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The sound of music and its link with mathematics

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Abstract

This article discusses a relatively unnoticed application of mathematics by describing its connection with an aspect of music, in particular, the musical scales. Stemming from a problem found in a Year 9 mathematics textbook commonly used in Singapore, the article illustrates the role of mathematics in musical scales by first considering the frequency ratios of consecutive musical notes in the 'just scale' and secondly explaining how an anomaly in the 'just scale' caused by the uneven frequency ratios is resolved with the help of mathematics, thereby leading to the development of the well-tempered scale. The article ends with an exploration of the frequency ratios of consecutive musical notes in the well-tempered scale. Not only does the article aim to broaden the teachers' horizons with such an introduction to the mathematical aspect of music, it also hopes to enrich their mathematical experiences as well.

1. Introduction

The application of mathematics can be widely found in many disciplines. In computer programming, mathematics is an important tool in creating computer games so that the animation can simulate real situations. In a game of pool, the players need to figure out the correct angle to hit a ball with a cue ball so that it will drop into a pocket. In an orchestral performance, the concert master is often seen leading the other musicians to tune their instruments prior to playing together. Some readers may begin to ask what the last example has got to do with mathematics. Indeed, not many people realize that mathematics also lends itself to the performing arts, in particular, music; and as a result of it, this article aims to raise their awareness of such an application of mathematics. The article begins with an introduction to a mathematics textbook problem which centres on the computation of frequency ratios, and follows with a description of the mathematics involved in two types of musical scales: the just scale and the well-tempered scale.

2. The mathematical problem

The textbook problem used in our discussion here is provided in Fig. 1. Found in one of the mathematics textbooks commonly used by Year 9 students in Singapore, this problem, set in the context of tuning piano keys, links mathematics and music.

9. Piano keys are tuned so that the ratio of the frequencies of consecutive notes is a constant r. (For example, the C (white) and $C^{\#}$ (black) are consecutive notes.)

(a) If the frequency of a C note is f, express in terms of f and r, the frequency of

(i) the next note, which is $C^{\#}$ (black key),

(ii) the next C note.

(b) Given that the frequencies of the C notes double as one goes from one to the other, find the value of r in index form.

(i) Use a calculator to show that its value is approximately 1.059 5.

(ii) If the frequency of the middle C (i.e. the first C note in the diagram) is 261.1 cycles per second, find the frequency of the E note.

Fig 1. The textbook problem. (*Source*: (1)).

Students can usually solve the part questions by following the given instructions and teachers' guidance, but it seems inadequate for them just to be able to derive the answers. For learning to be more meaningful, it appears essential for the teachers to discuss this problem in greater depth with their students. In view of a noteworthy characteristic of mathematics that its concepts are applied in many other disciplines, one of which is music, a discussion on the mathematical aspects of music amidst solving the textbook problem helps to emphasize and reinforce the relevance of certain mathematical concepts in real life. Topics for discussion can include an exploration of the reason why the black piano keys are grouped in twos and threes, and also why the frequency ratio of any consecutive note is a constant value of 1.0595. On certain occasions, teachers may even be caught by surprise when they are asked these questions during class and impromptu responses are needed. So to be able to deal with these questions or engage students in a purposeful classroom discussion, teachers need to have some background knowledge of the mathematics–music connection. In the following section, this connection is described with the aim of equipping teachers with some fundamental ideas about the mathematics behind music.

3. The mathematics—music connection

As mentioned in the textbook problem, the white C key and the black C# key produce notes that are said to be consecutive. However, a closer look at the given diagram of a segment of a piano keyboard in Fig. 1 shows that the C# key is not indicated as is the C key. So, which exactly is the black key producing the C# note? Readers without any music training may not be able to identify it correctly. Hence it would seem appropriate to begin this section with an introduction of the musical scale first.

Made up of seven white and five black keys, the present musical scale is a series of twelve notes, which is recursive on a keyboard instrument such as a piano. A segment of a piano keyboard, showing a fully-labelled octave, is presented in Fig. 2.

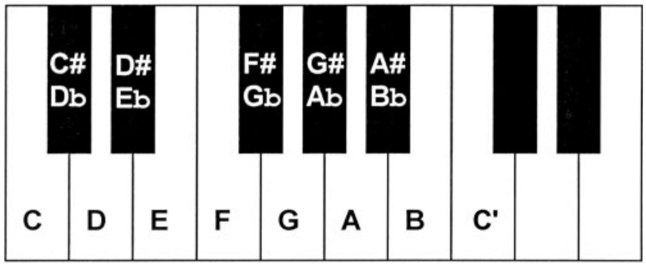

Fig 2. An octave on a piano keyboard.

