integral can be obtained from the New Calculus definition using only an x substitution for μ_s in the distance function which is used to determine the arithmetic mean of all the line segments. This substitution property is true for any function, not just the distance function.

Given that the mean value theorem is used, the finite sum is always equal to the arc length, regardless of the value in the summand, that is, k.

From this information, you can derive Green’s theorem and the Divergence theorem which uses vectors and has a parametric form in the integral.

The entire single variable New Calculus is captured in the following graphic:

$$f'(c) = \frac{f(c+n) - f(c-m)}{m+n}$$

$$f(c+n) - f(c-m) = \frac{m+n}{k}\sum_{s=0}^{k-1} f'(\mu_s) = \int_{c-m}^{c+n} f'(x)dx$$

$$f'(\mu_s) = \frac{f\left(c - m + \frac{(m+n)(s+1)}{k}\right) - f\left(c - m + \frac{(m+n)s}{k}\right)}{\frac{m+n}{k}}$$

Chapter 13: The Gabriel Polynomial

The ideas that led to the Gabriel polynomial were inspired by Newton's De Analysi, in particular his proposition 4.

P R O P. IV.

Si recta aliqua in partes quotcunque inæquales AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta divisionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. transibit.*

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Abscissam quamvis AA7 demitte Ordinatas perpendiculariter BA, B2A2, &c.

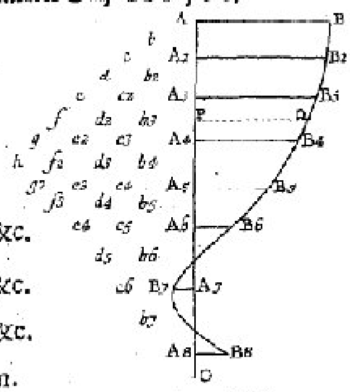
Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$, $\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$, $\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$, $\frac{-A7B7 - A8B8}{A7A8} = b7$.

Deinde $\frac{b - b2}{AA3} = c$, $\frac{b2 - b3}{A2A4} = c2$, $\frac{b3 - b4}{A3A5} = c3$, &c.

Tunc $\frac{c - c2}{AA4} = d$, $\frac{c2 - c3}{A2A5} = d2$, $\frac{c3 - c4}{A3A6} = d3$, &c.

Et $\frac{d - d2}{AA5} = e$, $\frac{d2 - d3}{A2A6} = e2$, $\frac{d3 - d4}{A3A7} = e3$, &c.

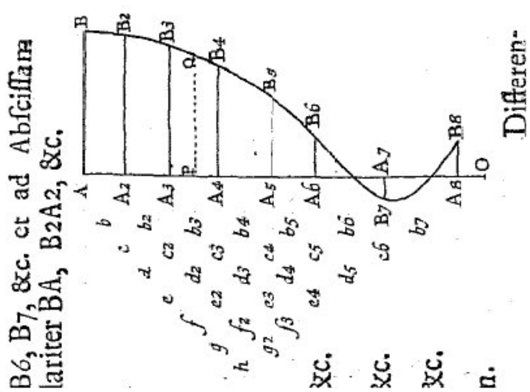
Sic pergendum est ad ultimam differentiam.



B b

Differen-

Rotating the diagram helps to understand what Newton was trying to do.



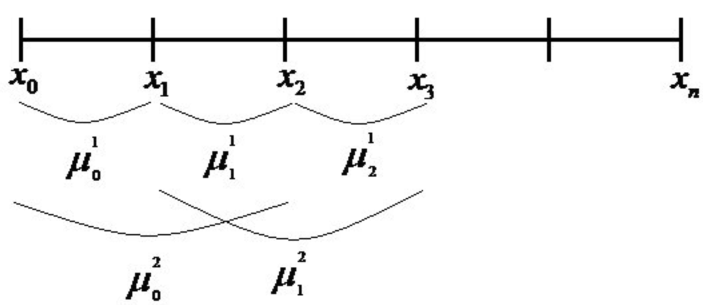
Through finite divided differences, Newton showed that it is possible to arrive at an interpolation polynomial. See formula [T1] on page 24 of HowWeGotCalculus.pdf.

