SURREY
MATHS SCHOOL

SuMS PROJECT QUALIFICATION

Project Briefs

Principles of the SPQ

The SPQ is a programme in which our passionate and ambitious students complete collaborative research projects. Guided by professionals in industry or by academics at universities, they will work in groups of three to five on a STEM related problem. The opportunity to complete a meaningful research project relating to these subject areas will help them to develop necessary practical skills that they will need in higher education, and indeed their careers, and will give them a chance to extend their interests beyond the curriculum. The main outcomes of the project are

- To develop students' academic literacy
- To provide students with a direct experience of research
- To develop students' skills of collaboration, communication and self-management
- To provide students with the experience of working with either an academic at a university or an industry professional

Project outcomes

The SPQ project is made up of three parts:

- A written literature review to introduce the problem and to showcase their research skills
- A written report, poster designed in LaTeX, a podcast or a documentary.
- A five to ten-minute presentation at the end of the school year to staff, parents, governors, mentors and other invited guests.

Acknowledgements

The SPQ project could not have gone ahead without the incredible support of the Faculty of Engineering and Physical Sciences at the University of Surrey. The projects the academics have provided and time they have committed to work with our students is hugely appreciated. Special thanks must go to Maths Education Committee who have coordinated the collaboration between the University and SuMS from the beginning.

We have a number of industry professionals to thank also. Their contributions have further increased the breath of the type of projects offered and have given the students the opportunity to apply the content they study to real-life applications.

Project 1: Population Modelling and the Logistic Map

Set by Dr Matt Turner

Introduction

Consider a population of animals living in the wild (pick your favourite animals!). We denote the population to be x_n in year n. Here we choose $0 \le x_n \le 1$, and so the value x_n is the ratio of the current population to the maximum population ($x_n = 1$). Or, if you prefer, think of x_n as meaning "x_n thousand animals in year n".

Being mathematicians, we might want to model the change in population from year n to year n+ 1 in order to predict future population levels in later years as the animals will breed (increasing the population), but also some animals will die (reducing the population). The simplest way to model this could be by a map or function f such that

$x_{n+1} = f(x_n)$

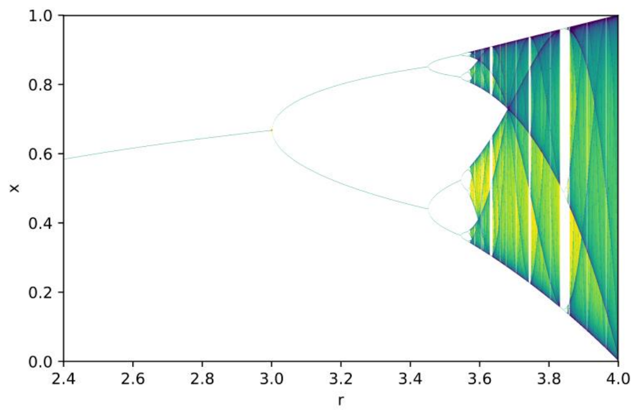
Here the population in year n+1 only depends upon the population in year n. Therefore, if we start with some population x_0 at year n = 0, then we calculate $x_1 = f(x_0)$ from x_0, $x_2 = f(x_1)$ from x_1, etc. This is called iterating the map f with the initial condition x_0. There are many choices of possible maps, but the most famous and well-studied is the logistic map

$f(x) = ax(1 - x)$

Here a is a parameter, which is a quantity we fix before iterating the map, and we take a to satisfy $0 \le a \le 4$. This choice of a ensures that if $0 \le x_n \le 1$ then $0 \le x_{n+1} \le 1$, in keeping with our modelling. Check this for yourself!

Problems to be investigated

- Developing computer programmes in a language such as Python, see what happens to x_n
- Research what a cobweb plot is, and write a computer script in Python to produce them for different logistic maps
- Bifurcation diagrams and how they behave as the parameters vary
- Mathematical chaos
- How logistic maps link to population modelling



Project 2: Methods of solving Polynomial Equations

Set by Dr David Fisher

Introduction

Methods of solving quadratic equations have been known since ancient times. By completing the square, we get a formula for the solutions of

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is a solution by radicals, i.e. it is obtained from the coefficients by adding, subtracting, multiplying, dividing and taking roots.

It is natural to ask whether higher-degree polynomial equations can be approached in a similar way, by finding formulae for the solutions of a cubic equation.

$$ax^3 + bx^2 + cx + d = 0$$

and a quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Computers can solve polynomial equations to any degree of accuracy that is required for practical purposes, but the problem of finding exact expressions for the solutions has inspired many important mathematical discoveries.

Questions to investigate

The aim of this project is to study some methods that have been developed for dealing with polynomials. In particular, you could investigate some or all of the following.

- Classical methods for solving quadratic equations, both by algebra and by geometry
- The extension of the number system to include negative numbers, and later complex numbers, in order to solve equations
- Algorithms for solving cubic and quartic equations by radicals
- The development of group theory to describe symmetries of the roots of equations
- The work of Abel, Galois and others on polynomials of degree 5 or more

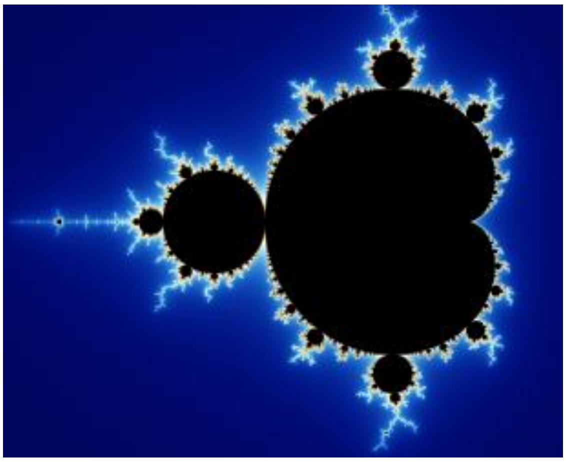
Project 3: Exploring Fractals: Sequences and the Mandelbrot set

Set by Dr Jan Gutowski

Introduction

Dynamical systems appear throughout nature, from understanding planetary orbits, to the weather, as well as in population modelling. The question of how well we can predict the future using mathematical equations is of critical importance. In the 20th century, the introduction of computers provided a very direct way to visualize the remarkable insight that even simple equations can have very complicated dynamics.

This project introduces the strange world of fractals, using one of the most famous examples, the Mandelbrot set. The first part of the project involves researching key mathematical ideas relating to complex numbers, and also some basic properties of sequences. The later parts of the project involve some numerical calculations, to produce an approximation of the Mandelbrot set, as well as investigating formal mathematical proofs which are used to understand key properties of the Mandelbrot set, and the relationship between the Mandelbrot set and some other interesting sequences.



Problems to be explored

The following sequence of complex numbers is used to generate the Mandelbrot set:

$$z_{n+1} = z_n^2 + c, \quad z_1 = 0$$

where c is a complex number. Depending on the choice of c, this sequence may converge, or diverge, or remain bounded without either converging or diverging. The Mandelbrot set consists of those choices of c such that the sequence stays bounded.