The white keys are given letter names from A to G following a cyclic order of A, B, C, D, E, F, G, A, B, C and so on, while the naming convention of the black keys derives from the surrounding white keys (2). For instance, the black key between C and D is named either C# or Db, depending on the harmonic context from which it is viewed. When the black key is viewed as a rise in pitch from C, then it is called C#. In a similar manner, it is named Db when viewed as a lowering of pitch from D. So in general, the notation '#' denotes a rise in pitch whereas 'b' signifies a lowering of pitch.

Before this present musical scale was developed, there was what is typically known as the 'just scale', which occurs naturally in vibrating strings. The development of the 'just scale' can be traced to ancient Greek contributions to mathematics and science (3). However, not many people realize that it was actually Pythagoras, the Greek mathematician whose name is often associated with the famous theorem relating the three sides of a right-angled triangle, who made a significant impact on the western music theory. He discovered that pleasant-sounding notes were produced by taking fractions of the frequency of the reference note.

From his experiment with vibrating strings, Pythagoras found that when two stretched strings, one being half the length of the other, were plucked, the notes that were produced sounded the same except that the one generated by the shorter string was higher in pitch. These two notes are then said to be an *octave* apart, which is an interval between twelve consecutive notes such as C and the next C (subsequently denoted by C' in the article). In terms of frequencies, the higher note produced by the shorter string has a frequency that is twice that of the lower note produced by the longer string. This observational 'proof' validates the given fact in the textbook problem that 'the frequencies of the C notes double as one goes from one to the other'.

Pythagoras also discovered that when the shorter string was two-thirds the length of the longer string, then the note produced is said to be a *fifth*[1] higher than the note generated by the longer string. In other words, if the note produced by the longer string is C, then this shorter string will produce the G note, and its frequency is 3/2 that of C. Similarly, when the length of the shorter string was three-quarters the length of the longer string which produced the C note, then the F note was heard, and its frequency is 4/3 that of C. In music terminology, the F note is said to be a *fourth*[2] higher than the C note. In summary, Pythagoras discovered that the ratios of 1/2, 2/3 and 3/4 for the lengths of the plucked strings under the same tension gave rise to the octave, the fifth and the fourth, respectively (3).

[1]A *fifth* refers to the musical interval, based on the just scale, between one note and another that is five notes away from it. For instance, E to B is a fifth because B is five notes away from E, and so is D to A.
[2]A *fourth* refers to the musical interval, based on the just scale, between one note and another that is four notes away from it. For instance, D to G is a fourth, and so are E to A and F to B.

Then in the 17th century, Galileo, the famous Italian astronomer who conjectured that the sun is located at the centre of the solar system, inquired into Pythagoras' discovery when he conducted experiments with music. He discovered that there was indeed a relationship between the length of the plucked string and the frequency of the note produced. This relationship simply reveals that the frequency of the note on a string varies inversely with the length of the string (4).

3.1. Part 1—Computing the frequency ratios of notes with reference to C

When the frequency ratios for these three intervals are known, the computation of the frequency ratios for the other intervals such as C to D, C to E, C to A, and C to B is then made possible. In this article, two examples are provided to illustrate how the computation is to be done. The first example involves determining the frequency ratio for C to C′ using the known facts that the interval from C to G has a frequency ratio of 3/2 and the interval from C to F has a ratio of 4/3. Suppose the frequency of C is x. Then the frequency of G above C is $(3/2)x$. Subsequently taking a fourth from G (that is, the next C above G) will yield the frequency of C′ as follows:

$$\text{Frequency} = \left(\frac{3}{2}x\right) \times \frac{4}{3} = 2x$$

This calculation confirms that sequentially taking a fifth (i.e. C to G) followed by a fourth (i.e. G to C′) results in an octave, and it has been demonstrated mathematically that the frequency of C′ is indeed twice that of C. This result is once again consistent with both Pythagoras' discovery and the given information in the textbook problem.

