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ON AN "OLD" CONCEPT OF QUANTITY

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The historical process of development of science and mathematics is certainly formidable. However, situations that lead to rethinking appear from time to time in this process. The rethinking that has deeply involved our group[1] originates from the long persistence of unsatisfactory results in teaching and learning fractions. It is a rethinking that we are carrying out paying attention to keep the reflective philosophical practice[2] at the heart of classroom practice. It has begun with the classroom practice of the ten situations proposed by Davydov and Tsvetkovich [1991] to familiarize primary school children with fractions. To this practice, we have associated a reflection of the historical type which refers to Pythagoreans.

Mixing practice and historical reflection has led us to focus our attention on a direct comparison between objects. On the one hand, the first situations proposed by Davydov and Tsvetkovich are situations of direct comparison between objects; because of the difficulties or the impossibility of carrying out the direct comparison, they bring the comparison between objects back to the comparison between their measures. On the other hand, historical reflection has allowed us to highlight how among Pythagoreans, but more generally throughout the ancient world, there was a method of direct comparison between objects: mutual subtraction – anthyphairesis. This method resulted in the Pythagorean mathematization of comparison: comparison is a logos.

This form of mathematization soon entered into a crisis both because of the discovery of incommensurable quantities and because of the ineffectiveness of the subtractive procedure

[1] Our "Research Group on the teaching of mathematics for primary school (University of Milan Bicocca)" has been active for twenty years.

[2] The expression "Reflective Philosophical practice" is taken from Radford. The meaning we give to it is explained in Bonissoni et al., 2021.

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that it entails. Already at the time of Platonic Academy, it was overshadowed by the extraordinary effectiveness of the multiplicative procedure of the Euclidean algorithm. Hence, the Pythagorean comparison was forgotten while the Euclidean algorithm has become one of the pillars on which Western mathematics and science are built.

Our rethinking dwells on Pythagorean forgotten practice, focusing on the act of mathematization: "comparison is a logos".

"Comparison is a logos": the direct comparison between objects – Reciprocity

Let's start with the direct comparison between objects. Classroom practice highlights two features: the reciprocity between the two compared quantities and their common unit. In the traditional process of measurement, these two features are bypassed by choosing one of the two compared quantities as the unit of measurement. Our rethinking of measurement instead starts from paying attention to reciprocity and to the three quantities involved in the comparison activities: the two compared quantities and the common unit. These features distinguish the first steps of our classroom practice on the comparison. The subsequent path towards the usual concept of measurement renounces reciprocity with the choice of one of the two compared quantities as reference quantities.[3] The concept of the common unit is however preserved, forcing to find a didactic mediation with the concept of a unit of measurement. This mediation enriches the reflection on the concept of measurement. If in fact, on the one hand, this way of accessing measurement is usually considered neither in teaching practice nor in the formal procedure of defining the concept, on the other hand, it highlights a widespread practice in everyday life.[4] According to the distinction between practical knowledge and academic knowledge (D'Ambrosio, 1994), a measurement based on the concept of the common unit is practical knowledge but not academic knowledge.

We have practiced this concept of measurement since primary school, with satisfactory results (Bonissoni et al., 2021).

[3]The term "reference quantity" is taken from Sierpinska, 2019.

[4]For example, given a basket of eggs, filling egg cartons of a given size is a measurement in which three quantities appear: the common unit (the egg), the quantity to be measured (the eggs in the basket), and the reference quantity (the eggs in the egg carton).

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“Comparison is a logos”: logos

Let’s now shift our attention to the word “logos”. “Logos” is a polysemic word that acquired different meanings in the historical course of ancient Greece. We have tried to free the word from the numerous meanings that surround it, and we have hypothesized that the meaning that the Pythagoreans attributed to it in the mathematization of comparison is contained in its root: “logos” has the same root as the verb “legéin”, that is “to bind”, “to relate”. So, we translate the Pythagorean act of mathematization into the utterance “comparison is a pair of related numbers”.