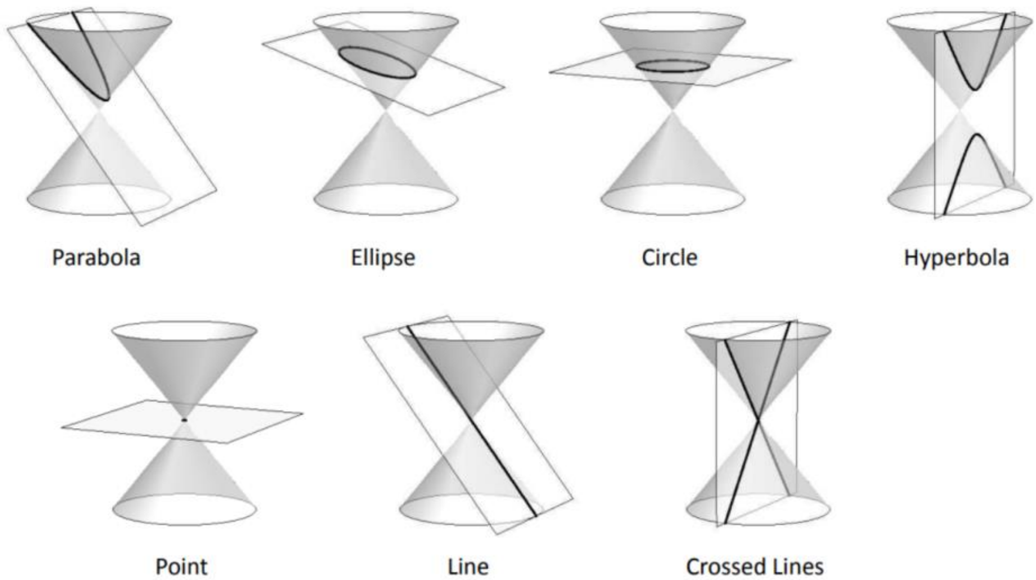
- By doing some numerical analysis, can you produce an approximation to the Mandelbrot set?
- More formally, can you prove (using mathematical induction) some results which can be used to show that a particular value of c is not an element of the Mandelbrot set?
- Can we say anything about those values of c for which the sequence converges?
- Can we say something about the geometric properties of the Mandelbrot set?
- How can we relate the Mandelbrot set sequence to some other sequences such as the logistic map (related to population modelling)?

Project 4: Conic sections and the Kepler problem in Classical Mechanics

Set by Dr Andrea Prinsloo

Introduction

Conic sections were obtained by ancient Greek mathematicians as the curves of intersection of a cone in 3-dimensions with various planes. The study of the orbits of planets and other celestial bodies is known as the Kepler problem in Classical Mechanics. These orbits form conic sections in the orbital plane. The planets, for instance, move in elliptic orbits with eccentricity $0 \leq \varepsilon < 1$ quantifying the deviation of the orbit from the circular (the case $\varepsilon = 0$).



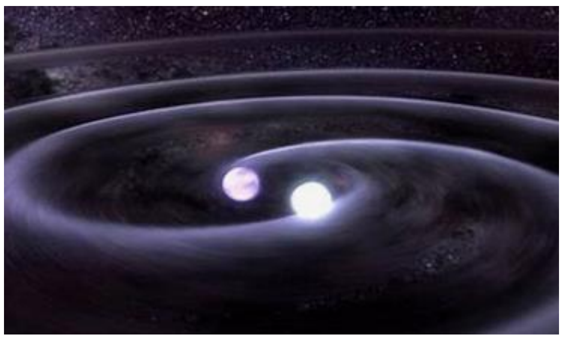
Problems to be investigated

- The geometry of conic sections, particularly circles, ellipses, hyperbolae and parabolae.

- Deriving the orbits of celestial bodies about the sun using Newton's law of gravitation in Classical Mechanics

- Use a computer programming language to plot the orbits of planets or other celestial bodies about the sun (extension)

- Consider a similar two-body system, for example two stars orbiting each other in a binary star system (extension)



Project 5: Designing IoT Solutions for Enhancing Urban Living in Smart Cities

Set by Rajaa Al Kiyumi

Context

The concept of Smart Cities revolves around using technology to improve the quality of life for citizens, increase the efficiency of urban services, and promote sustainability. The Internet of Things (IoT) plays a crucial role in achieving these goals by connecting various devices and systems, allowing them to communicate and share data. From smart traffic management to intelligent waste disposal, IoT applications have the potential to transform urban environments into more liveable and sustainable spaces. However, the implementation of these technologies also presents significant challenges, such as data privacy, security, and the integration of diverse systems.

Problem to be investigated

Traffic congestion is a significant urban issue affecting the efficiency of transportation systems, leading to increased travel times, fuel consumption, and air pollution. In rapidly growing cities, the demand for road space often exceeds capacity, resulting in frequent traffic jams, especially during peak hours. Conventional traffic management systems struggle to adapt to real-time conditions, leading to inefficiencies and frustration among commuters.

Research objective

The objective of this research is to design an IoT-based traffic management system that uses real-time data from connected vehicles, traffic sensors, and other urban infrastructure to optimize traffic flow in a smart city. This system will aim to reduce congestion by dynamically adjusting traffic signals, providing real-time route recommendations to drivers, and integrating public transportation schedules. The research will evaluate the potential impact of the proposed solution on reducing travel time, improving fuel efficiency, and minimizing environmental impact, while also assessing challenges such as data privacy, system scalability, and implementation costs.

Project 6: AI based Waste Sorting System

Set by Rajaa Al Kiyumi

Context

Proper waste management is a critical issue for environmental sustainability, with recycling playing a key role in reducing landfill waste and conserving natural resources. However, effective recycling relies on accurate sorting of waste into appropriate categories such as plastic, paper, metal, and organic materials. Manual sorting is labour-intensive and prone to errors, leading to contamination of recyclable materials and inefficiencies in waste management systems. With the rise of smart technologies, there is an increasing interest in developing automated solutions that can assist in sorting waste accurately and efficiently.

Problem to be investigated

The current waste sorting process is inefficient and often inaccurate, leading to reduced recycling rates and increased environmental pollution. There is a need for an automated system that can classify different types of waste using image recognition, thereby improving the efficiency and accuracy of the sorting process.

Research objectives

- To develop a machine learning model capable of classifying images of waste into categories such as plastic, paper, metal, and organic.
- To create a real-time classification system that can be integrated into automated waste sorting solutions.
- To evaluate the model's performance and identify ways to improve its accuracy and reliability.
- To explore the potential integration of this system with hardware in smart recycling bins or other waste management solutions.