Although there is nothing wrong with Newton's approach, my approach using the New Calculus makes it possible to derive important theorems (such as Taylor and McClaurin) without resorting to the use of limits or real analysis. The New Calculus is superior to Newton's calculus.

I define finite differences in terms of the *arithmetic mean* of each given interval.

The notation μ_0^1 means the first order mean (denoted by superscript) and the first difference (denoted by subscript). μ always refers to the abscissa of the mean.

The following diagram explains:



$[x_0 x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f'\left(\mu_0^1\right)$ also, $[x_1 x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'\left(\mu_1^1\right)$

There is no difference from Newton's method with regard to first order differences. Newton was unaware of the mean value theorem or may have had vague notions of it. The second order differences are where my method differs with Newton's approach.

Newton's second order divided difference is:

$$[x_0 x_1 x_2] = \frac{[x_0 x_1] - [x_1 x_2]}{x_0 - x_2}$$

Compare this with my definition:

$$[x_0 x_1 x_2] = \frac{[x_0 x_1] - [x_1 x_2]}{\mu_0^1 - \mu_1^1} = \frac{f'\left(\mu_0^1\right) - f'\left(\mu_1^1\right)}{\mu_0^1 - \mu_1^1} = f^2\left(\mu_0^2\right)$$

$\rightarrow\ [x_0 x_1 x_2]\left(\mu_0^1 - \mu_1^1\right) = f'\left(\mu_0^1\right) - f'\left(\mu_1^1\right)$

$\rightarrow\ [x_0 x_1 x_2]\left(\mu_0^1 - \mu_1^1\right) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} - f'\left(\mu_1^1\right)$

$\rightarrow\ \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f'\left(\mu_1^1\right) + [x_0 x_1 x_2]\left(\mu_0^1 - \mu_1^1\right)$

$\rightarrow f(x_0) = f(x_1) + (x_0 - x_1) f'\left(\mu_1^1\right) + (x_0 - x_1)\left(\mu_0^1 - \mu_1^1\right)[x_0 x_1 x_2]$

$\rightarrow f(x_0) = f(x_1) + (x_0 - x_1) f'\left(\mu_1^1\right) + (x_0 - x_1)\left(\mu_0^1 - \mu_1^1\right) f^2\left(\mu_0^2\right)$
[NT]

Let's see an example.

If $\quad f(x) = x^4 + x^3 - x^2 + x$

Then

$$f'(x) = 4x^3 + 3x^2 - 2x + 1$$

$$f^2(x) = 12x^2 + 6x - 2$$

Suppose we wish to estimate (3) , that is, $x_0 = 3$:

$$f(3) = f(x_1) + (3 - x_1) f'\left(\mu_1^1\right) + (3 - x_1)\left(\mu_0^1 - \mu_1^1\right) f^2\left(\mu_0^2\right)$$

$x_1 = 4,\quad x_2 = 5$

So,

$$f(3) = f(4) + (3 - 4) f'\left(\mu_1^1\right) + (3 - 4)\left(\mu_0^1 - \mu_1^1\right) f^2\left(\mu_0^2\right)$$

$$f(3) = f(4) - f'\left(\mu_1^1\right) - \left(\mu_0^1 - \mu_1^1\right) f^2\left(\mu_0^2\right)$$

But,

$$[x_0 x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f'\left(\mu_0^1\right) = \frac{f(3) - f(4)}{-1} = \frac{102 - 308}{-1} = \frac{-206}{-1} = 206$$

$$[x_1 x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'\left(\mu_1^1\right) = \frac{f(4) - f(5)}{-1} = \frac{308 - 730}{-1} = \frac{-422}{-1} = 422$$

So,

$$f(3) = f(4) - 422 - \left(\mu_0^1 - \mu_1^1\right) f^2\left(\mu_0^2\right)$$

Also,

$\mu_0^1 = 3.5224534198$ and $\mu_1^1 = 4.51765748556$

$\rightarrow \mu_0^1 - \mu_1^1 = -\ 0.99520406576$

These values of μ were obtained by an iterative method.