As for our practice on fractions in primary school classes, it is enough to take into account the part “comparison is a pair of numbers” of this utterance. It involves a leap from Davydov and Tsvetkovich’s proposal. This leap is manifested in the writing: M; R = 9;3, which reads: the comparison between M and R is the pair 9;3; that is, M contains the common unit 9 times and R contains it 3 times. This writing keeps the trace of a path towards the concept of the fraction that crosses different sub-constructs (ratio, measure, division) of the fraction construct, up to the Euclidean division. The children’s answer has been quiet, serene, and with adequate results (Longoni et al., 2016).

Pair of related numbers

We now come to the complete utterance: “the comparison is a pair of related numbers”. The search for the relationship between two numbers goes beyond the Cartesian idea of pair of numbers and seems to move in a direction not very frequented by research. However, its importance in understanding the concept of logos is also confirmed by Sierpinska’s statement: “A ratio is a relationship between two numbers, not just “two numbers””.

We have constructed an “originary” image of the relationship between two numbers by retracing step by step the procedure that Pythagoreans followed to determine the logos (Bonissoni et al., 2016). These concrete actions allow bringing to light something that has been forgotten.

The Pythagorean comparison searches the higher common unit between the two objects A and B, following a repetitive subtractive procedure. The Euclidean algorithm determinates the measure of a higher common unit between the measures of the two objects A and B, following a dividing procedure.

In the Pythagorean comparison (anthyphairesis), the higher common unit found (if it is possible), the measures of the two objects are constructed by back retracing the two columns: A=mU, B=nU. The distinguishing feature of this comparison is that it develops by “moving”

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The following table compares the two procedures:

Euclidean Algorithm	**Anthyphairesis**	
22 = measure of the object A ***6 = measure of the object B***	***Object A***	***Object B***
22:6 = 3 rest 4	**Subtraction** **Subtraction** **Subtraction**	
6:4 = 1 rest 2		**Subtraction**
4:2 = 2	**Subtraction**	
2 is the measure of higher common unit	**U is the *higher common unit* *between the objects A and B***	

between two columns. The relationship between the two final numbers is the "story" of this "movement".

The Euclidean algorithm proceeds in an extremely effective way, but it bypasses the relationship, the story between the two numbers, removing attention to the "movement" between the two columns that the subtractive procedure performs step by step.

Unique additive decomposition

One particularly significant feature of the subtractive Pythagorean procedure is the unique additive partition: of each pair of natural numbers, there is a unique additive partition. For example: the pair (11; 3) has the unique partition (1 + 1 + 3 + 3 + 3; 1 + 2). The number 11, when it belongs to another pair, has another partition: (11; 7) ® (1 + 3 + 7; 1 + 1 + 1 + 4). The partition is unique because it reproduces step by step the subtractive procedure of the corresponding Pythagorean comparison. This property seems to disagree with common knowledge: the fundamental theorem of arithmetic states that each natural number n can be prime factorized in a unique way, but a corresponding theorem for additive partition doesn't exist.

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A "new" concept of quantity

Attention to the relationship between pairs of numbers can open new horizons.

The question that most intrigues us is the following: does attention to Pythagorean comparison suggest rethinking foundations? Euclidean world is based on the unique "originary" act "counting is a number" and emerges from this act thanks to a continuous process of extension. The Pythagorean world is based on two "originary" acts: "Counting is a number" and "Comparison is a pair of related numbers". What possibilities can emerge from the hidden Pythagorean world? The reciprocity that characterizes the Pythagorean world deserves attention. The Pythagorean mathematization "Comparison is a logos" transforms comparison into a primitive measurement, endowed with reciprocity. This reciprocal measurement suggests a new concept of quantity.

The concept of quantity is mostly developed in close connection with the concept of number: the number 11 is a quantity. Davydov [1991] underlines the link of the concept of quantity with comparison. "A quantity is completely determined in mathematics when a set of elements and the criteria of comparison are determined [...]."

The comparison based on the old concept of measurement indicates a new concept of quantity, in which the focus is shifted from quantification to dual reciprocity: the number 11 is no longer a quantity; it takes on meaning thanks to the number with which it is paired.

Duality is the core of numerous phenomena of nature: Electromagnetism, Quantum physics, Living systems ... Is it possible that some paradoxes affecting these phenomena can be addressed by introducing a measurement with reciprocity?

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