Project 7: Why don’t plants get sunburn? A quantum chemistry and femtosecond laser spectroscopy study

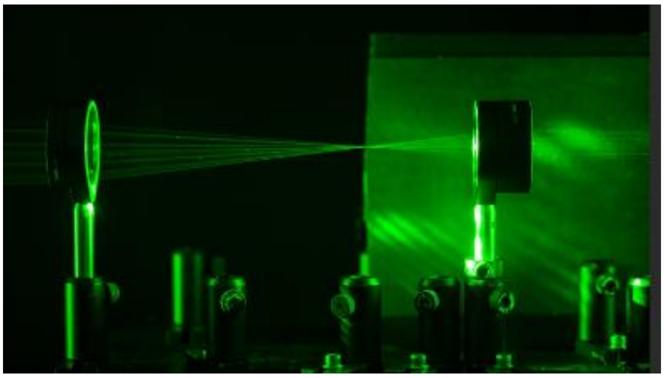
Set by Dr Lewis Baker

Introduction

Many plants have a sophisticated biochemical mechanism for producing natural sun screening molecules to protect themselves when overexposed to ultraviolet (UV) light. This process can be upregulated or downregulated in response to changing UV conditions via its phenylpropanoid pathway, maintaining a dynamic optimal equilibrium within the so-called ‘burden of disease’. These molecules belong to a class of molecules known as cinnamates, with the most well-known called sinapoyl malate. These molecules reside between the incoming UV light and sensitive cell organisms, such as the cell nucleus which contains a cell’s DNA, absorbing the UV light and dissipating the energy through pathways which are less harmful to the plant. The specific mechanism of this UV absorption and subsequent deactivation is the subject of this research and analysis project.

The project

You will be provided with an array of cutting-edge experimental and quantum chemistry simulations which you can explore in order to ascertain the identity of the deactivation pathway. You will have to compare gas-phase and solution-phase experimental data, and triangulate these with high-fidelity ab initio quantum chemistry calculations. You will also be expected to conduct you own calculations (with some guidance) to understand the molecular electronic arrangement in the molecules studied, as well as visualise all the data in publication-quality ways including: 2-D heat maps, kinetic fitting, geometry optimisation, 3-D electronic probability densities, identifying the chemical characteristics of electronic transitions.



Outcomes

- Prepare an initial geometry and conduct a simple force-field molecular mechanics initial optimisation.

- Conduct quantum mechanical calculations including: geometry optimisation using density functional theory and evaluate its accuracy through chemical intuition and a supporting frequency calculation.

- Calculate the vertical excitations of this optimised molecule to determine the likely UV absorption transition, and confirm the nature of the transition by generating surface plot files to visualise in a programme such as Visual Molecular Dynamics.

- Modify these calculations using a polarisable continuum model such as COSMO to simulate a solvent environment. Explain, if any, difference between the two systems (no solvent and solvent).

- Using the supporting data files: femtosecond transition absorption (gas and solution phase) data, along with additional calculations provided (potential energy surface scan and surface hopping QM/MM population dynamics), to confirm the likely deactivation mechanism.

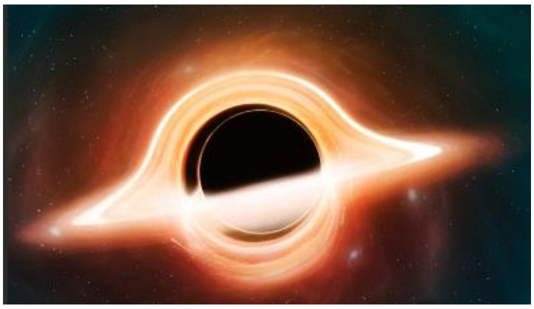
Project 8: Mysteries of the Universe

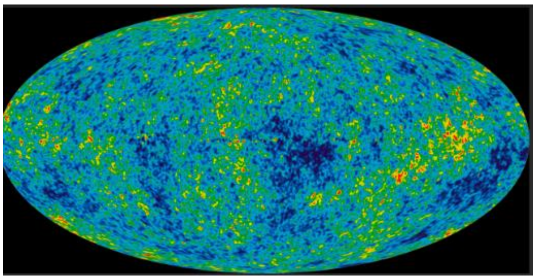
Set by Katie Johnson

Introduction

The universe is vast, complex, and full of mysteries that continue to baffle even the brightest minds in science. While we've made extraordinary progress in understanding the stars, galaxies, and the laws that govern everything, there's still so much we don't know. From the forces that hold galaxies together to the strange, invisible matter that makes up most of the universe, the cosmos is filled with unanswered questions.

This project, Mysteries of the Universe, invites you to explore these unknowns. What are the big cosmic puzzles that scientists are trying to solve? What theories have been proposed to answer these questions, and how are they tested through experiments and observations? Your task is to research and present the cutting-edge ideas and hypotheses that aim to explain the great mysteries of our universe.





In your exploration, consider how our current knowledge is both vast and limited—astronomers have mapped stars billions of light years away, yet we still can't fully explain fundamental forces like gravity. By investigating these mysteries, you'll gain a deeper appreciation for the exciting, ongoing journey of discovery that is astronomy.

Potential problems to be investigated

- Dark Matter: What is it, and why is it essential for understanding the structure of galaxies?
- Dark Energy: What role does it play in the expansion of the universe, and why is it one of the biggest mysteries in cosmology?
- Black Holes: How do they form, and what happens inside them? Do they hold the key to understanding gravity and quantum mechanics?
- The Nature of Gravity: We know how it works, but why does it exist? How does it fit into the larger framework of the universe?
- The Origin of the Universe: What was the Big Bang, and what came before it? Are there other universes beyond our own?
- The Fate of the Universe: How will the universe evolve? Will it expand forever, or will it collapse back into itself?
- Life Beyond Earth: What are the chances that life exists elsewhere in the universe, and how are we searching for it?

Project 9: The Power of Discounted Cashflows

Set by Katie Johnson

Introduction

In the world of finance, understanding the time value of money is crucial for making informed investment decisions. Discounted cashflows (DCF) are the backbone of financial analysis, used by professionals to determine the value of investments, forecast profits, and measure returns. Whether you're working in investment banking, real estate or corporate finance, mastering the principles behind DCFs will be invaluable.

In this project, you will explore the concept of discounted cashflows by creating a financial model based on a real estate investment scenario. The model will analyse the impact of lease events on the investment's profitability, focusing on key financial metrics such as total profit and Internal Rate of Return (IRR).

The situation

A large office building – Infinity Heights - is up for sale for a price of £25 million. It’s currently let to a single tenant, Paradox Partners who pay an annual rent of £2.25 million (on a quarterly basis). Their lease expires on 31st December 2025. Your task is to evaluate two scenarios and their effect on the investment returns:

Scenario A

Paradox Partners vacates at lease expiry, the property is vacant for 12 months and is re-let on a 10-year lease at a rent of £2.65 million per annum, subject to an 18-month rent-free period.

Scenario B

Paradox Partners renew on a 10-year lease at the current rent (£2.25 million per annum), with a 12-month rent-free period.

In both scenarios, assume that the property is sold 5 years after purchase at a yield of 9%.

For each scenario calculate the total profit (split between income profit and capital profit), and the IRR. Determine whether an investor should accept or reject each scenario based on a target discount rate of 9.5%.

As an extension, explore what the impact on your returns would be if you were to take out a loan to fund 50% of the purchase at an annual interest rate of 5%.

Questions to consider

- What does 'time value of money mean'? What is a discount rate? What is a discounted cashflow (DCF)? How does a company determine its discount rate?

- What is an IRR? What impact does debt (taking out a loan to pay for part of an investment) have on the IRR? What impact does receiving income more frequently have on the IRR (e.g. receiving income monthly rather than yearly)?