So,

$$f\,(3) = 308 - 422 - (-\ 0.99520406576)\, f^2 \left(\mu_0^2\right)$$

But,

$$f^2 \left(\mu_0^2\right) = \frac{f'\left(\mu_0^1\right) - f'\left(\mu_1^1\right)}{\mu_0^1 - \mu_1^1} = \frac{206 - 422}{-0.99520406576} = \frac{216}{0.99520406576}$$

Therefore,

$$f\,(3) = 308 - 422 + (0.99520406576)\left(\frac{216}{0.99520406576}\right)$$

$f\,(3) = 308 - 422 + 216 = 102$ which is exactly what we expected.

Naturally, it would have been far easier just to compute $f(3)$ in this case. However, the idea is used towards finding an easier method of approximation in terms of one abscissa, rather a mixture of abscissa and means. I wonder if Newton would have discovered his interpolation polynomial had he been aware of this exact method I described.

After all, using the first three terms of the Taylor series

$$f\,(x) \approx f\,(a) + (x-a)\,f'\,(a) + (x-a)^2\,\frac{f^2(a)}{2!} + \dots$$

With $x = 3$ and $a = 3.1$, we have:

$$f\,(3) \approx f\,(3.1) + (3-3.1)\,f'\,(3.1) + (3-3.1)^2\,\frac{f^2(3.1)}{2!} + \dots$$

$$f\,(3) \approx 115.6331 - (0.1)\,(142.794) + (0.1)^2\,\frac{131.92}{2!} + \dots$$

$$f\,(3) \approx 115.6331 - (0.1)\,(142.794) + (0.1)^2\,65.96$$

$$= 115.6331 - 14.2794 + 0.6596 = 102.0133$$

This approximate result differs from 102 by 0.0133 which is not surprising because Taylor's Series always results in an approximation, never an exact value.

Compare my equality,

$$f\,(x_0) = f\,(x_1) + (x_0 - x_1)\,f'\left(\mu_1^1\right) + (x_0 - x_1)\left(\mu_0^1 - \mu_1^1\right) f^2\left(\mu_0^2\right)$$

with Taylor's approximation:

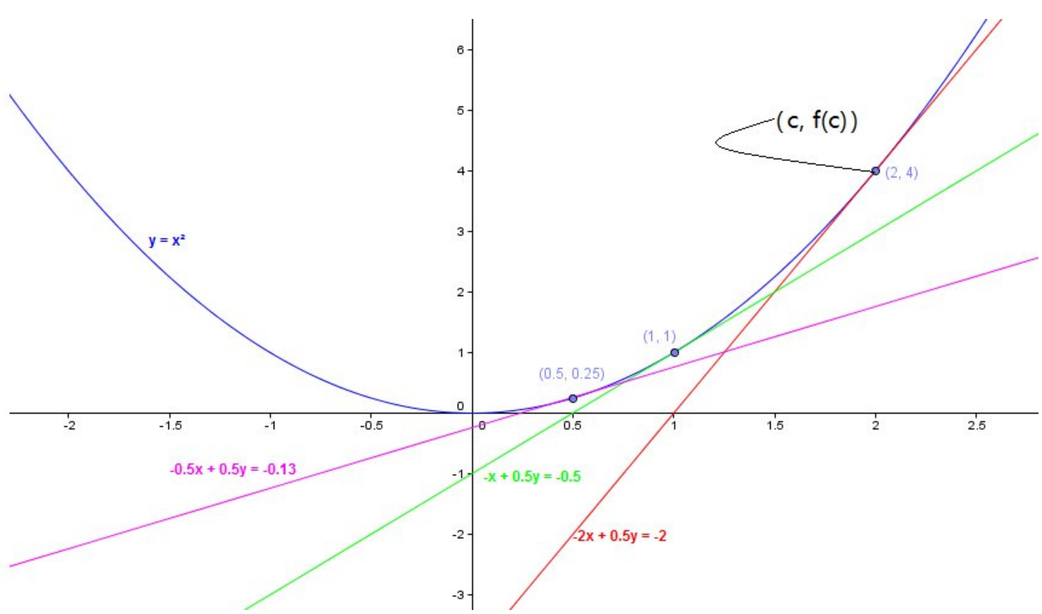
$$f\,(x) \approx f\,(a) + (x-a)\,f'\,(a) + (x-a)^2\,\frac{f^2(a)}{2!} + \dots$$

The Root approximation method.