The second example demonstrates the computation of the frequency ratio for C to D using a slightly different approach from the first. What is interesting here is that the frequency ratio for C to D is similar to that for F to G, and to calculate the frequency ratio for the interval from F to G, it is necessary to work from F to C, then to G using the transitivity property. The frequency ratio for C to F is 4/3, and so the frequency ratio for F to C is 3/4. Also given that the frequency ratio for C to G is 3/2, the frequency ratio for F to G is therefore $(3/4) \times (3/2) = 9/8$. Hence, the frequency ratio for C to D is found. Table 1 presents the frequency ratios for all the various intervals.

As clearly seen from Table 1, all the frequency ratios are rational numbers. These frequency ratios and the same transitivity approach can be used to find the frequency ratios of consecutive notes such as D to E, E to F, G to A, A to B, and B to C'. Again, two examples are provided to demonstrate how these ratios can be determined mathematically. First, the ratio for D to E can be found by working from D to C, and then to E. Given that the frequency ratios for D to C and for C to E are 8/9 and 5/4, respectively, it then follows that the frequency ratio for D to E is $(8/9) \times (5/4) = 10/9$. Next, a similar approach shows that the frequency ratio for A to B is $(3/5) \times (15/8) = 9/8$ by working from A to C, then to B. It will be helpful to remember that the ratios for C to D, as well as for F to G, have already been determined previously and so it is unnecessary to waste time in computing them again. All the expected frequency ratios of the consecutive notes are presented in Table 2.

Table 1. Frequency ratios of the notes with reference to C

C to:	D	E	F	G	A	B	C'
Frequency ratio	9/8	5/4	4/3	3/2	5/3	15/8	2/1

In contrast to what is stated in the textbook problem, Table 2 clearly reveals that the frequency ratios of the consecutive notes are not constant. Why is this so? This is because there is an anomaly in the 'just scale' which created several complications for musical composition and performance in the 17th century. This anomaly was not resolved until the beginning of the 18th century (5) with the introduction of the well-tempered scale. In the next section, the discussion will focus on how the problem in the 'just scale' leads to the development of the well-tempered scale.

3.2. Part 2—Developing the well-tempered scale

The previous section uncovers the problem of uneven distribution of the naturally occurring intervals between notes in the 'just scale'. Consequently, certain notes just seem to cause a dissonance when played together. To mitigate this problem so as to produce harmonious sound, the C to C′ interval has to be divided up in such a way that the frequency ratios of consecutive notes are equivalent. The way to go about doing it comes from a consideration of the frequency ratios in Table 2. What is interesting and noteworthy about Table 2, when examined carefully, is the fact that the frequency ratios take essentially three different values, namely 9/8, 10/9 and 16/15. Additionally, only the frequency ratios of the two intervals from E to F and B to C ($16/15 = 1.06$) are <1.10, whereas the other ratios of 9/8 ($=1.125$) and 10/9 ($=1.11$) are all >1.10. Now, some readers may wonder what is so important and special about classifying the intervals into these two broad categories of <1.10 and >1.10, because after all, all the intervals seem to have the same frequency ratio of 1.1 when rounded off to one decimal place. However, what they do not realize is that the crux of the whole idea of dividing up the C to C′ interval lies within such a classification.

In the 18th century, it was discovered that the division of the C to C′ interval into two sections by the consecutive notes E and F as well as B and C′, thus forming C, D and E in one part while F, G, A and B in another, appears to bring about a possibility of yielding the same frequency ratios for consecutive notes. This happens only when five new keys are created between the notes C and D, D and E, F and G, G and A, as well as A and B, but not between E and F, and B and C′. The resulting outcome of this situation gives rise to the development of a totally new musical scale, which is commonly known as the well-tempered scale nowadays (Fig. 2). It is worthy to highlight that these five newly-added keys correspond to the black keys in an octave. Now, having explained clearly the reason for arranging the black keys in groups of twos and threes, the study will then proceed to consider the frequency ratios of consecutive notes in the next section.

3.3. Part 3—Determining the constant factor

The previous section explains why the piano keyboard is arranged as a set of three white keys with two black keys and another set of four white with three black in an octave. It is easy to see from Fig. 2 that the C–C′ interval is divided up into twelve equal parts, with seven white keys and five black. So, what should the frequency ratio between any two consecutive notes of

Table 2. Frequency ratios of consecutive notes

	$C \to D$	$D \to E$	$E \to F$	$F \to G$	$G \to A$	$A \to B$	$B \to C'$
Frequency ratio	9/8	10/9	16/15	9/8	10/9	9/8	16/15

this twelve-tone scale be if there were to be a constant factor between successive frequencies? This section discusses this question and also explains why this constant has to be equivalent to the twelfth root of two, and not twelfths.