- What impact does vacant or rent-free periods have on returns?

- How does the yield on sale affect the final return on investment?

Use these questions to guide your analysis, but also feel free to consider other factors that may impact the investment decision. Your final submission should explain your findings clearly and logically, using financial models and sensitivity analysis, where relevant, to support your conclusions.

Project 10: Cosmic Mining

Set by IRIS

Introduction

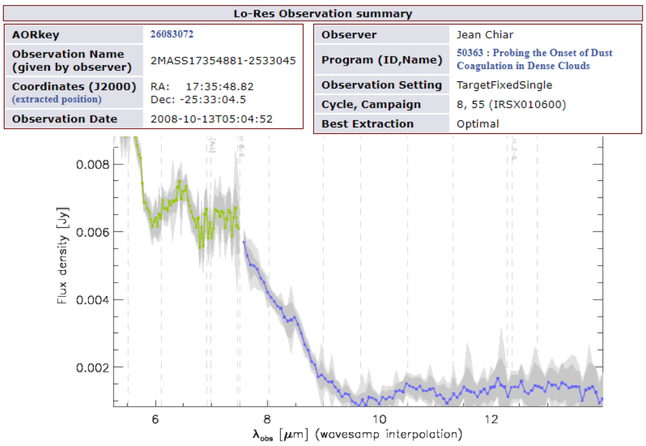
Most of the materials around us were formed by stars living, dying and colliding with each other. However, there are still many questions as to how these elements are made and distributed throughout galaxies like our own Milky Way. Cosmic Mining links to physics and astronomy topics such as electromagnetic spectrum, spectral analysis and stellar evolution.

Problems to be investigated

As part of Cosmic Mining, you will be analysing data from the Spitzer Space Telescope. You will learn how to examine and classify stellar objects based on the light they emit. Your work will contribute to the first fully classified catalogue of these sources, which will be an extremely valuable resource for astronomers.

The ultimate goal, however, is to assist astronomers with the identification and selection of potential targets for the James Webb Space Telescope – the most powerful and complex space telescope to ever be built. This will contribute to the development of an observing proposal which makes the scientific case for pointing the huge space observatory at these objects.

With access to the Combined Atlas of Sources with Spitzer IRS Spectra (CASSIS) you will develop the skills to identify aspects of the recorded spectra from the Spitzer IR satellite. Spizer was the pre cursor to the James Webb telescope and the results from its observations are helping scientists to identify potential targets for the James Webb telescope. Identifying absorption/emission spectra and the prominent elements of these spectra will allow you to identify the observed object as one of the following Galaxy/Young Stellar Object (1/2), Planetary Nebulae, Star, O-Rich evolved Star, Carbon Rich evolved star or more importantly unknown.



Example of an unidentified object observed within the galactic Bulge.

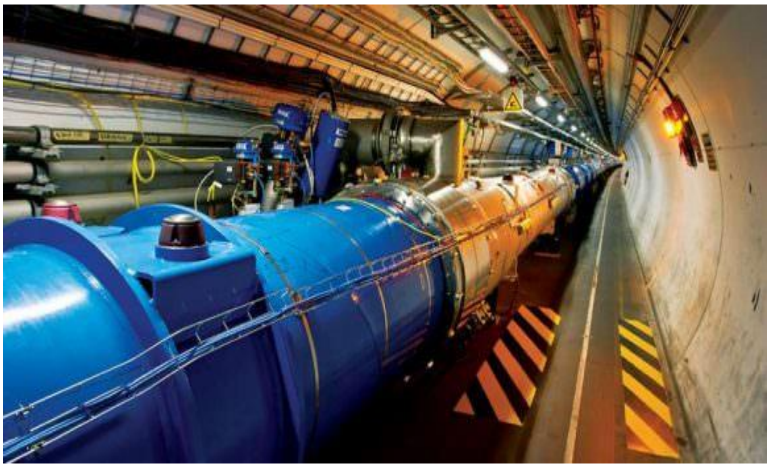
Following this you will be issued with previously unseen data to analyse, the results of which are passed to the James Webb team at STFC based at the Royal Observatory in Edinburgh. Finally, you might wish to undertake research around a specific target, cluster of targets that you have come across.

Project 11: Big Data - ATLAS

Set by IRIS

Introduction

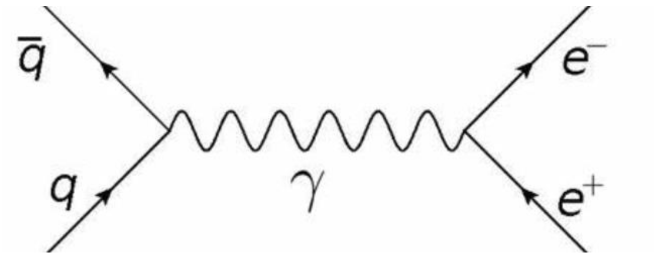
The ATLAS detector is used by particle physicists to observe the results of proton-proton collisions in the Large Hadron Collider at CERN in Geneva, the largest collider of its type in the world. This project will develop critical skills in statistical analysis, Python computer programming, data presentation and interpretation of ATLAS open-source data, including how to find the Higgs Boson.



Problems to be investigated

In this project you will be researching all of the following areas of particle physics

- Particle Accelerators
- Coding in Python
- Computing techniques in high energy physics analysis
- An introduction to Feynman diagrams and Lorentz vectors
- Quark interactions and production of Z^0
- The search for the Higgs Boson



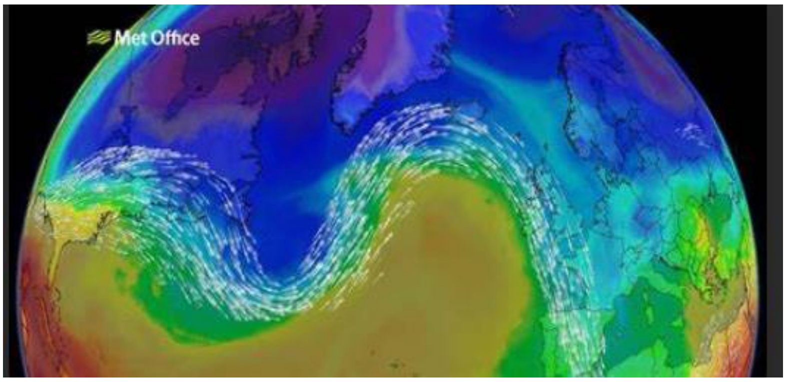
Richard Feynman and the Feynman diagram

Project 12: Mathematics of the Jet Stream

Set by Professor Ian Roulstone

Introduction

Weather forecasters refer frequently, in their television and online broadcasts, to jet streams and their impact on our changing weather patterns. A jet stream is like a river of fast-flowing air, around eight kilometres above the Earth's surface, blowing from west to east. Jet streams influence the wind and pressure at these high altitudes, and in turn these affect things nearer the surface, such as areas of high and low pressure. Therefore, jet streams influence the weather we experience on the ground. Sometimes, like in a fast-moving river, a jet stream's movement is very straight and smooth. However, its movement can buckle and loop, like a river's meander. This will slow things up, making areas of low pressure, along with their bands of rain and blustery winds, move less predictably.