A quick and simple derivation of the approximation method:

Newton used the tangent line in the design of his root approximation method. The general idea is this:

One can choose any point on a given curve and then theoretically "slide" the tangent line until it intersects the x-axis and the curve itself at the same point. Since the tangent line has nowhere else to go as you slide it along the curve, but gets noticeably closer to the root, it was obvious to Newton, that the tangent line found using the previous tangent line's x-intercept, would intercept the x-axis even closer to the root than its predecessor. The next diagram illustrates the idea:



Suppose the point chosen is $[c, f(c)]$. This is the initial guess.

In order to find the second guess, we need the equation of the tangent line at $x = c$, after which we can find where it intersects the x-axis to obtain the second guess.

Let $t(x) = f'(c)\,x + k$ where $t(x)$ is the equation of the tangent line.

We determine k as follows:

$$f(c) = f'(c)\,c + k$$

$$\rightarrow\ k = f(c) - f'(c)\,c$$

$$\rightarrow\ t(x) = f'(c)\,x + f(c) - f'(c)\,c$$

$$\rightarrow\ t(x) = f'(c)\,[x - c] + f(c)$$

We know the tangent line intersects the x-axis when $t(x) = 0$.

So,

$$0 = f'(c)\,[x - c] + f(c)$$

$$\rightarrow x - c = \frac{f(c)}{f'(c)}$$

$$\rightarrow x = c + \frac{f(c)}{f'(c)}$$

This last equation provides the next guess, that is, x as expected. Simply replace c with the first guess, say x_n and x with the next guess, that is, x_{n+1} and you have the formula:

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

This last formula is in fact an equality, not an approximation. It becomes an approximation once it is realised that x_{n+1} becomes the root:

$x_{n+1} \approx x_n - \frac{f(x_n)}{f'(x_n)}$

Newton's root approximation formula, $x_{n+1} \approx x_n - \frac{f(x_n)}{f'(x_n)}$ [NAP]

can be written as an equality using the mean value theorem:

$x_{n+1} = x_n - \frac{f(x_n) - f(x_{n+1})}{f'(c)}$ where

$f'(c) = \frac{1}{n}\sum_{s=0}^{n-1} f'\left(x_n + \frac{|x_{n+1} - x_n| s}{n}\right)$

$x_{n+1} = x_n - \frac{f(x_n) - f(x_{n+1})}{\frac{1}{n}\sum_{s=0}^{n-1} f'\left(x_n + \frac{|x_{n+1} - x_n| s}{n}\right)}$

Let's see an example.

To see how $\mu_1^1 = 4.51765748556$, we can use the Newton approximation formula [NAP] with

$f'(x) = 4x^3 + 3x^2 - 2x + 1 - 422$ and $f^2(x) = 12x^2 + 6x - 2$

From the equality,

$$x_2 = x_1 - \frac{f(x_1)-f(x_2)}{\frac{1}{n}\sum_{s=0}^{n-1} f'\left(x_n+\frac{|x_2-x_1|s}{n}\right)}$$

We have,

$$x_2 = x_1 - \frac{f(x_1) - f(x_2)}{f'(c)} \rightarrow x_1 f'(c) = x_2 f'(c) + f(x_1) - f(x_2)$$

$$\rightarrow f(x_1) - f(x_2) = x_1 f'(c) - x_2 f'(c)$$

$$\rightarrow f(x_1) - f(x_2) = f'(c)(x_1 - x_2)$$

$$\rightarrow \frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(c)$$

So,

$$\frac{308-730}{4-5} = 422 = f'(c) \quad \leftrightarrow \quad c = \mu_1^1 = 4.5176574856$$

We know that

$$\frac{1}{n}\sum_{s=0}^{n-1} f'\left(x_1 + \frac{|x_2-x_1|s}{n}\right) = 422 \quad \leftrightarrow \quad f'\left(\mu_1^1\right) = 422$$

So,

$x_2 = 4 - \frac{308-730}{422} = 5$ exactly as expected.

Appendix A:

A second rigorous formulation:

In January 2020, I discovered my historic geometric theorem as a result of the well-formed concepts of the New Calculus.

Appendix B:

How we got numbers, arithmetic and algebra from geometry is a fascinating story.