As evidenced by Pythagoras' discovery and the mathematical calculation shown previously, the frequency ratio of C to C′ is known to be 2. This means that if the frequency of C is taken to be f, then the frequency of C′ corresponds to $2f$. Furthermore, it is also known that there are twelve equal parts in the C to C′ interval. So piecing such information together, it will not be surprising to expect some readers to think and conjecture that the frequency of each note is one-twelfth more than that of the note directly below it, exactly like what Table 3 shows.

If this conjecture is true, then the frequency ratio between any two consecutive notes should be a constant. However, simple computations will easily disprove this conjecture. Take the following two examples, for instance:

$$\text{Frequency ratio of C to C\#} = \frac{(13/12)f}{f} = \frac{13}{12}$$

$$\text{Frequency ratio of E to F} = \frac{(17/12)f}{(16/12)f} = \frac{17}{16}$$

The two frequency ratios are obviously different. Hence, the frequency ratios of any two consecutive notes can never be a constant if there is a common difference of one-twelfth between successive frequencies.

So what frequencies within the range of f and $2f$ should all the notes in the C to C′ interval be assigned to so as to attain a constant for the frequency ratios of any two consecutive notes? Well, this crucial question is answered only when the frequencies of the notes form a geometric progression. In other words, the frequency of a note is r times as much as that of the note directly below it, where r is the constant ratio between the frequencies of the consecutive notes. Thus, suppose f is the frequency of the C note, then the frequency of the C# note is fr, that of the D note is fr^2, and so on. The frequencies of all the notes in an octave are listed in Table 4 below.

With the establishment of these frequencies, the value of the constant r can be easily determined. Since the frequencies of the C notes double from one to the next, the required equation in terms of f and r can readily be established: $fr^{12}/f = 2$. This equation reduces to $r^{12} = 2$, which then simplifies further to produce $r = 2^{1/12}$. The value $2^{1/12}$, which turns out to be an irrational number, is the twelfth root of two. When evaluated to four decimal places using

Table 3. Frequency of notes if the difference is constant

Note	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	next C
Frequency	f	$(13/12)f$	$(14/12)f$	$(15/12)f$	$(16/12)f$	$(17/12)f$	$(18/12)f$	$(19/12)f$	$(20/12)f$	$(21/12)f$	$(22/12)f$	$(23/12)f$	$2f$

Table 4. Frequency of notes if the ratio is constant

Note	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	next C
Frequency	f	fr	fr^2	fr^3	fr^4	fr^5	fr^6	fr^7	fr^8	fr^9	fr^{10}	fr^{11}	fr^{12}

a calculator, its value is 1.0595 and this is precisely what part (bi) of the textbook problem requires the students to compute.

Interestingly, the frequency ratios for those intervals listed in Table 2 seem to remain almost the same even under the well-tempered scale. For instance, the frequency ratio for the interval E to F under the well-tempered scale is found to be $2^{1/12}$ (≈ 1.0595), which is very close to the value of 16/15 ($=1.06$), found earlier in Part 2. Likewise, the frequency ratio for the interval C to D, which is $(2^{1/12})^2$ or 1.1225 when evaluated to four decimal places, also does not differ very much from 9/8 ($=1.125$).

4. Conclusion

This article describes three aspects of the mathematics–music connection, starting with an exploration of the frequency ratios of musical notes in the 'just scale', then an explanation of how the well-tempered scale was developed, and ending with a discussion of the constant frequency ratios of consecutive notes in the well-tempered scale. Its aims are not only to introduce the relationship between mathematics and music to teachers and raise their awareness, but, more importantly, also to ensure they are sufficiently knowledgeable about this relationship and the significant mathematical concepts involved in it to hold an engaging discussion with their students.

Apart from these aspects discussed here, there are still numerous worthwhile areas within the topic of music that deserve further investigation. A manifestation of the application of mathematics in music is demonstrated in the functioning of some brass instruments such as a bugle and a trumpet. For instance, a bugle, essentially a metal conical tube with a mouthpiece, has no valves at all, and yet can produce different notes. How is that possible? This is one area where teachers can explore and learn more about this phenomenon which makes use of the harmonic series, a particularly important application for brass instruments. Finally, it is hoped that they will show greater interest and initiative in spearheading the effort in exploring this relatively unnoticed mathematics–music connection further with their students, thereby enhancing their learning of mathematics.

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