The width of a jet stream is typically a few hundred kilometres and its vertical thickness often less than five kilometres. Wind speeds frequently exceed 90 kilometres per hour (kph), and speeds of up to 400 kph have been recorded!

Jet streams are influenced by temperature differences in the atmosphere, but their motion can be predicted using very basic physics and some novel mathematics. One of the primary mechanisms behind the motion of jet streams is the so-called Coriolis force, which affects the motion of everything on our rotating planet, from golf balls to global weather patterns. The Coriolis force varies in strength with latitude: it is zero along the Equator and reaches a maximum at the North and South Poles. This variability of the Coriolis force has a major impact on the motion of jet streams.

Problems to be Investigated

In this project you will learn how basic physics explains some of the largest-scale weather patterns we experience. Knowledge of the physical principles behind weather will be supplemented by the mathematical concepts that quantify our understanding of swirling patterns of air. When we then combine the physics with the mathematics, you will see immediately why jet streams can be anticipated and why they have such a major impact on the changes to our day-to-day weather.

You will also discover how energy from the Sun, combined with the rotation of our planet, drives the large-scale patterns observed in our weather. The physics of heat and moisture, together with Newton's laws of motion applied to the atmosphere, explain how this all works. You will then focus attention on the mathematics that describes the rotation, or swirl, of a fluids such as air and water (yes, we can think of air as a fluid!). This swirling is known as vorticity and weather systems are typically huge vortices in the atmosphere. When we describe the large-scale weather patterns using the concept of vorticity, we begin to see how to describe the motion of jet streams

Project 13: Convergence and divergence of infinite series

Set by Dr Jonathan Deane

Introduction

It won't surprise you to learn that it is possible to add together an infinite number of positive numbers to get an infinite result. For instance, if $S_1 = 1 + 1 + 1 + \ldots$ where we interpret the ... to mean 'carry on for ever', then S_1 is infinite. On the other hand, this is not always so: consider

$$S_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

which is finite. You may recognise this as an example of a geometric series. The infinite series S_1 is said to diverge, whereas S_2 is said to converge. In the general case, we define

$$S := a_1 + a_2 + a_3 + \ldots = \sum_{i=1}^{\infty} a_i.$$

Intuitively, in order for the series to converge, the a_i need, eventually, to decrease sufficiently fast.

Suggested areas for investigation

1. Work through some examples of series that converge, for example

$$\sum_{i=0}^{\infty} (-1)^i x^i \text{ where } 0 < x < 1; \qquad \sum_{i=1}^{\infty} \frac{i}{2^i}; \qquad \sum_{i=1}^{\infty} \frac{1}{i(i+1)}$$

plus variations. Since all these examples converge, find their sums. The third is an example of a telescoping series. When you see how it works, you may be able to think of other examples that can be summed in the same way.

2. Define the Harmonic series

$$H(n) := \sum_{i=1}^{n} 1/i.$$

and prove that

$$H(n) \to \infty \text{ as } n \to \infty.$$

In other words, prove that this series diverges. The standard proof of this result is a real classic.

3. What can you say about $\lim_{n\to\infty} (H(n) - \ln n)$?

4. The sum of the reciprocals of the prime numbers

Project 14: Continued fractions

Set by Dr James Grant

Introduction

A continued fraction is an expression either of the finite form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where a_0 is an integer, and $a_1, a_2, \ldots, a_n$ are positive integers, or the infinite form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where a_0 is an integer, and a_k are positive integers for all $k \geq 1$. For example,

$$5 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{3}}}. \qquad (1)$$

is a finite continued fractions, while

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}} \qquad \text{and} \qquad 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} \qquad (2)$$

are infinite continued fractions

Finite continued fractions can be evaluated as a standard fraction, beginning at the tail and "working upwards". For example,

$$5 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{3}}} = 5 + \cfrac{1}{4 + \cfrac{1}{\frac{22}{3}}} = 5 + \cfrac{1}{4 + \frac{3}{22}} = 5 + \cfrac{1}{\frac{91}{22}} = 5 + \frac{22}{91} = \frac{477}{91},$$

Which is the solution to continued fraction (1)

In general, the value of infinite continued fractions cannot be calculated explicitly. There are, however, special cases where it is possible. For example, the continued fractions in (2) are equal to the Golden Ratio $\varphi = \frac{1+\sqrt{2}}{2}$ and $\sqrt{2}$ respectively.

Continued fractions can be used to give the best rational approximation to irrational numbers. Given an irrational number x, its continued fraction expansion will be infinite. If, however, we cut off this expansion after a finite number of steps, then the resulting (finite) continued fraction will be the best rational approximation to x. For example

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\ddots}}}}}.$$

Terminating this expansion at the first fractional level gives $3 + \frac{1}{7} = \frac{22}{7}$ which is the best rational approximation to π with denominator less than or equal to 7. Similarly, terminating after the second term gives $3 + \frac{1}{7+\frac{1}{15}} = 3 + \frac{15}{306} = \frac{333}{106}$ which is the best approximation to π by a rational number with denominator $\leqslant 106$. Continuing, one generates successively more accurate rational approximations $\frac{355}{113}, \frac{103993}{33102}$ etc to π.

This project includes understanding in what sense continued fractions such as (2) represent the corresponding irrational number φ and $\sqrt{2}$, and finding an algorithm for constructing continued fractions that approximate a given real number.

Areas of investigation

- Express the following rational numbers as continued fractions

  $a)\,\frac{22}{7}$ $\qquad b)\,\frac{38}{11}$ $\qquad c)\,\frac{29}{24}$

- Show that any rational number can be expressed as a finite continued fraction. Find algorithms for expressing a given finite continued fraction as a single fraction and vice versa.

- Investigate the definition of a convergent sequence and some theorems that may be used to prove that a given sequence is convergent.

- Investigate the sense in which continued fractions yield best rational approximations to irrational numbers.

The core of the project is to:

- Prove that every positive real number has a terminating or convergent infinite continued fraction

- Prove that every continued fraction represents a real number.

- Implement algorithms for converting a continued fraction into a real number and vice versa.

Further work (some of which you may wish to do before completing the core) would be to prove that (2) do represent the Golden Ratio and $\sqrt{2}$, respectively, and, more generally, to characterise real numbers that can be expressed as 'repeating' continued fractions. You might also find the continued fraction representations of, for example, e, π, ...

Project 15: Latin Squares

Set by Professor Ed Godolphin

Introduction

A Latin square is an n×n array, containing n different symbols arranged so that each symbol occurs once in each row and once in each column. An example of a 5 × 5 Latin square is:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \end{matrix}$$

Two Latin squares of order n are said to be orthogonal if, when one square is superimposed on the other, every ordered pair of symbols occurs exactly once. An example of a pair of orthogonal Latin squares of order 3 is given by:

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}$$

A set $\{L_1, L_2, \ldots, L_t\}$ of $t \geq 2$ Latin squares of order n is said to be a set of Mutually Orthogonal Latin Squares (MOLS) if L_i is orthogonal to L_j for all $i \neq j$. Sets of MOLS have various uses including: in the design of experiments (statistics); to construct error detecting and correcting codes; for tournament scheduling.

Investigation

- Explore the history of Latin squares, including Euler's problem of arranging the 36 officers of six army regiments in a manner consistent with a pair of MOLS of order 6.

- Derive an upper bound for the number of squares, t, in a set of MOLS of order n.

- Find out how to construct sets of MOLS that achieve this upper bound for n prime.

- (Harder) For $n = p^r$, where p is prime and $r \geq 2$, demonstrate the construction of a maximal set of MOLS using an irreducible polynomial of degree r over the field F_p.

- Use the Kronecker product ($\otimes$) construction to obtain a pair of MOLS of order 12.

- Prove that a pair of MOLS can be found for every $n \equiv 0, 1, 3 \pmod 4$.

Project 16: Fresnel Integrals

Set by Dr Richard Harrison

Introduction

Fresnel integrals have wide ranging applications in mathematical physics, applied mathematics and many branches of engineering. Examples include applications to modelling wave-like behaviour such as electromagnetic wave propagation, particularly in optics and wave-particle duality models. In the computer science domain, they can be applied to image processing and reconstruction. Yet another use is optimal path planning in civil engineering, designing transition curves in rail and road infrastructure. The Fresnel integrals (with real arguments) are often defined as;

$$S(a) = \int_0^a \sin\left(\frac{\pi x^2}{2}\right) dx$$

$$C(a) = \int_0^a \cos\left(\frac{\pi x^2}{2}\right) dx$$

$$a, x \in \mathbb{R}$$

Although the scope of their applications is very wide, there are no analytical solutions to these integrals and they can only be evaluated using special functions and numerical integration.

Problems to investigate

The purpose of this task is to investigate the accuracy of some different numerical techniques for evaluating the Fresnel *sine* integral at the value $S\left(\frac{\pi}{4}\right)$. It is recommended that MATLAB is used since it has a dedicated function, fresnel(z) for evaluating this integral. You should investigate using;

- The trapezium rule
- Simpsons rule
- A simple Monte Carlo integration technique $\int_0^{\pi/4} f(u)du = \frac{1}{N}\sum_{i=1}^{N} f(u_i)$ where N is the number of random samples chosen on the domain of $\left[0, \frac{\pi}{4}\right]$
- A power series expansion based on a modification of the Maclaurin's series for $\sin x$.

Project 17: Reconstruction of the cerebrospinal fluid space geometry using analytical methods

Set by Dr Serge Cirovic

Introduction

Cerebrospinal fluid (CSF) is a water-like fluid which surrounds the brain and spinal cord and is also contained within the cavities (ventricles) of the brain shown in Figure 1 a).

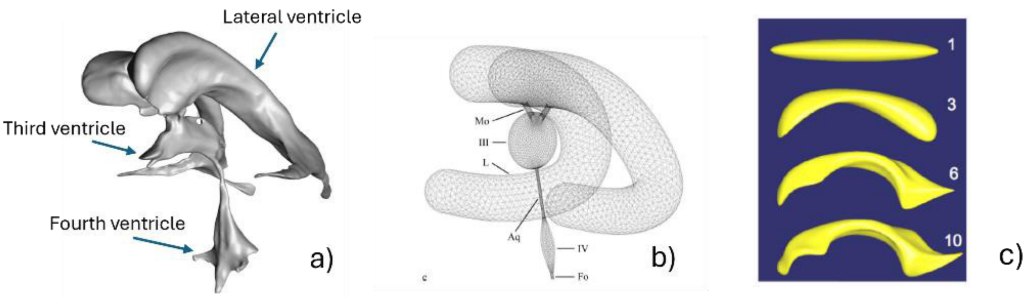


Figure 1.a) *Anatomically accurate geometry of the ventricular system of the brain [1].* ***b)*** *An equivalent idealized geometry of the ventricular system [1].* ***c)*** *Spherical harmonics representation of a lateral ventricle with 1, 3, 6, and 10 harmonics [2]*

Anomalies in the CSF space anatomy cause neurological disorders such as hydrocephalus. Due to the geometric complexity of this space, it is difficult to track down potential anomalies. The idea of this project is to develop ways of approximating patient-specific CSF space as a collection of mutually connected simpler geometries defined in analytical form (and therefore easily quantifiable). An example is shown in Figure 1 b) where the third ventricle is approximated to a sphere of the same volume as the actual anatomical structure shown in Figure 1 a). At this level of approximation coordinates of its centre and the radius fully quantify the ventricle. There are also more sophisticated approaches where orthogonal functions are used to represent portions of the CSF space to a different level of accuracy. Figure 1 c) shows the shape of a lateral ventricle approximated with 1,3,6, and 10 spherical harmonics.

Problems to investigate

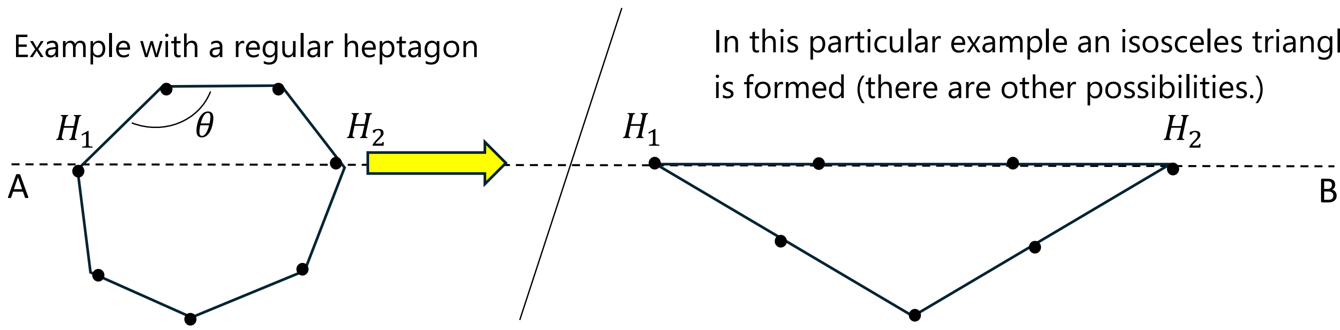
In this project the goal will be to explore different approaches for presenting the CSF space analytically. In simple terms you will: perform search of relevant literature, come up with a strategy for dividing CSF space into parts which can be associated with simpler shapes, and devise methods for fitting simple shapes to anatomically accurate geometry. Ideally you will obtain a single well-connected idealized volume for the entire CSF space, but you may decide to focus only on specific portions of the CSF system.

Project 18: Transforming polygons with hinged vertices

Set by Dr Pat Warner

Introduction

Imagine that a regular convex polygon is made from rigid metal wire and that there is a hinge at each vertex so that the internal angle $\theta : 0 \leq \theta \leq 180^o$ can change in the plane of the polygon. An example is the regular convex heptagon shown on the left of the figure below. The polygon cannot be bent out of the flat plane of the page. Choose any two *non-consecutive* vertices on the perimeter, H_1 and H_2. Holding H_1 in place pull H_2 away along a line *AB*, to elongate the polygon, as shown below. In this case, the wire frame of the heptagon is transformed to an isosceles triangle.



If a regular polygon has an even number of sides, it is always possible, depending on the choice of the two non-consecutive vertices, to transform it in a similar way to the above so that it diminishes to two coincident parallel lines of equal length, collinear with the line AB. For example, the diagonally opposing vertices of a square. Your task is to investigate the conditions under which it is possible transform an *irregular convex polygon* into two coincident parallel lines of equal length with the side lengths defined by an arithmetic progression with integral coefficients.

Problems to investigate

- An *irregular convex* polygon has 14 sides, $s_1, s_2, \ldots s_{14}$. The side lengths are in an arithmetic sequence with s_1 being the shortest side and s_{14} the longest. Given that $s_5 = 11$ cm and $s_{12} = 25$ cm, find the perimeter of the polygon. Is it possible to transform your irregular 14-sided polygon so that it diminishes to two coincident parallel lines?

- What happens by keeping the number of sides fixed at 14 but changing the integral coefficients of the arithmetic sequence, such as the shortest side length and the common difference between consecutive side lengths?

- The broadest generalisation is then to consider an *n*-sided irregular convex polygon where n is an integer. What is the least value of *n* that will enable the transformation, and the corresponding parameters of the arithmetic sequence? Can you determine a formula or algorithm to test whether an *n*-sided irregular convex polygon with side lengths defined by a particular arithmetic sequence can be transformed to two coincident parallel lines?

Project 19: Observations and Analysis of Avian Perching Manoeuvres

Set by Dr Olaf Marxen

Introduction

Despite the recent proliferation of computation, sensing, and actuation at the small scale, aerial robots are still not able to match the agility and adaptability of natural flight. This is due, in part, to our limited knowledge of the mechanics that characterise outdoor flight. The avian perching manoeuvre, during which a bird decelerates to land on a spot, such as a tree branch, is an example of a bird's superior capability. In this project, data characterising bird motion during perching manoeuvres will be collected and analysed.

The rationale behind this project is that we would like to understand, and be able to model, the aerodynamics underlying "bird landing", with a view to enable future fixed-wing unmanned aerial vehicles to perform such perching manoeuvres. In addition to biomimetic flight of aerial robots and a deeper understanding of biological flight behaviours, other applications can be envisioned, such as wind-turbines that learn best responses to alleviate gust loads during operation and hence avoiding costly showdowns.

Problems to investigate

Your tasks will be based around collecting and analysing qualitative and quantitative data of perching birds. You could find nest-cam streaming data on the internet or make your own outdoor observations of perching birds, taking your own video data if you have access to a mobile (e.g. phone) camera. You can also choose to work with video data that we took during a recent visit to the Hawk Conservancy. In the video footage, you would need to find those instances during which perching occurs and extract individual frames at fixed time intervals. You could look into automating this process using machine-learning tools available online. Using extracted frames, you could then perform statistical analysis to characterize the motion, such as the frequency of flapping during the final approach or the type of motion (bird wing rotation or wing folding). You could also attempt to deduce trajectory data from videos.

Possible outcomes

One project outcome may be, for instance a report containing a set of statistical charts representing the frequency of a particular motion of the bird during perching, for example the flapping frequency. Alternatively, if you developed a machine-learning model for tracking the motion of the bird in video data, you could demonstrate the use of this model during a presentation. Your conclusions should highlight specific features of avian perching deduced from your analysis.

Project 20: Green energy from waste: quantifying and modelling the biogas potential from waste in the UK.

Set by Dr Michael Short

Introduction

Anaerobic digesters are now common across the UK and provide an effective waste-to-energy route for biogenic waste. Anaerobic digestion's 2 primary products are biogas and digestate (a nitrogen-rich fertiliser). Biogas is a mixture of around 60 % methane and 40 % CO2 that is highly versatile as a feedstock. Biogas can be combusted to directly supply heat or electricity, used to produce combined heat and power (CHP), upgraded to biomethane for injection into the gas grid or to produce transportation fuels, or further processed to produce flexible feedstocks such as syngas or hydrogen. The UK government targets high amounts of green gas, but it is not clear how much suitable waste there is and where these collection and processing facilities will be.

Problems to investigate

In this project, using publicly available data, you will be asked to quantify and estimate the overall potential for biogas in the UK from waste streams. You will be tasked with finding data from various sources (mostly government websites) on the potential waste feedstocks that may be used to produce biogas across the UK and look at the current production, future production plans from the UK Net Zero and Biomass strategy documents and assess whether the targets are realistic and identify key challenges with attaining these targets. You should have or be in the process of developing, skills and knowledge in statistics and be able to work with (or learn to work with) Microsoft Excel or similar data processing applications.

Possible outcomes

One project outcome may be, for instance a report containing a set of statistical charts representing the availability of different waste feedstocks in different locations, with their current and future potential gas yields. Alternatively, you may want to develop a model based on population sizes and/or farming practices for predicting gas availability. There should also be statistical visualisations such as confidences on estimates and comparisons between them.

Project 21: Predicting hourly gas demands in the grid

Set by Dr Michael Short

Introduction

The UK has a bold hydrogen strategy seeking to inject hydrogen into the natural gas grid to reduce our reliance on fossil fuels with a cleaner-burning alternative. However, this shift will require large infrastructure investments to generate the hydrogen, energy, and the water required to do so. Using predictions on costs of various technologies over time, this project will look at the potential demands for hydrogen and cost to deliver such large-scale infrastructure along different timelines.

Problems to investigate

In this project, using publicly available data, academic literature, and government and International Energy Agency reports, you will be asked to quantify and estimate the overall hydrogen demand and cost in the UK, considering different potential hydrogen generation technologies, hydrogen colours, and the potential for large-scale delivery of cost-effective hydrogen. You will be tasked with finding data from various sources (mostly government websites) on the different technologies, their costs and potential for scale-up and look at the current production, future production plans from the UK Net Zero strategy documents and assess whether the targets are realistic and identify key challenges with attaining these targets. You should have or be in the process of developing, skills and knowledge in statistics and be able to work with (or learn to work with) Microsoft Excel or similar data processing applications.

Possible outcomes

One project outcome may be, for instance a report containing a set of statistical charts representing the availability of different feedstocks and demands in different locations, with their current and future potential costs and delivery methods. Alternatively, you may want to develop a model based on population sizes, feedstock availability, and technology sizing and costing. There should also be statistical visualisations such as confidences on estimates and comparisons between them.

Project 22: Securing the future: how to protect against quantum attackers

Set by Dr Daniel Gardham

Introduction

Cryptography is the practice of using mathematical codes to protect information and keep it secure. It underpins the authentication protocols that keep our bank accounts, emails and even social media secure. However, recent advancements in quantum computing threaten the cryptographic protocols we have in place today. As a result, much research has been undertaken in the last 15 years to prepare our computer systems to be resilient to these attacks. This has resulted in rapid growth of an area called post-quantum cryptography, that uses new areas of maths to build cryptographic primitives and protocols.

However, these new protocols can vary wildly compared to their classical counterparts, for instance they can have much higher or lower communication and storage costs, they may be faster or slower to execute, and other more context-dependent constraints. Understanding these trade-offs is critical before post-quantum cryptography can be globally adopted, as they may not be fit for all purposes.

Problems to investigate

Your tasks will be based around comparing the "cost" of upgrading to post-quantum security. You will identify various classical (i.e. quantum insecure) protocols and establish the performance changes between them and post-quantum variations. This can either be done in a broader style where you perform a literature review over many different primitives, and synthesize a generalised cost model, or you could implement and benchmark your own results for a single protocol. A different path could be to look at the theoretical properties of various schemes and investigate whether high-security is achievable as we transition to a post-quantum society.

To be successful in this project, you should have a keen interest in developing applied mathematical skills in the area of cyber security. Some knowledge programming (ideally python) is also required if you wish to benchmark your own implementations. Post-quantum cryptography comes in many flavours, but the style most suited to this project utilises matrix algebra. You should therefore have an interest in advancing your knowledge in this area.

Project 23: Articulatory phonetics GenAI with Natural Language Prompts

Set by Dr Xiatian Zhu

Introduction

Recent advancements in Generative AI have significantly impacted text-to-speech (TTS) synthesis. However, generating audio that authentically conveys human-like emotions and expressiveness remains a challenge. This project aims to investigate and develop a series of innovative techniques for emotion-rich audio generation from natural language style prompts with articulatory phonetics information, enhancing emotional expressiveness, richness, and diversity in speech generation.

Problems to investigate

1. Literature Review: Study recent advancements in expressive TTS, focusing on natural language prompts and articulatory phonetics

2. Dataset Exploration: Identify and analyse datasets containing emotional speech samples, style descriptions, and articulatory measurements.

3. Model Development: Create a generative AI model that combines instruction-guided emotional synthesis with articulatory control.

4. Performance Evaluation: Assess the model's ability to generate high-quality, natural-sounding speech across various emotional styles and articulatory configurations.

5. User Interaction Design: Develop a user interface for real-time interaction using natural language prompts and articulatory controls

Possible outcomes

- A demonstration system showcasing the model's capabilities in generating emotionally expressive speech.

- A novel AI model for emotion-rich TTS that integrates natural language prompts and articulatory phonetics

- Documentation detailing the model architecture, training process, and potential applications.

- An analysis of the model's performance compared to existing TTS systems, including both advantages and limitations.

Project 24: The Mathematics of Change Ringing

Set by Mr Colin Wyld

Introduction

The origins of what we call change ringing, the uniquely British way of ringing church bells, lie in the sixteenth century when church bells began to be hung with a full wheel enabling the bell to swing in a full circle and back again. This gave ringers control of their bell, which allowed sets of bells (rings) to be rung in a continuously changing pattern. Music is created by moving bells up and down the ringing order to a defined sequence known as a method.

Change Ringing is based on permutations of sets of integers. Each bell is given a number, starting with 1 for the bell with the highest note, 2 for the next one down the scale and so on until the one with the deepest note has a number "N" equal to the number of bells being rung. The way in which bells are rung separates naturally into two distinct actions known as "handstroke" and "backstroke" with each bell sounding only once during each stroke. All the bells are rung at handstroke then all at backstroke then at handstroke again and so on. This naturally divides the ringing into discrete sequences of N notes in which every bell rings once and only once. The order in which they ring each time is defined by a permutation of the numbers 1 to N. The position, within the sequence of notes, when each bell sounds is the same as the position of the corresponding number in the permutation (written from left to right). By working out sequences of permutations on paper or in a computer we can compose pieces that can be rung on church, or hand, bells.

The way bells are hung, and rung, also restricts how each discrete sequence of notes can be derived from its predecessor. In conventional ringing no bell will change its position by more than one place at a time. For example, the bell ringing in 2nds place in one sequence may, in the next, ring one place earlier in 1st place, in the same place i.e. 2nds or one place later in 3rd place. Meeting this requirement limits the operations we can use to change (hence change ringing) a sequence, or its corresponding permutation, into the one that follows. An example is given below, reading each change ("1234" etc.) from left to right

1234 2143 2413 4231 4321 3412 3142 1324 1342 3124 3214 2341 2431 4213 4123 1432 1423 4132 4312 3421 3241 2314 2134 1243 1234

The sequences of permutations to be rung are known as touches (fewer than 5000 changes – the touch above has 24 different changes) or peals (5000 or more changes). The object that all composers of peals and touches try to achieve, is to create a sequence of permutations starting and finishing with the same permutation, usually a descending scale with the bells ringing in numerical order from 1 to N known as rounds, as in the example above, 1234. Each permutation is derived from its predecessor using a ringable transposition and no permutation within the piece is repeated until the permutation at the start is repeated at the end. This is more challenging if all the permutations of a set of integers are to be included.

That is not however the whole story. For the last four centuries or so ringers have been looking for ever more interesting ways of ringing peals etc. Often this means something that is more challenging physically or mentally, but it also includes greater musical content or more elegant solutions to problems that have already been solved. The advent of the electronic computer has enabled us to tackle problems that were previously intractable because of the amount of arithmetic required for their solution. There remain however problems where even with a computer the amount of arithmetic is so great that they can only be tackled by combining an understanding of the symmetries within sets of permutations and efficient computer programming.

Projects and Methodology

The following problems, from which you are invited to choose one, are of this type. You will need to work out how to break them down into manageable components and how to minimise the processing required to analyse them, using computer programming.

1. In 1997 Andrew Johnson composed, using only common bobs, a peal of Stedman Triples exploiting a 10-fold symmetry to reduce the field to a manageable size 2^{84} rather than 2^{840} Andrew's original search was based on a limited analysis of one of the potential sets of solutions. Your project will be to analyse more fully the set that Andrew analysed using his solution to verify the effectiveness of your code and, if time, analyse many other sets

2. There are other symmetries that might yield similar results. Adapt the analysis developed for project 1 to work on these other symmetries.

3. The complete set of permutations of any set of integers 1 to N is a mathematical group known as the symmetric group S_N. In 1906 a famous ringer, and composer, John Carter published a composition for 7 bells which he called Scientific Triples. This was based on a subgroup of S_7 although Carter kept this secret. The puzzle he set was not solved for nearly 70 years and that solution was not published until 2018. Other subgroups of S_7 may form the basis of other families of compositions. Your project will be to analyse S_5, S_6 and S_7 to identify all the different subgroups and how they relate to each other.

Project 25: Can a simple computer program be used to predict musical preferences?

Set by Alexia Beale

Introduction

The choice of music to play for a target audience is an important decision for many situations, for example personal auto-generated playlists, DJing and musical therapy.

In order to predict which music would be best to play, it is necessary to understand which features of music have the greatest influence on musical preferences. A computer program may be written to aid in this decision-making process, using fuzzy logic, which is a tool used to combine many numerical values to give a single value output.

Problems to investigate

- Investigating features of a music track, and determining which of these are the most important and have the greatest impact on whether a user will like specific pieces of music
- Creating a computational tool which inputs test data and uses this to predict how much a user will like a particular music track or alternatively ranks several music tracks into order of preference.
- Making use of computational functions to identify properties of a music track to fully automate the code
- Researching alternative computational methods to predict musical preferences

Possible outcomes

- The creation of the computational tool and the discovery of which features of music have the strongest effect on musical preferences
- Analysis of the effectivity of the computational tool

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